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Analysis of two-operator boundary-domain integral equations for variable-coefficient Dirichlet and Neumann problems in 2D with general right-hand side

Markos F. Yimer¹ and Tsegaye G. Ayele^{1*}

¹Department of Mathematics, College of Natural and Computational Sciences,
Addis Ababa University, P. O. Box 1176, Addis Ababa Ethiopia.

E-mail: markos.fisseha@aau.edu.et; tsegaye.ayele@aau.edu.et

ABSTRACT: The Dirichlet and Neumann boundary value problems for the linear second-order scalar elliptic differential equation with variable coefficients in a bounded two-dimensional domain are considered. The right-hand side the PDE belongs to $H^{-1}(\Omega)$ or $\tilde{H}^{-1}(\Omega)$, when neither classical nor canonical conormal derivatives of solutions are well defined. The two-operator approach and appropriate parametrix (Levi function) are used to reduce each of the problems to two different systems of two-operator boundary-domain integral equations (BDIEs). Although the theory of BDIEs in 3D is well developed, the BDIEs in 2D need a special consideration due to their different equivalence properties. As a result, we need to set conditions on the domain or on the associated Sobolev spaces to ensure the invertibility of the corresponding parametrix-based integral layer potentials and hence the unique solvability of BDIEs. The equivalence of the two-operator BDIE systems to the original problems, BDIE system solvability, solution uniqueness/nonuniqueness and invertibility BDIE system are analyzed in the appropriate Sobolev spaces. It is shown that the BDIE operators for the Neumann BVP are not invertible, and appropriate finite-dimensional perturbations are constructed leading to invertibility of the perturbed operators.

Keywords/phrases: Analysis, Boundary-Domain Integral Equations, Parametrix, Partial Differential Equations, Variable Coefficients.

INTRODUCTION

Partial differential equations (PDEs) with variable coefficients often arise in mathematical modelling of inhomogeneous media (e.g. functionally graded materials or materials with damage induced inhomogeneity) in solid mechanics, electromagnetics, thermal conductivity, fluid flows through porous media, and other areas of physics and engineering. Generally, explicit fundamental solutions are not available if the PDE coefficients are not constant, preventing reduction of boundary value problems (BVPs) for such PDEs to explicit boundary integral equations (BIEs), which could be effectively solved numerically. Nevertheless, for a rather wide class of variable-coefficient PDEs it is possible to use instead an explicit parametrix (Levi function) associated with the fundamental solution of the corresponding frozen-coefficient PDEs, and reduce BVPs for such PDEs to systems of boundary-domain integral equations (BDIEs) for further numerical solution of the latter, see for example Chkadua *et al.* (2009,

2011); Mikhailov (2002, 2005b, 2006). Still this (one-operator) approach does not work when the fundamental solution of the frozen-coefficient PDE is not known explicitly (as e.g. in the Lamé system of anisotropic elasticity). To overcome this difficulty, one can apply the so-called two-operator approach, formulated in Mikhailov (2005a) for a certain nonlinear problem, that employs a parametrix of another (second) PDE, not related with the PDE in question, for reducing the BVP to a BDIE system. Since the second PDE is rather arbitrary, one can always choose it in such a way, that its parametrix is known explicitly. The simplest choice for the second PDE is the one with an explicit fundamental solution. For a function from the Sobolev space $H^1(\Omega)$, a classical conormal derivative in the sense of traces may not exist (Mikhailov, 2015, Appendix A). However, when this function satisfies a second order PDE with a right-hand side from $H^{-1}(\Omega)$, the generalized conormal derivative can be defined in the weak sense, associated with the first Green identity

*Author to whom correspondence should be addressed.

and an extension of the PDE right-hand side to $\tilde{H}^{-1}(\Omega)$, see (McLean, 2000, Lemma 4.3) and (Mikhailov, 2011, Definition 3.1). Since the extension is not unique, the conormal derivative appears to be an operator that is not unique, which is also nonlinear in u unless a linear relation between u and the PDE right-hand side extension is enforced. This creates some difficulties in formulating the BDIEs. These difficulties are addressed in Mikhailov (2015, 2018) presenting formulation and analysis of direct segregated BDIE systems equivalent to the Dirichlet and Neumann problems for the divergent-type PDE with a variable scalar coefficient and a general right-hand side. This needed a non-trivial generalization of the third Green identity and its conormal derivative for such functions, which extends the approach implemented in Mikhailov (2006); Chkadua *et al.* (2009, 2010, 2011); Mikhailov (2013) for the PDE right-hand side from $L_2(\Omega)$. In Ayele Tsegaye (2021b), using the two-operator approach in settings different from those in Ayele Tsegaye and Mikhailov (2010, 2011), a generalization of the two-operator third Green identity and its conormal derivative is derived and the two-operator BDIEs for variable-coefficient Dirichlet, Neumann and mixed BVPs are analyzed in 3D.

Nowadays, the theory of BDIEs in 3D is well developed (Mikhailov, 2002, 2005a; Chkadua *et al.*, 2009, 2010, 2011), but the BDIEs in 2D need a special consideration due to their different equivalence properties. As a result, we need to set conditions on the domain or on the associated Sobolev spaces to ensure the invertibility of the corresponding parametrix-based integral layer potentials and hence the unique solvability of BDIEs (Dufera Tamirat and Mikhailov, 2015; Ayele Tsegaye *et al.*, 2017, 2019; Ayele Tsegaye, 2021a; Ayele Tsegaye and Bekele Solomon, 2023; Ayele Tsegaye and Dagnaw Mulugeta, 2021a).

In this paper, we extend the results in (Ayele Tsegaye and Yimer Markos, 2023; Ayele Tsegaye, 2021a), and consider the Dirichlet and Neumann BVPs for the linear second-order scalar elliptic differential equation with variable coefficient in a two-dimensional bounded domain. The PDE right-hand side belongs to $H^{-1}(\Omega)$ or $\tilde{H}^{-1}(\Omega)$ when neither classical nor canonical conormal derivatives of solutions are well defined. The two-operator approach and appropriate parametrix (Levi function) are used to reduce each problem into two different systems of BDIEs. The properties of the corresponding potential operators are investigated.

The equivalence of the two-operator BDIE systems to the original problems, BDIE system solvability, solution uniqueness/nonuniqueness and invertibility BDIE system are analyzed in the appropriate Sobolev spaces. It is shown that the BDIE operators for the Neumann BVP are not invertible, and appropriate finite-dimensional perturbations are constructed leading to invertibility of the perturbed operators.

PRELIMINARIES

Conormal derivatives. Let Ω be a domain in \mathbb{R}^2 bounded by a smooth curve $\partial\Omega$. Consider the scalar elliptic differential equation, which for sufficiently smooth function u and $x \in \Omega$ has the following strong form,

$$Au(x) := A(x, \partial_x)u(x) = \sum_{i=1}^2 \frac{\partial}{\partial x_i} \left(a(x) \frac{\partial u(x)}{\partial x_i} \right) = \tilde{f}(x), \quad (1)$$

where u is unknown function and \tilde{f} is a given function in Ω . We assume that $a \in C^\infty(\mathbb{R}^2)$ and

$$0 < a_{\min} \leq a(x) \leq a_{\max} < \infty, \quad \forall x \in \mathbb{R}^2.$$

In what follows $\mathcal{D}(\Omega) = C_0^\infty(\Omega)$, $H^s(\Omega) = H_2^s(\Omega)$, $H^s(\partial\Omega) = H_2^s(\partial\Omega)$ are the Bessel potential spaces, where $s \in \mathbb{R}$ is an arbitrary real number (McLean, 2000; Lions and Magenes, 1972). We recall that H^s coincides with the Sobolev-Slobodetski spaces W_2^s for any nonnegative s . We denote by $\tilde{H}^s(\Omega)$ the subspace of $H^s(\mathbb{R}^2)$,

$$\tilde{H}^s(\Omega) := \{g : g \in H^s(\mathbb{R}^2), \text{supp}(g) \subset \bar{\Omega}\}$$

while $H^s(\Omega)$ denotes the space of restriction on Ω of distributions from $H^s(\mathbb{R}^2)$,

$$H^s(\Omega) = \{r_\Omega g : g \in H^s(\mathbb{R}^2)\}$$

where r_Ω denotes the restriction operator on Ω . We will also use the notation $g|_\Omega := r_\Omega g$. We denote by $H_{\partial\Omega}^s$ the following subspace of $H^s(\mathbb{R}^2)$ (and $\tilde{H}^s(\Omega)$),

$$H_{\partial\Omega}^s := \{g : g \in H^s(\mathbb{R}^2), \text{supp}(g) \subset \partial\Omega\}. \quad (2)$$

From the trace theorem (Lions and Magenes, 1972; Dautray and Lions, 1990; McLean, 2000) for $u \in H^1(\Omega)$, it follows that $\gamma^+ u \in H^{\frac{1}{2}}(\partial\Omega)$, where $\gamma^+ = \gamma_{\partial\Omega}^+$ is the trace operator on $\partial\Omega$ from Ω . Let also $\gamma^{-1} : H^{\frac{1}{2}}(\partial\Omega) \rightarrow H^1(\Omega)$ denote a (non-unique) continuous right inverse to the trace operator γ^+ , i.e., $\gamma_{\partial\Omega}^+ \gamma_{\partial\Omega}^{-1} w = \gamma^+ \gamma^{-1} w = w$ for any $w \in H^{\frac{1}{2}}(\partial\Omega)$, and $(\gamma^{-1})^* : \tilde{H}^{-1}(\Omega) \rightarrow H^{-\frac{1}{2}}(\partial\Omega)$ is continuous operator dual to γ^{-1} ,

i. e., $\langle (\gamma^{-1})^* \tilde{f}, w \rangle_{\partial\Omega} := \langle \tilde{f}, \gamma^{-1}w \rangle_{\Omega}$ for any $\tilde{f} \in \tilde{H}^{-1}(\Omega)$ and $w \in H^{\frac{1}{2}}(\partial\Omega)$.

For $u \in H^2(\Omega)$, we denote by T_a^+ the corresponding canonical (strong) conormal derivative operator on $\partial\Omega$ in the sense of traces,

$$\begin{aligned} T_a^+ u &:= \sum_{i=1}^2 a(x)n_i(x)\gamma^+ \frac{\partial u(x)}{\partial x_i} \\ &= a(x)\gamma^+ \frac{\partial u(x)}{\partial n(x)}, \end{aligned} \tag{3}$$

where $n(x)$ is the outward to Ω unit normal vector at the point $x \in \partial\Omega$. However, the classical conormal derivative operator is generally, not well defined if $u \in H^1(\Omega)$, see (Mikhailov, 2015, Appendix A). For $u \in H^1(\Omega)$, the PDE Au in (1) is understood in the sense of distributions,

$$\langle Au, v \rangle_{\Omega} := -\mathcal{E}_a(u, v), \quad \forall v \in \mathcal{D}(\Omega), \tag{4}$$

where

$$\mathcal{E}_a(u, v) := \int_{\Omega} a(x)\nabla u(x) \cdot \nabla v(x) dx$$

is a symmetric bilinear form and the duality brackets $\langle g, \cdot \rangle_{\Omega}$ denote the value of a linear functional (distribution) g , extending the usual L_2 inner product. Since the set $\mathcal{D}(\Omega)$ is dense in $\tilde{H}^1(\Omega)$, the above formula defines a continuous operator $A : H^1(\Omega) \rightarrow H^{-1}(\Omega) = [\tilde{H}^1(\Omega)]^*$,

$$\begin{aligned} \langle Au, v \rangle_{\Omega} &:= -\mathcal{E}_a(u, v), \\ &\forall u \in H^1(\Omega), \forall v \in \tilde{H}^1(\Omega). \end{aligned} \tag{5}$$

Let us consider also the operator

$$\check{A} : H^1(\Omega) \rightarrow \tilde{H}^{-1}(\Omega) = [H^1(\Omega)]^*$$

such that

$$\begin{aligned} \langle \check{A}u, v \rangle_{\Omega} &:= -\mathcal{E}_a(u, v) \\ &= - \int_{\Omega} a(x)\nabla u(x) \cdot \nabla v(x) dx \\ &= - \int_{\mathbb{R}^2} \mathring{E}[a\nabla u](x) \cdot \nabla V(x) dx \\ &= \langle \nabla \cdot \mathring{E}[a\nabla u], V \rangle_{\mathbb{R}^2} = \langle \nabla \cdot \mathring{E}[a\nabla u], v \rangle_{\Omega}, \\ &\forall u \in H^1(\Omega), \quad \forall v \in H^1(\Omega), \end{aligned} \tag{6}$$

which is evidently continuous and can be written as:

$$\check{A}u = \nabla \cdot \mathring{E}[a\nabla u]. \tag{7}$$

Here $V \in H^1(\mathbb{R}^2)$ is such that $r_{\Omega}V = v$ and \mathring{E} denotes the operator of extension of the functions, defined in Ω , by zero outside Ω in \mathbb{R}^2 . For any $u \in H^1(\Omega)$, the functional $\check{A}u$ belongs to $\tilde{H}^{-1}(\Omega)$ and is the extension of the functional $Au \in H^{-1}(\Omega)$, which domain is thus extended from $\tilde{H}^1(\Omega)$ to the domain $H^1(\Omega)$ for $\check{A}u$.

Inspired by the first Green identity for smooth functions, we can define the *generalized conormal derivative* as in (McLean, 2000, Lemma 4.3), (Mikhailov, 2011, Definition 3.1) and (Kohr et al., 2015, Lemma 2.2).

Definition 1. Let $u \in H^1(\Omega)$ and $Au = r_{\Omega}\tilde{f}$ in Ω for some $\tilde{f} \in \tilde{H}^{-1}(\Omega)$. Then the *generalized conormal derivative* $T_a^+(\tilde{f}, u) \in H^{-\frac{1}{2}}(\partial\Omega)$ is defined as

$$\begin{aligned} \langle T_a^+(\tilde{f}, u), w \rangle_{\partial\Omega} &:= \langle \tilde{f}, \gamma^{-1}w \rangle_{\Omega} + \mathcal{E}_a(u, \gamma^{-1}w) \\ &= \langle \tilde{f} - \check{A}u, \gamma^{-1}w \rangle_{\Omega}, \quad \forall w \in H^{\frac{1}{2}}(\partial\Omega), \end{aligned}$$

that is

$$T^+(\tilde{f}, u) := (\gamma^{-1})^*(\tilde{f} - \check{A}u). \tag{8}$$

Due to (McLean, 2000, Lemma 4.3) and (Mikhailov, 2011, Theorem 3.2), we have the estimate

$$\begin{aligned} \|T_a^+(\tilde{f}, u)\|_{H^{-\frac{1}{2}}(\partial\Omega)} &\leq C_1 \|u\|_{H^1(\Omega)} \\ &\quad + C_2 \|\tilde{f}\|_{\tilde{H}^{-1}(\Omega)}, \end{aligned} \tag{9}$$

and for $u \in H^1(\Omega)$ such that $Au = r_{\Omega}\tilde{f}$ in Ω for some $\tilde{f} \in \tilde{H}^{-1}(\Omega)$ the first Green identity holds in the following form:

$$\begin{aligned} \langle T_a^+(\tilde{f}, u), \gamma^+v \rangle_{\partial\Omega} &:= \langle \tilde{f}, v \rangle_{\Omega} + \mathcal{E}_a(u, v) \\ &= \langle \tilde{f} - \check{A}u, v \rangle_{\Omega}, \quad \forall v \in H^1(\Omega). \end{aligned} \tag{10}$$

As follows from Definition 1, the generalised conormal derivative is nonlinear with respect to u for a fixed \tilde{f} , but linear with respect to the couple (\tilde{f}, u) , i.e.,

$$\begin{aligned} \alpha_1 T_a^+(\tilde{f}_1, u_1) + \alpha_2 T_a^+(\tilde{f}_2, u_2) \\ &= T_a^+(\alpha_1 \tilde{f}_1, \alpha_1 u_1) + T_a^+(\alpha_2 \tilde{f}_2, \alpha_2 u_2) \\ &= T_a^+(\alpha_1 \tilde{f}_1 + \alpha_2 \tilde{f}_2, \alpha_1 u_1 + \alpha_2 u_2) \end{aligned} \tag{11}$$

for any real numbers α_1, α_2 .

Let us also define some subspaces of $H^s(\Omega)$, cf. Mikhailov (2011, 2013); Costabel (1988); Grisvard (1985).

Definition 2. Let $s \in \mathbb{R}$ and $A_* : H^s(\Omega) \rightarrow \mathcal{D}^*(\Omega)$ be a linear operator. For $t \geq -\frac{1}{2}$ we introduce the space

$$\begin{aligned} H^{s,t}(\Omega; A_*) &:= \{g \in H^s(\Omega) : \text{there exists} \\ &\quad \tilde{f}_g \in \tilde{H}^t(\Omega) \text{ such that } A_*g|_{\Omega} = \tilde{f}_g|_{\Omega}\} \end{aligned}$$

endowed with the norm

$$\|g\|_{H^{s,t}(\Omega; A_*)} := \left(\|g\|_{H^s(\Omega)}^2 + \|\tilde{f}_g\|_{\tilde{H}^t(\Omega)}^2 \right)^{\frac{1}{2}}$$

and the inner product

$$(g, h)_{H^{s,t}(\Omega; A_*)} = (g, h)_{H^s(\Omega)} + (\tilde{f}_g, \tilde{f}_h)_{\tilde{H}^t(\Omega)}.$$

The distribution $\tilde{f}_g \in \tilde{H}^t(\Omega)$, $t \geq -\frac{1}{2}$, in the above definition is an extension of the distribution $A_*g|_\Omega \in H^t(\Omega)$, and the extension is unique (if it does exist) since any distribution from the space $H^t(\mathbb{R}^2)$ with support in $\partial\Omega$ is identically zero if $t \geq -\frac{1}{2}$, see (McLean, 2000, Lemma 3.39) and (Mikhailov, 2011, Theorem 2.10). We denote this extension as an operator \tilde{A}_* , i.e., $\tilde{A}_*g = \tilde{f}_g$. The uniqueness implies that the norm $\|g\|_{H^{s,t}(\Omega; A_*)}$ is well defined. We will mostly use the operators A, B or Δ as A_* in the above definition. Note that since $Au - a\Delta u = \nabla a \cdot \nabla u \in L_2(\Omega)$, for $u \in H^1(\Omega)$, we have $H^{1,0}(\Omega; A) = H^{1,0}(\Omega; \Delta)$.

Definition 3. For $u \in H^{1,-\frac{1}{2}}(\Omega; A)$, we define the canonical conormal derivative $T_a^+u \in H^{-\frac{1}{2}}(\partial\Omega)$ as

$$\begin{aligned} \langle T_a^+u, w \rangle_{\partial\Omega} &:= \langle \tilde{A}u, \gamma^{-1}w \rangle_\Omega + \mathcal{E}_a(u, \gamma^{-1}w) \\ &= \langle \tilde{A}u - \check{A}u, \gamma^{-1}w \rangle_\Omega, \quad \forall w \in H^{\frac{1}{2}}(\partial\Omega), \quad i. e., \\ T_a^+u &:= (\gamma^{-1})^*(\tilde{A}u - \check{A}u). \end{aligned} \quad (12)$$

The canonical conormal derivative T_a^+u is independent of (non-unique) choice of the operator γ^{-1} , the operator $T_a^+ : H^{1,-\frac{1}{2}}(\Omega; A) \rightarrow H^{-\frac{1}{2}}(\partial\Omega)$ is continuous, and the first Green identity holds in the following form,

$$\langle T_a^+u, \gamma^+v \rangle_{\partial\Omega} := \langle \tilde{A}u, v \rangle_\Omega + \mathcal{E}_a(u, v), \quad (13)$$

$$\forall v \in H^1(\Omega).$$

The operator $T_a^+ : H^{1,t}(\Omega; A) \rightarrow H^{-\frac{1}{2}}(\partial\Omega)$ in Definition 3 is continuous for $t \geq -\frac{1}{2}$. The canonical conormal derivative is defined by the function u and the operator A and does not depend separately on the right-hand side \tilde{f} (i.e. its behavior on the boundary), unlike the generalised conormal derivative defined in (8), and the operator T_a^+ is linear. Note that the canonical conormal derivative coincides with classical conormal derivative $T_a^+u = a \frac{\partial u}{\partial n}$ if the latter does exist in the trace sense, see (Mikhailov, 2011, Corollary 3.14 and Theorem 3.16).

Let $u \in H^{1,-\frac{1}{2}}(\Omega; A)$. Then Definitions 1 and 3 imply that the generalised conormal derivative for arbitrary extension $\tilde{f} \in \tilde{H}^{-1}(\Omega)$ of the distribution Au can be expressed as

$$\begin{aligned} \langle T_a^+(\tilde{f}, u), w \rangle_{\partial\Omega} &:= \langle T_a^+u, w \rangle_{\partial\Omega} \\ &+ \langle \tilde{f} - \check{A}u, \gamma^{-1}w \rangle_\Omega, \quad \forall w \in H^{\frac{1}{2}}(\partial\Omega). \end{aligned} \quad (14)$$

Let us consider the auxiliary linear elliptic partial differential operator B defined by

$$\begin{aligned} Bu(x) &:= B(x, \partial_x)u(x) \\ &= \sum_{i=1}^2 \frac{\partial}{\partial x_i} \left(b(x) \frac{\partial u(x)}{\partial x_i} \right), \end{aligned} \quad (15)$$

where $b \in C^\infty(\bar{\Omega})$ and $b(x) > 0$ for $x \in \bar{\Omega}$. Since for $u \in H^{1,0}(\Omega, \Delta)$, $Au - Bu = (a - b)\Delta u + \nabla(a - b) \cdot \nabla u \in L_2(\Omega)$, we have, $H^{1,0}(\Omega; A) = H^{1,0}(\Omega; B)$. Let $u \in H^1(\Omega)$ and $v \in H^{1,0}(\Omega; B)$. Then we write the first Green identity for operator B in the form

$$\begin{aligned} \mathcal{E}_b(u, v) + \int_\Omega u(x)Bv(x)dx \\ = \langle T_b^+v, \gamma^+u \rangle_{\partial\Omega}, \end{aligned} \quad (16)$$

where

$$\mathcal{E}_b(u, v) = \int_\Omega b(x)\nabla u(x) \cdot \nabla v(x)dx.$$

If, in addition, $Au = \tilde{f}$ in Ω , where $\tilde{f} \in \tilde{H}^{-1}(\Omega)$, then according to the definition of $T_a^+(\tilde{f}, u)$, in (8), the two-operator second Green identity can be written as

$$\begin{aligned} \langle \tilde{f}, v \rangle_\Omega - \int_\Omega u(x)Bv(x)dx \\ + \int_\Omega [a(x) - b(x)]\nabla u(x) \cdot \nabla v(x)dx \\ = \langle T_a^+(\tilde{f}, u), \gamma^+v \rangle_{\partial\Omega} \\ - \langle T_b^+v, \gamma^+u \rangle_{\partial\Omega}. \end{aligned} \quad (17)$$

Moreover, for $u, v \in H^{1,0}(\Omega; A) = H^{1,0}(\Omega; B)$ (17) becomes

$$\begin{aligned} \int_\Omega [v(x)Au(x) - u(x)Bv(x)]dx \\ + \int_\Omega [a(x) - b(x)]\nabla u(x) \cdot \nabla v(x)dx \\ = \langle T_a^+u, \gamma^+v \rangle_{\partial\Omega} - \langle T_b^+v, \gamma^+u \rangle_{\partial\Omega}. \end{aligned}$$

Parametrix, remainder and potential type operators.

Definition 4. A function $P_b(x, y)$ of two variables $x, y \in \Omega$ is a parametrix (Levi function) for the operator $B(x, \partial_x)$ in \mathbb{R}^2 (Miranda, 1970; Hellwig, 1977; Pomp, 1998a,b; Mikhailov, 2002) if

$$B(x, \partial_x)P_b(x, y) = \delta(x - y) + R_b(x, y), \quad (18)$$

where δ is the Dirac-delta distribution, while $R_b(x, y)$ is a remainder possessing at most a weak singularity at $x = y$.

For some positive constant r_0 and $x, y \in \mathbb{R}^2$, the parametrix and hence the corresponding remainder in 2D can be chosen as in Mikhailov (2002),

$$P_b(x, y) = \frac{1}{2\pi b(y)} \ln \left(\frac{|x - y|}{r_0} \right), \quad (19)$$

$$R_b(x, y) = \sum_{i=1}^2 \frac{x_i - y_i}{2\pi b(y)|x - y|^2} \frac{\partial b(x)}{\partial x_i}. \quad (20)$$

The parametrix $P_b(x, y)$ in (19) is the fundamental solution to the operator $B(y, \partial_x) := b(y)\Delta_x$ with "frozen" coefficient $b(x) = b(y)$, and

$$B(y, \partial_x)P_b(x, y) = \delta(x - y). \quad (21)$$

Let $b \in C^\infty(\mathbb{R}^2)$ and $b(x) > 0$ a.e. in \mathbb{R}^2 . For some scalar function g the parametrix-based Newtonian and the remainder volume potential operators, corresponding to the parametrix (19) and the remainder (20) are given by

$$\mathbf{P}_b g(y) := \int_{\mathbb{R}^2} P_b(x, y)g(x)dx, \quad y \in \mathbb{R}^2, \quad (22)$$

$$\mathcal{P}_b g(y) := \int_{\Omega} P_b(x, y)g(x)dx, \quad y \in \Omega, \quad (23)$$

$$\mathcal{R}_b g(y) := \int_{\Omega} R_b(x, y)g(x)dx, \quad y \in \Omega. \quad (24)$$

For $g \in H^s(\mathbb{R}^2)$, $s \in \mathbb{R}$, (22) is understood as $\mathbf{P}_b g = \frac{1}{b}\mathbf{P}_\Delta g$, where the Newtonian potential operator \mathbf{P}_Δ for Laplacian Δ is well defined in terms of the Fourier transform (i.e., as pseudo-differential operator), on any space $H^s(\mathbb{R}^2)$. For $g \in \tilde{H}^s(\Omega)$, and any $s \in \mathbb{R}$, definitions in (23) and (24) can be understood as

$$\mathcal{P}_b g = \frac{1}{b}r_\Omega \mathbf{P}_\Delta g, \quad \mathcal{P}_b g = \frac{a}{b}r_\Omega \mathbf{P}_a g, \quad (25)$$

$$\mathcal{R}_b g = -\frac{1}{b}r_\Omega \nabla \cdot \mathbf{P}_\Delta (g \nabla b),$$

while for $g \in H^s(\Omega)$, $-\frac{1}{2} < s < \frac{1}{2}$, as (25) with g replaced by $\tilde{E}g$, where $\tilde{E} : H^s(\Omega) \rightarrow \tilde{H}^s(\Omega)$, $-\frac{1}{2} < s < \frac{1}{2}$, is the unique extension operator related with the operator \hat{E} of extension by zero, cf. (Mikhailov, 2011, Theorem 16).

For $y \notin \partial\Omega$, the single layer and the double layer surface potential operators, corresponding to the parametrix (19) are defined as

$$V_b g(y) := - \int_{\partial\Omega} P_b(x, y)g(x)dS_x, \quad (26)$$

$$W_b g(y) := - \int_{\partial\Omega} [T_b(x, n(x), \partial_x)P_b(x, y)]g(x)dS_x, \quad (27)$$

where g is some scalar density function. The integrals are understood in the distributional

sense if g is not integrable, while V_Δ and W_Δ are the single layer and double layer potentials corresponding to the Laplacian Δ . The corresponding boundary integral (pseudodifferential) operators of direct surface values to the single and the double layer potentials, \mathcal{V}_b and \mathcal{W}_b when $y \in \partial\Omega$, are

$$\mathcal{V}_b g(y) := - \int_{\partial\Omega} P_b(x, y)g(x)dS_x, \quad (28)$$

$$\mathcal{W}_b g(y) := - \int_{\partial\Omega} T_b(x, n(x), \partial_x)P_b(x, y)g(x)dS_x, \quad (29)$$

where \mathcal{V}_Δ and \mathcal{W}_Δ are respectively the direct values of the single and double layer potentials corresponding to the Laplacian Δ .

We can also calculate at $y \in \partial\Omega$ the conormal derivatives, associated with the operator A , of the single layer potential and of the double layer potential:

$$T_a^\pm V_b g(y) = \frac{a(y)}{b(y)} T_b^\pm V_b g(y), \quad (30)$$

$$\mathcal{L}_{ab}^\pm g(y) := T_a^\pm W_b g(y) = \frac{a(y)}{b(y)} T_b^\pm W_b g(y). \quad (31)$$

The direct value operators associated with (30) are:

$$\mathcal{W}'_{ab} g(y) := - \int_{\partial\Omega} [T_a(y, n(y), \partial_y)P_b(x, y)]g(x)dS_x, \quad (32)$$

$$\mathcal{W}'_b g(y) := - \int_{\partial\Omega} [T_b(y, n(y), \partial_y)P_b(x, y)]g(x)dS_x. \quad (33)$$

From equations (22)-(33) we can deduce representations of the parametrix-based surface potential boundary operators in terms of their counterparts for $b = 1$, that is, associated with the fundamental solution $P_\Delta = \frac{1}{2\pi} \ln \left(\frac{|x-y|}{r_0} \right)$ of the Laplace operator Δ .

$$\begin{aligned} \mathbf{P}_a g &= \frac{1}{a}\mathbf{P}_\Delta g, & \mathbf{P}_b g &= \frac{1}{b}\mathbf{P}_\Delta g, \\ \mathcal{P}_a g &= \frac{1}{a}\mathcal{P}_\Delta g, & \mathcal{P}_b g &= \frac{1}{b}\mathcal{P}_\Delta g. \end{aligned} \quad (34)$$

$$\begin{aligned} \frac{a}{b}V_a g &= V_b g = \frac{1}{b}V_\Delta g, \\ \frac{a}{b}W_a \left(\frac{bg}{a} \right) &= W_b g = \frac{1}{b}W_\Delta (bg). \end{aligned} \quad (35)$$

$$\begin{aligned} \frac{a}{b}\mathcal{V}_a g &= \mathcal{V}_b g = \frac{1}{b}\mathcal{V}_\Delta g, \\ \frac{a}{b}\mathcal{W}_a \left(\frac{bg}{a} \right) &= \mathcal{W}_b g = \frac{1}{b}\mathcal{W}_\Delta (bg). \end{aligned} \quad (36)$$

$$\begin{aligned}\mathcal{W}'_{ab}g &= \frac{a}{b}\mathcal{W}'_bg \\ &= \frac{a}{b}\left\{\mathcal{W}'_{\Delta}g + \left[b\frac{\partial}{\partial n}\left(\frac{1}{b}\right)\right]\mathcal{V}_{\Delta}g\right\},\end{aligned}\quad (37)$$

$$\begin{aligned}\mathcal{L}_{ab}^{\pm}g &= \frac{a}{b}\mathcal{L}_b^{\pm}g \\ &= \frac{a}{b}\left\{\mathcal{L}_{\Delta}(bg) + \left[b\frac{\partial}{\partial n}\left(\frac{1}{b}\right)\right]\gamma^{\pm}W_{\Delta}(bg)\right\},\end{aligned}\quad (38)$$

$$\begin{aligned}\hat{\mathcal{L}}_bg &:= T_{\Delta}^{+}W_{\Delta}(bg) = T_{\Delta}^{-}W_{\Delta}(bg) \\ &= \hat{\mathcal{L}}_{\Delta}(bg) \quad \text{on } \partial\Omega.\end{aligned}\quad (39)$$

It is taken into account that b and its derivatives are continuous in \mathbb{R}^2 and

$$\mathcal{L}_{\Delta}(bg) := \mathcal{L}_{\Delta}^{+}(bg) = \mathcal{L}_{\Delta}^{-}(bg)$$

by the Liapunov-Tauber theorem. Hence,

$$\begin{aligned}\Delta(bV_bg) &= 0, \quad \Delta(bW_bg) = 0 \quad \text{in } \Omega, \\ \forall g &\in H^s(\partial\Omega), \quad \forall s \in \mathbb{R},\end{aligned}\quad (40)$$

$$\begin{aligned}\Delta(b\mathcal{P}_bg) &= g \quad \text{in } \Omega, \\ \forall g &\in \tilde{H}^s(\Omega), \quad \forall s \in \mathbb{R}.\end{aligned}\quad (41)$$

The mapping properties of the operators (22)-(33) follow from relations (34)-(39) and are described in detail in (Ayele Tsegaye and Mikhailov, 2011, Appendix A). Particularly, we have the following jump relations:

Theorem 1. For $g_1 \in H^{-\frac{1}{2}}(\partial\Omega)$, and $g_2 \in H^{\frac{1}{2}}(\partial\Omega)$. Then there hold the following relations on $\partial\Omega$,

$$\gamma^{\pm}V_bg_1 = \mathcal{V}_bg_1, \quad (42)$$

$$\gamma^{\pm}W_bg_2 = \mp\frac{1}{2}g_2 + \mathcal{W}_bg_2, \quad (43)$$

$$T_a^{\pm}V_bg_1 = \pm\frac{1}{2}\frac{a}{b}g_1 + \mathcal{W}'_{ab}g_1. \quad (44)$$

THE TWO-OPERATOR THIRD GREEN IDENTITY AND INTEGRAL RELATIONS

Applying some limiting procedures (Miranda, 1970) and (Hellwig, 1977, Section 3.8), we obtain the parametrix based third Green identities.

Theorem 2. (i) If $u \in H^1(\Omega)$, then the following third Green identity holds,

$$u + \mathcal{Z}_bu + \mathcal{R}_bu + W_b\gamma^+u = \mathcal{P}_b\check{A}u \quad \text{in } \Omega, \quad (45)$$

where the operator \check{A} is defined in (7), and for $u \in C^1(\bar{\Omega})$,

$$\begin{aligned}\mathcal{P}_b\check{A}u(y) &:= \langle \check{A}u, P_b(\cdot, y) \rangle_{\Omega} = -\mathcal{E}_a(u, P_b(\cdot, y)) \\ &= -\int_{\Omega} a(x)\nabla u(x) \cdot \nabla_x P_b(x, y)dx\end{aligned}\quad (46)$$

and

$$\begin{aligned}\mathcal{Z}_bu &= -\int_{\Omega} [a(x) - b(x)]\nabla_x P_b(x, y) \cdot \nabla u(x)dx \\ &= \frac{1}{b(y)}\sum_{j=1}^2 \partial_j \mathcal{P}_{\Delta} [(a-b)\partial_j u] \quad \text{in } \Omega.\end{aligned}\quad (47)$$

(ii) If $Au = r_{\Omega}\tilde{f}$ in Ω , where $\tilde{f} \in \tilde{H}^{-1}(\Omega)$, then recalling the definition of $T_a^+(\tilde{f}, u)$, in (8), we arrive at the generalised two-operator third Green identity in the following form,

$$\begin{aligned}u + \mathcal{Z}_bu + \mathcal{R}_bu - V_bT_a^+(\tilde{f}, u) + W_b\gamma^+u \\ = \mathcal{P}_b\tilde{f} \quad \text{in } \Omega,\end{aligned}\quad (48)$$

where it was taken into account that

$$\begin{aligned}\langle T_a^+(\tilde{f}, u), P_b(x, y) \rangle_{\partial\Omega} &= -V_bT_a^+(\tilde{f}, u) \quad \text{and} \\ \langle \tilde{f}, P_b(x, y) \rangle_{\Omega} &= \mathcal{P}_b\tilde{f}.\end{aligned}\quad (49)$$

Proof. (i) Let first $u \in D(\bar{\Omega})$. Let $y \in \Omega$, $B_{\epsilon}(y) \subset \Omega$ be a ball centred at y with sufficiently small radius ϵ , and $\Omega_{\epsilon} := \Omega \setminus \bar{B}_{\epsilon}(y)$. For the fixed y , evidently, $P_b(\cdot, y) \in D(\bar{\Omega}_{\epsilon}) \subset H^{1,0}(A; \Omega_{\epsilon})$ and has the coinciding classical and canonical conormal derivatives on $\partial\Omega_{\epsilon}$. Then from (19) and the first Green identity (16) employed for Ω_{ϵ} with $v = P_b(\cdot, y)$ we obtain

$$\begin{aligned}-\int_{\partial B_{\epsilon}(y)} T_x^+ P_b(x, y)\gamma^+u(x)ds_x \\ -\int_{\partial\Omega} T_x P_b(x, y)\gamma^+u(x)ds_x \\ +\int_{\Omega_{\epsilon}} u(x)R_b(x, y)dx \\ = -\int_{\Omega_{\epsilon}} b(x)\nabla u(x) \cdot \nabla_x P_b(x, y)dx,\end{aligned}$$

which we rewrite as

$$\begin{aligned}-\int_{\partial B_{\epsilon}(y)} T_x^+ P_b(x, y)\gamma^+u(x)ds_x \\ -\int_{\partial\Omega} T_x P_b(x, y)\gamma^+u(x)ds_x \\ -\int_{\Omega_{\epsilon}} [a(x) - b(x)]\nabla u(x) \cdot \nabla_x P_b(x, y)dx \\ +\int_{\Omega_{\epsilon}} u(x)R_b(x, y)dx \\ = -\int_{\Omega_{\epsilon}} a(x)\nabla u(x) \cdot \nabla_x P_b(x, y)dx.\end{aligned}\quad (50)$$

Taking the limit as $\epsilon \rightarrow 0$, (50) reduces to the third Green identity (45)–(46) for any $u \in \mathcal{D}(\bar{\Omega})$. Taking into account the density of $\mathcal{D}(\bar{\Omega})$ in $H^1(\Omega)$, and the mapping properties of

the integral potentials, see (Ayele Tsegaye and Mikhailov, 2011, Appendix A), we obtain that (45)–(46) hold true also for any $u \in H^1(\Omega)$.

(ii) Let $\{\tilde{f}_k\} \in \mathcal{D}(\Omega)$ be a sequence converging to \tilde{f} in $\tilde{H}^{-1}(\Omega)$ as $k \rightarrow \infty$. Then, according to (Mikhailov, 2015, Theorem B.1) there exists a sequence $\{u_k\} \in \mathcal{D}(\bar{\Omega})$ converging to u in $H^1(\Omega)$ such that $Au_k = r_\Omega \tilde{f}_k$ and $T_a^+(u_k) = T_a^+(\tilde{f}_k, u_k)$ converge to $T_a^+(\tilde{f}, u)$ in $H^{-\frac{1}{2}}(\partial\Omega)$. For such u_k by (46) and (8), we have

$$\begin{aligned} \mathcal{P}_b \tilde{A}u_k(y) &= \frac{1}{b(y)} \nabla_y \cdot \int_\Omega a(x) P_\Delta(x, y) \nabla u_k(x) dx \\ &= -\frac{1}{b(y)} \lim_{\epsilon \rightarrow 0} \int_{\Omega_\epsilon} a(x) \nabla u_k(x) P_\Delta(x, y) dx \\ &= -\lim_{\epsilon \rightarrow 0} \mathcal{E}_{\Omega_\epsilon}(u_k, P_b(\cdot, y)) = -\lim_{\epsilon \rightarrow 0} \int_{\Omega_\epsilon} \tilde{f}_k P_b(x, y) dx \\ &\quad + \lim_{\epsilon \rightarrow 0} \int_{\partial B_\epsilon(y)} P_b(x, y) T_a^+ u_k(x) dS(x) \\ &\quad + \lim_{\epsilon \rightarrow 0} \int_{\partial\Omega} P_b(x, y) T_a^+ u_k(x) dS(x) \\ &= \mathcal{P}_b \tilde{f}_k + V_b T_a^+ u_k(y). \end{aligned} \tag{51}$$

Taking the limits as $k \rightarrow \infty$ in (51), we obtain $\mathcal{P}_b \tilde{A}u(y) = \mathcal{P}_b \tilde{f} + V_b T_a^+(\tilde{f}, u)$, and substitution to (45) gives (48). \square

Below we shall state and prove Corollary 3 from Ayele Tsegaye and Bekele Solomon (2023) for completeness.

Corollary 3. *Using the Gauss divergence theorem, we can rewrite $\mathcal{Z}_b u(y)$ in the form that does not involve derivatives of u ,*

$$\mathcal{Z}_b u(y) := \left[\frac{a(y)}{b(y)} - 1 \right] u(y) + \hat{\mathcal{Z}}_b u(y), \tag{52}$$

$$\begin{aligned} \hat{\mathcal{Z}}_b u(y) &:= \frac{a(y)}{b(y)} W_a \gamma^+ u(y) - W_b \gamma^+ u(y) \\ &\quad + \frac{a(y)}{b(y)} \mathcal{R}_a u(y) - \mathcal{R}_b u(y), \end{aligned} \tag{53}$$

which allows to call \mathcal{Z}_b integral operator in spite of its integro-differential representation (47).

Proof. As in the proof of Theorem 2 item (i), let first $u \in D(\bar{\Omega})$. Let $y \in \Omega$, $B_\epsilon(y) \subset \Omega$ be a ball centred at y with sufficiently small radius ϵ , and $\Omega_\epsilon := \Omega \setminus \bar{B}_\epsilon(y)$. For the fixed y , evidently, $P_b(\cdot, y) \in D(\bar{\Omega}_\epsilon) \subset H^{1,0}(A; \Omega_\epsilon)$ and has the coinciding classical and canonical conormal derivatives on $\partial\Omega_\epsilon = \partial\Omega \cup \partial B_\epsilon(y)$. Further, let us denote

$$\mathcal{Z}_b^\epsilon u(y) = - \int_{\Omega_\epsilon} [a(x) - b(x)] \nabla_x P_b(x, y) \cdot \nabla u(x) dx,$$

which can be rewritten as

$$\begin{aligned} \mathcal{Z}_b^\epsilon u(y) &= \int_{\Omega_\epsilon} [\nabla(a(x) - b(x)) \cdot \nabla_x P_b(x, y)] u(x) dx \\ &\quad - \int_{\Omega_\epsilon} \nabla[(a(x) - b(x))u(x)] \cdot \nabla_x P_b(x, y) dx. \end{aligned}$$

Observe that

$$\begin{aligned} I_1(y, \epsilon) &= \int_{\Omega_\epsilon} [\nabla(a(x) - b(x)) \cdot \nabla_x P_b(x, y)] u(x) dx \\ &= \int_{\Omega_\epsilon} [\nabla a(x) \cdot \nabla_x P_b(x, y)] u(x) dx \\ &\quad - \int_{\Omega_\epsilon} [\nabla b(x) \cdot \nabla_x P_b(x, y)] u(x) dx \\ &= \frac{a(y)}{b(y)} \int_{\Omega_\epsilon} [\nabla a(x) \cdot \nabla_x P_a(x, y)] u(x) dx \\ &\quad - \int_{\Omega_\epsilon} [\nabla b(x) \cdot \nabla_x P_b(x, y)] u(x) dx \end{aligned}$$

and

$$\begin{aligned} I_2(y, \epsilon) &= - \int_{\Omega_\epsilon} \nabla[(a(x) - b(x))u(x)] \nabla_x P_b(x, y) dx \\ &= \int_{\Omega_\epsilon} [a(x) - b(x)] u(x) \Delta_x P_b(x, y) dx \\ &\quad - \int_{\partial\Omega_\epsilon} [a(x) - b(x)] \gamma^+ u(x) \nabla_x P_b(x, y) \cdot n(x) dS_x \\ &= -\frac{a(y)}{b(y)} \int_{\partial\Omega} a(x) \nabla_x P_a(x, y) \cdot n(x) \gamma^+ u(x) dS_x \\ &\quad + \int_{\partial\Omega} b(x) \nabla_x P_b(x, y) \cdot n(x) \gamma^+ u(x) dS_x \\ &\quad - \frac{a(y)}{b(y)} \int_{\partial B(y, \epsilon)} a(x) \nabla_x P_a(x, y) \cdot n(x) u(x) dS_x \\ &\quad + \int_{\partial B_\epsilon(y)} b(x) \nabla_x P_b(x, y) \cdot n(x) u(x) dS_x \\ &= \frac{a(y)}{b(y)} W_a \gamma^+ u(y) - W_b \gamma^+ u(y) \\ &\quad - \frac{1}{b(y)} \int_{\partial B_\epsilon(y)} a(x) \nabla_x P_\Delta(x, y) \cdot n(x) \gamma^+ u(x) dS_x \\ &\quad + \frac{1}{b(y)} \int_{\partial B_\epsilon(y)} b(x) \nabla_x P_\Delta(x, y) \cdot n(x) \gamma^+ u(x) dS_x. \end{aligned}$$

Taking the limit as $\epsilon \rightarrow 0$ we obtain

$$\begin{aligned} \mathcal{Z}_b u(y) &= \lim_{\epsilon \rightarrow 0} \mathcal{Z}_b^\epsilon u(y) = \lim_{\epsilon \rightarrow 0} [I_1(y, \epsilon) + I_2(y, \epsilon)] \\ &= \frac{a(y)}{b(y)} \mathcal{R}_a u(y) - \mathcal{R}_b u(y) + \frac{a(y)}{b(y)} W_a \gamma^+ u(y) \\ &\quad - W_b \gamma^+ u(y) + \left[\frac{a(y)}{b(y)} - 1 \right] u(y) \end{aligned}$$

which is as in (52) and (53). \square

Note that the operator \mathcal{Z}_b does not vanish unless operators A and B are equal. For some functions

\tilde{f} , Ψ , Φ let us consider a more general "indirect" integral relation, associated with (48).

$$\begin{aligned} u + \mathcal{Z}_b u + \mathcal{R}_b u - V_b \Psi + W_b \Phi \\ = \mathcal{P}_b \tilde{f} \quad \text{in } \Omega. \end{aligned} \quad (54)$$

Lemma 4. *Let $u \in H^1(\Omega)$, $\Psi \in H^{-\frac{1}{2}}(\partial\Omega)$, $\Phi \in H^{\frac{1}{2}}(\partial\Omega)$ and $\tilde{f} \in \tilde{H}^{-1}(\Omega)$ satisfy (54). Then*

$$Au = r_\Omega \tilde{f} \quad \text{in } \Omega, \quad (55)$$

$$\begin{aligned} r_\Omega V_b(\Psi - T_a^+(\tilde{f}, u)) - r_\Omega W_b(\Phi - \gamma^+ u) \\ = 0 \quad \text{in } \Omega, \end{aligned} \quad (56)$$

$$\begin{aligned} \gamma^+ u + \gamma^+ \mathcal{Z}_b u + \gamma^+ \mathcal{R}_b u - \mathcal{V}_b \Psi - \frac{1}{2} \Phi + \mathcal{W}_b \Phi \\ = \gamma^+ \mathcal{P}_b \tilde{f} \quad \text{on } \partial\Omega, \end{aligned} \quad (57)$$

$$\begin{aligned} T_a^+(\tilde{f}, u) + T_a^+ \mathcal{Z}_b u + T_a^+ \mathcal{R}_b u - \frac{a}{2b} \Psi - \mathcal{W}'_{ab} \Psi \\ + \mathcal{L}_{ab}^+ \Phi = T_a^+(\tilde{f} + \mathring{E} \mathcal{R}_*^b \tilde{f}, \mathcal{P}_b \tilde{f}) \quad \text{on } \partial\Omega, \end{aligned} \quad (58)$$

where

$$\mathcal{R}_*^b \tilde{f}(y) := - \sum_{j=1}^2 \partial_j [(\partial_j b) \mathcal{P}_b \tilde{f}]. \quad (59)$$

Proof. Subtracting (54) from identity (45), we obtain

$$\begin{aligned} V_b \Psi(y) - W_b(\Phi - \gamma^+ u)(y) \\ = \mathcal{P}_b [\tilde{A}u(y) - \tilde{f}](y), \quad y \in \Omega. \end{aligned} \quad (60)$$

Multiplying equality (60) by $b(y)$, applying the Laplace operator Δ and taking into account equations (40) and (41), we get $r_\Omega \tilde{f} = r_\Omega (\tilde{A}u) = Au$ in Ω . This means \tilde{f} is an extension of the distribution $Au \in H^{-1}(\Omega)$ to $\tilde{H}^{-1}(\Omega)$, and u satisfies (55). Then (8) implies

$$\begin{aligned} \mathcal{P}_b [\tilde{A}u - \tilde{f}](y) &= \langle \tilde{A}u - \tilde{f}, P_b(\cdot, y) \rangle_\Omega \\ &= - \langle T_a^+(\tilde{f}, u), P_b(\cdot, y) \rangle_{\partial\Omega} \quad (61) \\ &= V_b T_a^+(\tilde{f}, u), \quad y \in \Omega. \end{aligned}$$

Substituting (61) into (60) leads to (56). Equation (57) follows from (54) and jump relations in (42) and (43). To prove (58), let us first remark that for $u \in H^1(\Omega)$, we have $H^1(\Omega; A) = H^1(\Omega; \Delta) = H^1(\Omega; B)$ and

$$B \mathcal{P}_b \tilde{f} = \tilde{f} + \mathcal{R}_*^b \tilde{f} \quad \text{in } \Omega, \quad (62)$$

due to (55), which implies $B(\mathcal{P}_b \tilde{f} - u) = \mathcal{R}_*^b \tilde{f}$ in Ω , with $\mathcal{R}_*^b \tilde{f}$ given by (59), and thus $\mathcal{R}_*^b \tilde{f} \in L_2(\Omega)$. Then $B(\mathcal{P}_b \tilde{f} - u)$ can be canonically extended (by zero) to

$$\tilde{B}(\mathcal{P}_b \tilde{f} - u) = \mathring{E} \mathcal{R}_*^b \tilde{f} \in \tilde{H}^0(\Omega) \subset \tilde{H}^{-1}(\Omega).$$

Thus there exists a canonical conormal derivative $T_b^+(\mathcal{P}_b \tilde{f} - u)$ written as (see, e.g., (Mikhailov, 2015, Eq. (4.14)), (Mikhailov, 2018, Eq. (4.23)).

$$\begin{aligned} T_b^+(\mathcal{P}_b \tilde{f} - u) &= T_b^+(\tilde{f} + \mathring{E} \mathcal{R}_*^b \tilde{f}, \mathcal{P}_b \tilde{f}) \\ &\quad - T_b^+(\tilde{f}, u), \end{aligned} \quad (63)$$

and hence

$$\begin{aligned} T_a^+(\mathcal{P}_b \tilde{f} - u) &= \frac{a}{b} T_b^+(\mathcal{P}_b \tilde{f} - u) \\ &= \frac{a}{b} [T_b^+(\tilde{f} + \mathring{E} \mathcal{R}_*^b \tilde{f}, \mathcal{P}_b \tilde{f}) - T_b^+(\tilde{f}, u)] \quad (64) \\ &= T_a^+(\tilde{f} + \mathring{E} \mathcal{R}_*^b \tilde{f}, \mathcal{P}_b \tilde{f}) - T_a^+(\tilde{f}, u). \end{aligned}$$

From (54) it follows that

$$\mathcal{P}_b \tilde{f} - u = \mathcal{Z}_b u + \mathcal{R}_b u - V_b \Psi + W_b \Phi \quad \text{in } \Omega.$$

Substituting this on the left-hand side of (63) and taking into account (38) and the jump relation (44), we arrive at (58). \square

Remark 5. *If $\tilde{f} \in \tilde{H}^{-\frac{1}{2}}(\Omega) \subset \tilde{H}^{-1}(\Omega)$, then $\tilde{f} + \mathring{E} \mathcal{R}_*^b \tilde{f} \in \tilde{H}^{-\frac{1}{2}}(\Omega)$ as well, which implies*

$$\tilde{f} + \mathring{E} \mathcal{R}_*^b \tilde{f} = \tilde{A} \mathcal{P}_b \tilde{f}$$

and

$$\begin{aligned} T_a^+(\tilde{f} + \mathring{E} \mathcal{R}_*^b \tilde{f}, \mathcal{P}_b \tilde{f}) \\ = T_a^+(\tilde{B} \mathcal{P}_b \tilde{f}, \mathcal{P}_b \tilde{f}) = T_a^+ \mathcal{P}_b \tilde{f}. \end{aligned} \quad (65)$$

Furthermore, if the hypotheses of Lemma 4 are satisfied, then (55) implies $u \in H^{1, -\frac{1}{2}}(\Omega; A)$ and $T_a^+(\tilde{f}, u) = T_a^+(\tilde{A}u, u) = T_a^+ u$. Henceforth (58), takes the familiar form, cf. (Ayele Tsegaye and Mikhailov, 2011, equation (3.23)),

$$\begin{aligned} T_a^+ u + T_a^+ \mathcal{Z}_b u + T_a^+ \mathcal{R}_b u - \frac{a}{2b} \Psi - \mathcal{W}'_{ab} \Psi + \mathcal{L}_{ab}^+ \Phi \\ = T_a^+ \mathcal{P}_b \tilde{f} \quad \text{on } \partial\Omega. \end{aligned}$$

Remark 6. *Let $\tilde{f} \in \tilde{H}^{-1}(\Omega)$ and a sequence $\{\phi_i\} \in \tilde{H}^{-1}(\Omega)$ converge to \tilde{f} in $\tilde{H}^{-1}(\Omega)$. By the continuity of operators, cf. (Mikhailov, 2015, C.1 and C.2), estimate (9) and relation (65) for ϕ_i , we obtain that*

$$\begin{aligned} T_a^+(\tilde{f} + \mathring{E} \mathcal{R}_*^b \tilde{f}, \mathcal{P}_b \tilde{f}) &= \lim_{i \rightarrow \infty} T_a^+(\phi_i + \mathring{E} \mathcal{R}_*^b \phi_i, \mathcal{P}_b \phi_i) \\ &= \lim_{i \rightarrow \infty} T_a^+ \mathcal{P}_b \phi_i, \quad \text{in } H^{-\frac{1}{2}}(\partial\Omega), \end{aligned}$$

cf. (Mikhailov, 2015, Theorem B.1).

Lemma 4 and the third Green identity (48) imply, the following assertion.

Corollary 7. *If $u \in H^1(\Omega)$ and $\tilde{f} \in \tilde{H}^{-1}(\Omega)$ are such that $Au = r_\Omega \tilde{f}$ in Ω , then*

$$\frac{1}{2} \gamma^+ u + \gamma^+ \mathcal{Z}_b u + \gamma^+ \mathcal{R}_b u - \mathcal{V}_b T_a^+(\tilde{f}, u) + \mathcal{W}_b \gamma^+ u = \gamma^+ \mathcal{P}_b \tilde{f}, \text{ on } \partial\Omega, \quad (66)$$

$$\begin{aligned} & \left(1 - \frac{a}{2b}\right) T_a^+(\tilde{f}, u) + T_a^+ \mathcal{Z}_b u \\ & + T_a^+ \mathcal{R}_b u - \mathcal{W}'_{ab} T_a^+(\tilde{f}, u) + \mathcal{L}'_{ab} \gamma^+ u \\ & = T_a^+(\tilde{f} + \mathring{E} \mathcal{R}_*^b \tilde{f}, \mathcal{P}_b \tilde{f}) \text{ on } \partial\Omega. \end{aligned} \quad (67)$$

Note that if \mathcal{P}_b is not only the parametrix but also the fundamental solution of the operator B , then the remainder operator \mathcal{R}_b vanishes in (48) and (66)-(67) (and everywhere in the paper), while the operator \mathcal{Z}_b stays unless $A = B$. The following statement is proved in (Mikhailov, 2015, Lemma 4.6):

Theorem 8. *Let $\tilde{f} \in \tilde{H}^{-1}(\Omega)$. A function $u \in H^1(\Omega)$ is a solution of PDE $Au = r_\Omega \tilde{f}$ in Ω if and only if it is a solution of boundary-domain integro-differential equation (BDIDE) (48).*

Proof. If $u \in H^1(\Omega)$ solves the PDE $Au = r_\Omega \tilde{f}$ in Ω , then it satisfies (48). On the other hand, if u solves BDIDE (48), then using Lemma 4 for $\Psi = T_a^+(\tilde{f}, u)$, $\Phi = \gamma^+ u$ completes the proof. \square

INVERTIBILITY OF SINGLE LAYER POTENTIAL OPERATOR

The boundary integral operator, $\mathcal{V}_\Delta : H^{-\frac{1}{2}}(\partial\Omega) \rightarrow H^{\frac{1}{2}}(\partial\Omega)$ is a Fredholm operator of index zero, see, e.g., (McLean, 2000, Theorem 7.6). Thus the first relation in (36) leads to the same result for the single layer potential \mathcal{V}_b . For the case of 3D, (Ayele Tsegaye and Mikhailov, 2011, Lemma 3.2(i)) asserts that for $\Psi^* \in H^{-\frac{1}{2}}(\partial\Omega)$, if $V_b \Psi^* = 0$ in Ω , then $\Psi^* = 0$ in Ω . This fact implies the invertibility of single layer potential operator V_b mapping from $H^{-\frac{1}{2}}(\partial\Omega)$ to $H^{\frac{1}{2}}(\partial\Omega)$. But this is not the case for 2D. It is well-known, see e.g. (Constanda, 2000, Remark 1.42(ii)) and (Steinbach, 2007, Theorem 6.22) that for some 2D domains the kernel of the operator \mathcal{V}_Δ is nontrivial, thus due to the first relation in (36), the kernel of operator \mathcal{V}_b is nontrivial as well for the same domains. To ensure the invertibility of the single layer potential operator in 2D, for $s \in \mathbb{R}$, let us define the subspace of $H^s(\partial\Omega)$, cf. e.g., (Steinbach, 2007, p. 147),

$$H_{**}^s(\partial\Omega) := \{g \in H^s(\partial\Omega) : \langle g, 1 \rangle_{\partial\Omega} = 0\}. \quad (68)$$

The following result is proved in (Dufera Tamirat and Mikhailov, 2015, Theorem 4), see also (Ayele Tsegaye and Bekele Solomon, 2019, Theorem 1).

Theorem 9. *If $\psi \in H_{**}^{-\frac{1}{2}}(\partial\Omega)$ satisfies $\mathcal{V}_b \psi = 0$ on $\partial\Omega$, then $\psi = 0$.*

Following (McLean, 2000, Theorem 8.15), there exists a unique real-valued distribution $\psi_{eq} \in H^{-\frac{1}{2}}(\partial\Omega)$ called equilibrium density for $\partial\Omega$ such that $\mathcal{V}_\Delta \psi_{eq}$ is constant on $\partial\Omega$, and $(1, \psi_{eq})_{\partial\Omega} = 1$. For $n = 2$ the constant $\mathcal{V}_\Delta \psi_{eq}$ is not always positive and one introduces the *logarithmic capacity*, $\text{Cap}_{\partial\Omega}$ using the relation

$$\mathcal{V}_\Delta \psi_{eq} = \frac{1}{2\pi} \ln \left(\frac{r_0}{\text{Cap}_{\partial\Omega}} \right),$$

for some positive constant r_0 as in equation (19). In particular $\mathcal{V}_\Delta \psi_{eq} = 0$ if and only if $r_0 = \text{Cap}_{\partial\Omega}$.

The following statement is proved in (McLean, 2000, Theorem 8.16).

Theorem 10. *Let r_0 be some positive constant.*

- (i) *The operator $\mathcal{V}_\Delta : H^{-\frac{1}{2}}(\partial\Omega) \rightarrow H^{\frac{1}{2}}(\partial\Omega)$, is $H^{-\frac{1}{2}}(\partial\Omega)$ -elliptic, i.e., $\langle \mathcal{V}_\Delta \psi, \psi \rangle_{\partial\Omega} \geq c \|\psi\|_{H^{-\frac{1}{2}}(\partial\Omega)}^2$ for all $\psi \in H^{-\frac{1}{2}}(\partial\Omega)$, if and only if $r_0 > \text{Cap}_{\partial\Omega}$.*
- (ii) *The operator $\mathcal{V}_\Delta : H^{-\frac{1}{2}}(\partial\Omega) \rightarrow H^{\frac{1}{2}}(\partial\Omega)$, has a bounded inverse if and only if $r_0 \neq \text{Cap}_{\partial\Omega}$.*

The following theorem ensures the invertibility of the single layer potential operator \mathcal{V}_b in 2D.

Theorem 11. *Let $\Omega \subset \mathbb{R}^2$ with $r_0 > \text{diam}(\Omega)$. Then the single layer potential $\mathcal{V}_b : H^{-\frac{1}{2}}(\partial\Omega) \rightarrow H^{\frac{1}{2}}(\partial\Omega)$ is invertible.*

Proof. Since $\text{Cap}_{\partial\Omega} \leq \text{diam}(\Omega)$, see, (Yan and Sloan, 1988, p.553, properties 1 and 3), then $r_0 > \text{diam}(\Omega)$ implies $r_0 > \text{Cap}_{\partial\Omega}$. For the case $a = b$ the assertion is proved in (Dufera Tamirat and Mikhailov, 2015, Theorem 5). Due to the first relation in (36) and Theorem 10(ii) follows the invertibility of the single layer potential operator \mathcal{V}_b for the case $a \neq b$ as well, see also (Ayele Tsegaye and Bekele Solomon, 2019, Theorem 2). \square

As in Ayele Tsegaye and Bekele Solomon (2023), we shall restrict ourselves to Theorem 11 while discussing about the invertibility of single layer potential V_b in 2D. On the other hand, choosing an appropriate parameter r_0 , one can get

the zero kernel for \mathcal{V}_b not only on the subspace $H_{**}^{-1/2}(\partial\Omega)$ but also on the entire space $H^{-1/2}(\partial\Omega)$. The proof to the following result is due to (Ayele Tsegaye and Bekele Solomon, 2019, Lemma 1) and (Ayele Tsegaye, 2021b, Lemma 2).

Lemma 12. (i) Let $r_0 > \text{diam}(\Omega)$. If $\Psi^* \in H^{-\frac{1}{2}}(\partial\Omega)$ and $r_\Omega V_b \Psi^* = 0$ in Ω , then $\Psi^* = 0$.
(ii) If $\Phi^* \in H^{\frac{1}{2}}(\partial\Omega)$ and $r_\Omega W_b \Phi^* = 0$ in Ω , then $\Phi^* = 0$.

TWO-OPERATOR BDIE SYSTEMS FOR DIRICHLET PROBLEM

Let Ω be a domain in \mathbb{R}^2 bounded by a smooth curve $\partial\Omega$. We shall derive and investigate the two-operator BDIE systems for the following Dirichlet problem: for $\varphi_0 \in H^{\frac{1}{2}}(\partial\Omega)$ and $f \in H^{-1}(\Omega)$, find a function $u \in H^1(\Omega)$ satisfying

$$Au = f \quad \text{in } \Omega, \quad (69)$$

$$\gamma^+ u = \varphi_0 \quad \text{on } \partial\Omega. \quad (70)$$

Here equation (69) is understood in the distributional sense (4) and the Dirichlet boundary condition (70) is understood in the trace sense. The following assertion is well-known and can be proved e.g. using variational settings and the Lax-Milgram lemma:

Theorem 13. *The Dirichlet problem (69)-(70) is uniquely solvable in $H^1(\Omega)$. The solution is $u = (\mathcal{A}^D)^{-1}(f, \varphi_0)^\top$, where the inverse operator, $(\mathcal{A}^D)^{-1} : H^{-1}(\Omega) \times H^{\frac{1}{2}}(\partial\Omega) \rightarrow H^1(\Omega)$, to the left-hand side operator, $\mathcal{A}^D : H^1(\Omega) \rightarrow H^{-1}(\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$, of the Dirichlet problem (69)-(70), is continuous.*

BDIE system formulation to the Dirichlet problem. Following similar procedure as in Mikhailov (2015), let us reduce the Dirichlet problem (69)-(70) with $f \in H^{-1}(\Omega)$, for $u \in H^1(\Omega)$, to two different systems of segregated two-operator BDIEs.

Let $\tilde{f} \in \tilde{H}^{-1}(\Omega)$ be an extension of $f \in H^{-1}(\Omega)$ (i.e., $f = r_\Omega \tilde{f}$), which always exists, see, (Mikhailov, 2015, Lemma 2.15 and Theorem 2.16). We represent in (48), (66) and (67) the generalized conormal derivative and the trace of the function u as

$$T_a^+(\tilde{f}, u) = \psi, \quad \gamma^+ u = \varphi_0$$

respectively, and will regard the new unknown function $\psi \in H^{-\frac{1}{2}}(\partial\Omega)$ as formally segregated of u . Thus we will look for the couple

$$(u, \psi) \in H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega).$$

BDIE system (D1). To reduce BVP (69)-(70) to one of BDIE systems we will use equation (48) in Ω and equation (66) on $\partial\Omega$. Then we arrive at the system of BDIEs (D1),

$$u + \mathcal{Z}_b u + \mathcal{R}_b u - V_b \psi = \mathcal{F}_1^{D1} \quad \text{in } \Omega, \quad (71)$$

$$\gamma^+ \mathcal{Z}_b u + \gamma^+ \mathcal{R}_b u - \mathcal{V}_b \psi = \mathcal{F}_2^{D1} \quad \text{on } \partial\Omega, \quad (72)$$

where $F_0 := \mathcal{P}_b \tilde{f} - W_b \varphi_0$ and

$$\mathcal{F}^{D1} := \begin{bmatrix} \mathcal{F}_1^{D1} \\ \mathcal{F}_2^{D1} \end{bmatrix} = \begin{bmatrix} F_0 \\ \gamma^+ F_0 - \varphi_0 \end{bmatrix}. \quad (73)$$

For $\varphi_0 \in H^{\frac{1}{2}}(\partial\Omega)$, we have the inclusions $\mathcal{F}_1^{D1} = F_0 \in H^1(\Omega)$ if $\tilde{f} \in \tilde{H}^{-1}(\Omega)$ and due to the mapping properties of operators involved in (73), we have the inclusion $\mathcal{F}^{D1} \in H^1(\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$.

Remark 14. Let $\tilde{f} \in \tilde{H}^{-1}(\Omega)$ and $\varphi_0 \in H^{\frac{1}{2}}(\partial\Omega)$. Then $\mathcal{F}^{D1} = 0$ if and only if $(\tilde{f}, \varphi_0) = 0$.

Proof. The later equality implies the former. Conversely, let $\mathcal{F}^{D1} = 0$, that is, $F_0 = \mathcal{P}_b \tilde{f} - W_b \varphi_0 = 0$ in Ω and $\gamma^+ F_0 - \varphi_0 = 0$ on $\partial\Omega$. Then $\varphi_0 = 0$ on $\partial\Omega$ and $\mathcal{P}_b \tilde{f} = 0$ in Ω . Multiplying the later by b , we get $\mathcal{P}_\Delta \tilde{f} = 0$ in Ω and applying Laplace operator gives $\tilde{f} = 0$ in \mathbb{R}^2 . \square

BDIE system (D2). To obtain a BDIE system of the second kind, we will use equation (48) in Ω and equation (67) on $\partial\Omega$. Then we arrive at the system of BDIEs (D2),

$$u + \mathcal{Z}_b u + \mathcal{R}_b u - V_b \psi = \mathcal{P}_b \tilde{f} - W_b \varphi_0 \quad \text{in } \Omega, \quad (74)$$

$$\begin{aligned} \left(1 - \frac{a}{2b}\right) \psi + T_a^+ \mathcal{Z}_b u + T_a^+ \mathcal{R}_b u - \mathcal{W}'_{ab} \psi \\ = T_a^+(\tilde{f} + \mathring{E} \mathcal{R}_*^b \tilde{f}, \mathcal{P}_b \tilde{f}) - \mathcal{L}_{ab}^+ \varphi_0 \end{aligned} \quad \text{on } \partial\Omega, \quad (75)$$

where

$$\mathcal{F}^{D2} := \begin{bmatrix} \mathcal{F}_1^{D2} \\ \mathcal{F}_2^{D2} \end{bmatrix} = \begin{bmatrix} \mathcal{P}_b \tilde{f} - W_b \varphi_0 \\ T_a^+(\tilde{f} + \mathring{E} \mathcal{R}_*^b \tilde{f}, \mathcal{P}_b \tilde{f}) - \mathcal{L}_{ab}^+ \varphi_0 \end{bmatrix}. \quad (76)$$

Due to the mapping properties of operators involved in (76), we have the inclusion $\mathcal{F}^{D2} \in H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega)$. In similar way as in Remark 14, we can prove the following remark.

Remark 15. Let $\tilde{f} \in \tilde{H}^{-1}(\Omega)$ and $\varphi_0 \in H^{\frac{1}{2}}(\partial\Omega)$. Then $\mathcal{F}^{D2} = 0$ if and only if $(\tilde{f}, \varphi_0) = 0$.

Proof. The later equality implies the former. Conversely, let $\mathcal{F}^{D2} = 0$, that is, $F_0 = \mathcal{P}_b \tilde{f} - W_b \varphi_0 = 0$ in Ω and $T_a^+(\tilde{f} + \tilde{\mathcal{R}}_*^b \tilde{f}, \mathcal{P}_b \tilde{f}) - \mathcal{L}_{ab}^+ \varphi_0$ on $\partial\Omega$. Multiplying the first relation by b , we get $\mathcal{P}_\Delta \tilde{f} - W_\Delta(b\varphi_0) = 0$ in Ω . Taking into account that $bW_b(\varphi_0) = W_\Delta(b\varphi_0)$ is harmonic and applying Laplace operator gives $\tilde{f} = 0$ in \mathbb{R}^2 , and hence $W_b \varphi_0 = 0$ in Ω . Then by Lemma 12(ii), $\varphi_0 = 0$ on $\partial\Omega$. \square

BDIE systems equivalence to the Dirichlet problem.

Theorem 16. Let $\varphi_0 \in H^{\frac{1}{2}}(\partial\Omega)$, $f \in H^{-1}(\Omega)$ and $\tilde{f} \in \tilde{H}^{-1}(\Omega)$ is such that $r_\Omega \tilde{f} = f$. Then

(i) If $u \in H^1(\Omega)$ solves the BVP (69)-(70), then the couple $(u, \psi) \in H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega)$, where

$$\psi = T_a^+(\tilde{f}, u), \quad \text{on } \partial\Omega, \quad (77)$$

solves the BDIE systems (D1) and (D2).

(ii) If a couple $(u, \psi) \in H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega)$ solves BDIE system (D1) and $r_0 > \text{diam}(\Omega)$, then this solution is unique and solves BDIEs (D2), while u solves the Dirichlet problem (69)-(70), and ψ satisfies (77).

(iii) If a couple $(u, \psi) \in H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega)$ solves BDIE system (D2), then this solution is unique and solves BDIEs (D1), while u solves the Dirichlet problem (69)-(70), and ψ satisfies (77).

Proof. (i) Let $u \in H^1(\Omega)$ be a solution to BVP (69)-(70). Due to Theorem 13 it is unique. Setting ψ by (77) evidently implies, $\psi \in H^{-\frac{1}{2}}(\partial\Omega)$. From Theorem 8 and relations (66)-(67) follows that the couple (u, ψ) satisfies the BDIE systems (D1) and (D2), with the right-hand sides (73) and (76) respectively, which completes the proof of item (i).

Let now the couple $(u, \psi) \in H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega)$ solve BDIE system (D1) or (D2). Due to Theorem 8, the hypothesis of Lemma 4 are satisfied implying that u solves PDE (69) in Ω , while relations in (55) and (56) also hold.

(ii) Let the couple $(u, \psi) \in H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega)$ solve BDIE system (D1). Taking trace of (71) on $\partial\Omega$ and subtracting (72) from it we obtain

$$\gamma^+ u = \varphi_0 \quad \text{on } \partial\Omega, \quad (78)$$

that is, u satisfies the Dirichlet condition (70). (71) and Lemma 4 with $\Psi = \psi$, $\Phi = \varphi_0$ imply that $V_b \Psi^* - W_b \Phi^* = 0$, in Ω , where $\Psi^* = \psi - T_a^+(\tilde{f}, u)$ and $\Phi^* = \varphi_0 - \gamma^+ u$. Due to (78), $\Phi^* = 0$. Then Lemma 12(i) implies $\Psi^* = 0$, which proves condition (77). Thus u obtained from the solution of BDIE system (D1) solves the Dirichlet problem and hence, by item (i) of the theorem, (u, ψ) solve also BDIE system (D2).

(iii) Let now the couple $(u, \psi) \in H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega)$ solve BDIE system (D2). Taking generalized conormal derivative of (74) and subtracting (75) from it, we get $\psi = T_a^+(\tilde{f}, u)$ on $\partial\Omega$. Then substituting this in (56) gives $W_b(\varphi_0 - \gamma^+ u) = 0$ in Ω and Lemma 12(ii) then implies $\varphi_0 = \gamma^+ u$ on $\partial\Omega$. Due to (73), the BDIE system (71)-(72) with zero right-hand side can be considered as obtained for $\tilde{f} = 0$, $\varphi_0 = 0$, where $\tilde{f} \in \tilde{H}^{-1}(\Omega)$ is an extension of $f \in H^{-1}(\Omega)$, that is, $f = r_\Omega \tilde{f}$, implying that its solution is given by a solution of the homogeneous problem (69)-(70), which is zero by Theorem 13. This implies uniqueness of the solution of the inhomogeneous BDIE system (D1). Similar arguments work for the BDIE system (D2). \square

BDIE system operators invertibility for the Dirichlet problem. The BDIE systems (D1) and (D2) can be written as

$$\mathfrak{D}^1 \mathcal{U}^D = \mathcal{F}^{D1} \quad \text{and} \quad \mathfrak{D}^2 \mathcal{U}^D = \mathcal{F}^{D2},$$

respectively. Here $\mathcal{U}^D := (u, \psi)^\top \in H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega)$,

$$\mathfrak{D}^1 := \begin{bmatrix} I + \mathcal{Z}_b + \mathcal{R}_b & -V_b \\ \gamma^+ \mathcal{Z}_b + \gamma^+ \mathcal{R}_b & -\mathcal{V}_b \end{bmatrix}, \quad (79)$$

$$\mathfrak{D}^2 := \begin{bmatrix} I + \mathcal{Z}_b + \mathcal{R}_b & -V_b \\ T_a^+ \mathcal{Z}_b + T_a^+ \mathcal{R}_b & (1 - \frac{a}{2b}) I - \mathcal{W}'_{ab} \end{bmatrix}, \quad (80)$$

while \mathcal{F}^{D1} and \mathcal{F}^{D2} are given by (73) and (76) respectively. Due to the mapping properties of the operators involved in the definitions of the operators \mathfrak{D}^1 and \mathfrak{D}^2 as well as the right-hand sides \mathcal{F}^{D1} and \mathcal{F}^{D2} (see, e.g., (Ayele Tsegaye and Mikhailov, 2011, Appendix A), we have $\mathcal{F}^{D1} \in H^1(\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$, $\mathcal{F}^{D2} \in H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega)$, while the operators

$$\mathfrak{D}^1 : H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega) \rightarrow H^1(\Omega) \times H^{\frac{1}{2}}(\partial\Omega) \quad (81)$$

$$\mathfrak{D}^2 : H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega) \rightarrow H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega) \quad (82)$$

are continuous. Due to Theorem 16(ii)-(iii), operators (81) and (82) are injective.

Lemma 17. *For any couple $(\mathcal{F}_1, \mathcal{F}_2) \in H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega)$, there exists a unique couple $(\tilde{f}_{**}, \Phi_*) \in \tilde{H}^{-1}(\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$ such that*

$$\mathcal{F}_1 = \mathcal{P}_b \tilde{f}_{**} - W_b \Phi_* \quad (83)$$

$$\mathcal{F}_2 = T_a^+(\tilde{f}_{**} + \mathring{E}\mathcal{R}_*^b \tilde{f}_{**}, \mathcal{P}_b \tilde{f}_{**}) - \mathcal{L}_{ab}^+ \Phi_* \quad (84)$$

Moreover, $(\tilde{f}_{**}, \Phi_*) = \mathcal{C}_{**}(\mathcal{F}_1, \mathcal{F}_2)$ with $\mathcal{C}_{**} : H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega) \rightarrow \tilde{H}^{-1}(\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$ a linear continuous operator given by

$$\tilde{f}_{**} = \check{\Delta}(b\mathcal{F}_1) + \gamma^*(\mathcal{F}_2 + (\gamma^+ \mathcal{F}_1)\partial_n b) \quad (85)$$

$$\begin{aligned} \Phi_* = \frac{1}{b} \left(-\frac{1}{2}I + \mathcal{W}_\Delta \right)^{-1} \gamma^+ \left\{ -b\mathcal{F}_1 + \mathcal{P}_\Delta \left[\check{\Delta}(b\mathcal{F}_1) \right. \right. \\ \left. \left. + \gamma^* \left(\frac{b}{a}\mathcal{F}_2 + (\gamma^+ \mathcal{F}_1)\partial_n b \right) \right] \right\} \quad (86) \end{aligned}$$

where $\check{\Delta}(b\mathcal{F}_1) = \nabla \cdot \mathring{E}\nabla(b\mathcal{F}_1)$.

Proof. Let us first assume that there exist $(\tilde{f}_{**}, \Phi_*) \in \tilde{H}^{-1}(\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$ satisfying equations (83)-(84) and find their expression in terms of \mathcal{F}_1 and \mathcal{F}_2 . Let us rewrite (83) as

$$\mathcal{F}_1 - \mathcal{P}_b \tilde{f}_{**} = -W_b \Phi_* \quad \text{in } \Omega. \quad (87)$$

Multiplying (87) by b and applying Laplacian to it, we obtain,

$$\begin{aligned} \Delta(b\mathcal{F}_1 - \mathcal{P}_\Delta \tilde{f}_{**}) &= \Delta(b\mathcal{F}_1) - \tilde{f}_{**} \\ &= -\Delta(W_\Delta(b\Phi_*)) = 0 \quad \text{in } \Omega, \quad (88) \end{aligned}$$

which means

$$\Delta(b\mathcal{F}_1) = r_\Omega \tilde{f}_{**} \quad \text{in } \Omega, \quad (89)$$

and $b\mathcal{F}_1 - \mathcal{P}_\Delta \tilde{f}_{**} \in H^{1,0}(\Omega, \Delta)$ and hence $\mathcal{F}_1 - \mathcal{P}_b \tilde{f}_{**} \in H^{1,0}(\Omega, B) = H^{1,0}(\Omega, A)$. The latter implies that the canonical conormal derivatives $T_b^+(\mathcal{F}_1 - \mathcal{P}_b \tilde{f}_{**})$ and $T_a^+(\mathcal{F}_1 - \mathcal{P}_b \tilde{f}_{**})$ are well defined and can be also written in terms of their generalized conormal derivatives

$$\begin{aligned} \frac{b}{a} T_a^+(\mathcal{F}_1 - \mathcal{P}_b \tilde{f}_{**}) &= T_b^+(\mathcal{F}_1 - \mathcal{P}_b \tilde{f}_{**}) \\ &= T_b^+(\check{B}(\mathcal{F}_1 - \mathcal{P}_b \tilde{f}_{**}), \mathcal{F}_1 - \mathcal{P}_b \tilde{f}_{**}) \\ &= T_b^+(\mathring{E}\nabla \cdot (b\nabla(\mathcal{F}_1 - \mathcal{P}_b \tilde{f}_{**})), \mathcal{F}_1 - \mathcal{P}_b \tilde{f}_{**}) \\ &= T_b^+(\mathring{E}\Delta(b\mathcal{F}_1 - \mathcal{P}_\Delta \tilde{f}_{**}) \\ &\quad - \mathring{E}\nabla \cdot ((\mathcal{F}_1 - \mathcal{P}_b \tilde{f}_{**})\nabla b), \mathcal{F}_1 - \mathcal{P}_b \tilde{f}_{**}) \\ &= T_b^+(-\mathring{E}\nabla \cdot (\mathcal{F}_1 \nabla b) - \mathring{E}\mathcal{R}_*^b \tilde{f}_{**}, \mathcal{F}_1 - \mathcal{P}_b \tilde{f}_{**}) \end{aligned}$$

and therefore,

$$\begin{aligned} T_a^+(\mathcal{F}_1 - \mathcal{P}_b \tilde{f}_{**}) &= T_a^+(-\mathring{E}\nabla \cdot (\mathcal{F}_1 \nabla b) \\ &\quad - \mathring{E}\mathcal{R}_*^b \tilde{f}_{**}, \mathcal{F}_1 - \mathcal{P}_b \tilde{f}_{**}) \quad (90) \end{aligned}$$

where (62) and (89) were taken into account. Applying the conormal derivative operator T_a^+ to both sides of equation (87), substituting their (90), taking into account (11), we obtain,

$$\begin{aligned} T_a^+(\tilde{f}_{**} - \mathring{E}\nabla \cdot (\mathcal{F}_1 \nabla b), \mathcal{F}_1) \\ - T_a^+(\tilde{f}_{**} + \mathring{E}\mathcal{R}_*^b \tilde{f}_{**}, \mathcal{P}_b \tilde{f}_{**}) \\ = -\mathcal{L}_{ab}^+ \Phi_*, \quad \text{on } \partial\Omega. \quad (91) \end{aligned}$$

Subtracting (91) from (84), we get,

$$\mathcal{F}_2 = T_a^+(\tilde{f}_{**} - \mathring{E}\nabla \cdot (\mathcal{F}_1 \nabla b), \mathcal{F}_1) \quad \text{on } \partial\Omega. \quad (92)$$

Due to (89), we can represent

$$\begin{aligned} \tilde{f}_{**} &= \check{\Delta}(b\mathcal{F}_1) + \tilde{f}_{1*} \\ &= \nabla \cdot \mathring{E}\nabla(b\mathcal{F}_1) - \gamma^* \Psi_{**}, \quad (93) \end{aligned}$$

where $\tilde{f}_{1*} \in H_{\partial\Omega}^{-1}$ is defined by (2) and hence, due to e.g. (Mikhailov, 2011, Theorem 2.10) can be in turn represented as $\tilde{f}_{1*} = -\gamma^* \Psi_{**}$, with some $\Psi_{**} \in H^{-\frac{1}{2}}(\partial\Omega)$. Then (89) is satisfied and

$$\begin{aligned} \frac{b}{a} T_a^+(\tilde{f}_{**} - \mathring{E}\Delta \cdot (\mathcal{F}_1 \nabla b), \mathcal{F}_1) \\ = T_b^+(\tilde{f}_{**} - \mathring{E}\Delta \cdot (\mathcal{F}_1 \nabla b), \mathcal{F}_1) \\ = (\gamma^{-1})^* [\tilde{f}_{**} - \mathring{E}\nabla \cdot (\mathcal{F}_1 \nabla b) - \check{B}\mathcal{F}_1] \\ = (\gamma^{-1})^* [\tilde{f}_{**} - \mathring{E}\nabla \cdot (\mathcal{F}_1 \nabla b) - \nabla \cdot \mathring{E}(b\nabla \mathcal{F}_1)] \\ = (\gamma^{-1})^* [\nabla \cdot \mathring{E}\nabla(b\mathcal{F}_1) - \nabla \cdot \mathring{E}(b\nabla \mathcal{F}_1) \\ - \gamma^* \Psi_{**} - \mathring{E}\nabla \cdot (\mathcal{F}_1 \nabla b)] \\ = (\gamma^{-1})^* [\nabla \cdot \mathring{E}(\mathcal{F}_1 \nabla b) - \gamma^* \Psi_{**} - \mathring{E}\nabla \cdot (\mathcal{F}_1 \nabla b)] \\ = -\Psi_{**} - (\gamma^+ \mathcal{F}_1)\partial_n b \end{aligned}$$

for which

$$\begin{aligned} T_a^+(\tilde{f}_{**} - \mathring{E}\Delta \cdot (\mathcal{F}_1 \nabla b), \mathcal{F}_1) \\ = \frac{a}{b} [-\Psi_{**} - (\gamma^+ \mathcal{F}_1)\partial_n b] \quad (94) \end{aligned}$$

because

$$\begin{aligned} \langle (\gamma^{-1})^* [\nabla \cdot \mathring{E}(\mathcal{F}_1 \nabla b) - \gamma^* \Psi_{**} - \mathring{E}\nabla \cdot (\mathcal{F}_1 \nabla b)], w \rangle_{\partial\Omega} \\ = \langle [\nabla \cdot \mathring{E}(\mathcal{F}_1 \nabla b) - \gamma^* \Psi_{**} - \mathring{E}\nabla \cdot (\mathcal{F}_1 \nabla b)], \gamma^{-1} w \rangle_{\Omega} \\ = \langle [\nabla \cdot \mathring{E}(\mathcal{F}_1 \nabla b), \gamma^{-1} w]_{\mathbb{R}^2} \\ - \gamma^* \Psi_{**} - \langle \mathring{E}\nabla \cdot (\mathcal{F}_1 \nabla b), \gamma^{-1} w \rangle_{\Omega} \\ = -\langle [\mathring{E}(\mathcal{F}_1 \nabla b), \nabla(\gamma^{-1} w)]_{\mathbb{R}^2} - \gamma^* \Psi_{**} + \\ + \langle (\mathcal{F}_1 \nabla b), \nabla(\gamma^{-1} w) \rangle_{\Omega} - \langle n \cdot \gamma^+ (\mathcal{F}_1 \nabla b), \gamma^+ \gamma^- w \rangle_{\Omega} \\ = -\langle (\gamma^+ (\mathcal{F}_1 \nabla b), w) \rangle_{\partial\Omega} - \Psi_{**}. \quad (95) \end{aligned}$$

Hence (92) reduces to

$$\begin{aligned} \Psi_{**} &= -\frac{b}{a}\mathcal{F}_2 - (\gamma^+\mathcal{F}_1)\partial_nb \\ &= -T_b^+\mathcal{F}_1 - (\gamma^+\mathcal{F}_1)\partial_nb, \end{aligned} \tag{96}$$

and (93) to (85).

Now (87) can be written in the form

$$W_\Delta(b\Phi_*) = \mathcal{F}_\Delta \quad \text{in } \Omega, \tag{97}$$

where

$$\begin{aligned} \mathcal{F}_\Delta &:= -b\mathcal{F}_1 + \mathcal{P}_\Delta\tilde{f}_{**} \\ &= -b\mathcal{F}_1 + \mathcal{P}_\Delta\left[\check{\Delta}(b\mathcal{F}_1) + \gamma^*\left(\frac{b}{a}\mathcal{F}_2 + (\gamma^+\mathcal{F}_1)\partial_nb\right)\right] \end{aligned} \tag{98}$$

is harmonic function in Ω due to (88). The trace of Eq. (98) gives

$$-\frac{1}{2}b\Phi_* + \mathcal{W}_\Delta(b\Phi_*) = \gamma^+\mathcal{F}_\Delta \quad \text{on } \partial\Omega. \tag{99}$$

It is well known that the operator $[-\frac{1}{2}I + \mathcal{W}_\Delta]$ is an isomorphism, see (Steinbach, 2007, Lemmas 6.10 and 6.11), this implies

$$\begin{aligned} \Phi_* &= \frac{1}{b}\left(-\frac{1}{2}I + \mathcal{W}_\Delta\right)^{-1}\gamma^+\mathcal{F}_\Delta \\ &= \frac{1}{b}\left(-\frac{1}{2}I + \mathcal{W}_\Delta\right)^{-1}\gamma^+\left\{-b\mathcal{F}_1 + \mathcal{P}_\Delta\left[\check{\Delta}(b\mathcal{F}_1) + \gamma^*\left(\frac{b}{a}\mathcal{F}_2 + (\gamma^+\mathcal{F}_1)\partial_nb\right)\right]\right\}, \end{aligned}$$

which is Eq. (86). Relations (85), (86) can be written as $(\tilde{f}_{**}, \Phi_*) = \mathcal{C}_{**}(\mathcal{F}_1, \mathcal{F}_2)$, where $\mathcal{C}_{**} : H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega) \rightarrow \tilde{H}^{-1}(\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$ is a linear continuous operator, as required. We still have to check that the functions \tilde{f}_{**} and Φ_* , given by (85) and (86), satisfy equations (83) and (84). Indeed, Φ_* given by (86) satisfies equation (99) and thus $\gamma^+W_\Delta(a\Phi_*) = \gamma^+\mathcal{F}_\Delta$. Since both $W_\Delta(a\Phi_*)$ and \mathcal{F}_Δ are harmonic functions, this implies (97)-(98) and by (85) also (83). Finally, (85) implies by (94) that (92) is satisfied, and adding (91) to it leads to (84). Let us prove that the operator \mathcal{C}_{**} is unique. Indeed, let a couple $(\tilde{f}_{**}, \Phi_*) \in \tilde{H}^{-1}(\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$ be a solution of linear system (83)-(84) with $\mathcal{F}_1 = 0$ and $\mathcal{F}_2 = 0$. Then (89) implies that $r_\Omega\tilde{f}_{**} = 0$ in Ω , that is $\tilde{f}_{**} \in H_{\partial\Omega}^{-1} \subset \tilde{H}^{-1}(\Omega)$. Hence (92) reduces to

$$0 = T_a^+(\tilde{f}_{**}, 0) \quad \text{on } \partial\Omega. \tag{100}$$

By the first Green identity (10), this gives,

$$\begin{aligned} 0 &= \langle T_a^+(\tilde{f}_{**}, 0), \gamma^+v \rangle_{\partial\Omega} = \langle \tilde{f}_{**}, v \rangle_\Omega, \\ &\quad \forall v \in H^1(\Omega), \end{aligned} \tag{101}$$

which implies $\tilde{f}_{**} = 0$ in \mathbb{R}^2 . Finally, (86) gives $\Phi_* = 0$. Hence any solution of non-homogeneous linear system (83) – (84) has only one solution, which implies the uniqueness of the operator \mathcal{C}_{**} . \square

The following assertion is (Mikhailov, 2005b, Lemma 19) generalized to a wider space in 2D.

Lemma 18. *For any couple $(\tilde{\mathcal{F}}_1, \tilde{\mathcal{F}}_2) \in H^1(\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$, there exists a unique couple $(\tilde{f}_{**}, \Phi_*) \in \tilde{H}^{-1}(\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$ such that*

$$\tilde{\mathcal{F}}_1 = \mathcal{P}_b\tilde{f}_{**} - W_b\Phi_* \tag{102}$$

$$\tilde{\mathcal{F}}_2 = \gamma^+(\mathcal{P}_b\tilde{f}_{**} - W_b\Phi_*) \tag{103}$$

Moreover, $(\tilde{f}_{**}, \Phi_*) = \tilde{\mathcal{C}}_{**}(\tilde{\mathcal{F}}_1, \tilde{\mathcal{F}}_2)$ with $\tilde{\mathcal{C}}_{**} : H^1(\Omega) \times H^{\frac{1}{2}}(\partial\Omega) \rightarrow \tilde{H}^{-1}(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega)$ a linear continuous operator is given by

$$\tilde{f}_{**} = \check{\Delta}(b\tilde{\mathcal{F}}_1) + \gamma^*(T_b^+\tilde{\mathcal{F}}_1 + \tilde{\mathcal{F}}_2)\partial_nb \tag{104}$$

$$\begin{aligned} \Phi_* &= \frac{1}{b}\left(-\frac{1}{2}I + \mathcal{W}_\Delta\right)^{-1}\left(-b\tilde{\mathcal{F}}_2 + \gamma^+\mathcal{P}_\Delta[\check{\Delta}(b\tilde{\mathcal{F}}_1) + \gamma^*(T_b^+\tilde{\mathcal{F}}_1 + \tilde{\mathcal{F}}_2)\partial_nb]\right) \end{aligned} \tag{105}$$

where $\check{\Delta}(b\tilde{\mathcal{F}}_1) = \nabla \cdot \check{E}\nabla(b\tilde{\mathcal{F}}_1)$.

Proof. Let us first assume that there exist $(\tilde{f}_{**}, \Phi_*) \in \tilde{H}^{-1}(\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$ satisfying equations (102)-(103) and find their expression in terms of $\tilde{\mathcal{F}}_1$ and $\tilde{\mathcal{F}}_2$. Let us re write (102) as

$$\tilde{\mathcal{F}}_1 - \mathcal{P}_b\tilde{f}_{**} = -W_b\Phi_* \quad \text{in } \Omega. \tag{106}$$

Multiplying (106) by b and applying Laplacian to it, we obtain,

$$\begin{aligned} \Delta(b\tilde{\mathcal{F}}_1 - \mathcal{P}_\Delta\tilde{f}_{**}) &= \Delta(b\tilde{\mathcal{F}}_1) - \tilde{f}_{**} \\ &= -\Delta(W_\Delta(b\Phi_*)) = 0 \quad \text{in } \Omega, \end{aligned} \tag{107}$$

which means

$$\Delta(b\tilde{\mathcal{F}}_1) = r_\Omega\tilde{f}_{**} \quad \text{in } \Omega, \tag{108}$$

and $b\tilde{\mathcal{F}}_1 - \mathcal{P}_\Delta\tilde{f}_{**} \in H^{1,0}(\Omega, \Delta)$, while $\tilde{\mathcal{F}}_1 - \mathcal{P}_b\tilde{f}_{**} \in H^{1,0}(\Omega, B) = H^{1,0}(\Omega, A)$. The latter imply that the canonical conormal derivatives $T_b^+(\tilde{\mathcal{F}}_1 - \mathcal{P}_b\tilde{f}_{**})$ and $T_a^+(\tilde{\mathcal{F}}_1 - \mathcal{P}_b\tilde{f}_{**})$ are well defined and $T_a^+(\tilde{\mathcal{F}}_1 - \mathcal{P}_b\tilde{f}_{**}) = \frac{b}{a}T_b^+(\tilde{\mathcal{F}}_1 - \mathcal{P}_b\tilde{f}_{**})$. Due to (108) and using $\tilde{f}_{1*} = -\gamma^*\Psi_{**}$ with some $\Psi_{**} \in H^{-\frac{1}{2}}(\partial\Omega)$ as in (96), we can represent

$$\begin{aligned} \tilde{f}_{**} &= \check{\Delta}(b\tilde{\mathcal{F}}_1) + \tilde{f}_{1*} \\ &= \nabla \cdot \check{E}\nabla(b\tilde{\mathcal{F}}_1) - \gamma^*\Psi_{**}, \end{aligned} \tag{109}$$

where $\tilde{f}_{1*} \in H_{\partial\Omega}^{-1}$. Then (108) is satisfied. Replacing \mathcal{F}_2 by $T_a^+(\tilde{\mathcal{F}}_1, u)$ in Lemma 17, relation

(96) yields,

$$\begin{aligned}\Psi_{**} &= -\frac{b}{a}T_a^+\tilde{\mathcal{F}}_1 - (\gamma^+\tilde{\mathcal{F}}_1)\partial_n b \\ &= -T_b^+\tilde{\mathcal{F}}_1 - \tilde{\mathcal{F}}_2\partial_n b \quad (110)\end{aligned}$$

and (109) reduces to (104). Now (106) can be written in the form

$$W_\Delta(b\Phi_*) = \mathcal{Q}_\Delta \quad \text{in } \Omega, \quad (111)$$

where

$$\begin{aligned}\mathcal{Q}_\Delta &:= -b\tilde{\mathcal{F}}_1 + \mathcal{P}_\Delta\tilde{f}_{**} = -b\tilde{\mathcal{F}}_1 + \mathcal{P}_\Delta[\check{\Delta}(b\tilde{\mathcal{F}}_1) \\ &\quad + \gamma^*(T_b^+\tilde{\mathcal{F}}_1 + (\gamma^+\tilde{\mathcal{F}}_1)\partial_n b)] \quad (112)\end{aligned}$$

is harmonic function in Ω due to (107). The trace of equation (112) gives

$$-\frac{1}{2}b\Phi_* + \mathcal{W}_\Delta(b\Phi_*) = \gamma^+\mathcal{Q}_\Delta \quad \text{on } \partial\Omega. \quad (113)$$

By similar argument as in Lemma 17, the operator $-\frac{1}{2}I + \mathcal{W}_\Delta : H^{\frac{1}{2}}(\partial\Omega) \rightarrow H^{-\frac{1}{2}}(\partial\Omega)$ is an isomorphism this implies

$$\begin{aligned}\Phi_* &= \frac{1}{b}\left(-\frac{1}{2}I + \mathcal{W}_\Delta\right)^{-1}\gamma^+\mathcal{Q}_\Delta \\ &= \frac{1}{b}\left(-\frac{1}{2}I + \mathcal{W}_\Delta\right)^{-1}\gamma^+\{-b\tilde{\mathcal{F}}_1 + \mathcal{P}_\Delta[\check{\Delta}(b\tilde{\mathcal{F}}_1) \\ &\quad + \gamma^*(T_b^+\tilde{\mathcal{F}}_1 + (\gamma^+\tilde{\mathcal{F}}_1)\partial_n b)]\}\end{aligned}$$

which is Eq. (105). Relations (104), (105) can be written as $(\tilde{f}_{**}, \Phi_*) = \tilde{\mathcal{C}}_{**}(\tilde{\mathcal{F}}_1, \tilde{\mathcal{F}}_2)$, where $\tilde{\mathcal{C}}_{**} : H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega) \rightarrow \tilde{H}^{-1}(\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$ is a linear continuous operator, as required. We still have to check that the functions \tilde{f}_{**} and Φ_* , given by (104) and (105), satisfy equations (102) and (103). Indeed, Φ_* given by (105) satisfies equation (113) and thus $\gamma^+W_\Delta(a\Phi_*) = \gamma^+\mathcal{Q}_\Delta$. Since both $W_\Delta(a\Phi_*)$ and \mathcal{Q}_Δ are harmonic functions, this implies (111)-(112) and by (104) also (102) while (103) follows from Eqs.(104) and (111).

Let us prove that the operator $\tilde{\mathcal{C}}_{**}$ is unique. Indeed, let a couple $(\tilde{f}_{**}, \Phi_*) \in \tilde{H}^{-1}(\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$ be a solution of linear system (102)-(103) with $\tilde{\mathcal{F}}_1 = 0$ and $\tilde{\mathcal{F}}_2 = 0$. Then (108) implies that $r_\Omega\tilde{f}_{**} = 0$ in Ω , that is $\tilde{f}_{**} \in H_{\partial\Omega}^{-1} \subset \tilde{H}^{-1}(\Omega)$. Hence (92) reduces to

$$0 = T_a^+(\tilde{f}_{**}, 0) \quad \text{on } \partial\Omega. \quad (114)$$

By the first Green identity (10), this gives relation (101), which implies $\tilde{f}_{**} = 0$ in \mathbb{R}^2 . Finally, (105) gives $\Phi_* = 0$. Hence any solution of nonhomogeneous linear system (102)–(103) has only one solution, which implies the uniqueness of the operator $\tilde{\mathcal{C}}_{**}$. \square

Theorem 19. *Let $r_0 > \text{diam}(\Omega)$. The operators (81) and (82) are continuous and continuously invertible.*

Proof. The continuity of operators (81) and (82) is proved above. To prove the invertibility of operator (81), let us consider the BDIE system (D1) with arbitrary right-hand side

$$\mathcal{F}_*^{D1} = (\mathcal{F}_{*1}^{D1}, \mathcal{F}_{*2}^{D1})^T \in H^1(\Omega) \times H^{\frac{1}{2}}(\partial\Omega).$$

Take $\tilde{\mathcal{F}}_1 = \mathcal{F}_{*1}^{D1}$ and $\Phi_* = \gamma^+\mathcal{F}_{*1}^{D1} - \mathcal{F}_{*2}^{D1}$ in Lemma 18, to obtain the representation of \mathcal{F}_*^{D1} as:

$$\mathcal{F}_{*1}^{D1} = \tilde{\mathcal{F}}_1 \quad \mathcal{F}_{*2}^{D1} = \gamma^+\tilde{\mathcal{F}}_1 - \Phi_*$$

where the couple

$$(\tilde{f}_*, \Phi_*) = \tilde{\mathcal{C}}_{**}(\tilde{\mathcal{F}}_1, \tilde{\mathcal{F}}_2) \in \tilde{H}^{-1}(\Omega) \times H^{\frac{1}{2}}(\partial\Omega) \quad (115)$$

is unique and the operator

$$\tilde{\mathcal{C}}_{**} : H^1(\Omega) \times H^{\frac{1}{2}}(\partial\Omega) \rightarrow \tilde{H}^{-1}(\Omega) \times H^{\frac{1}{2}}(\partial\Omega) \quad (116)$$

is linear and continuous. If $r_0 > \text{diam}(\Omega)$, then taking into account (Mikhailov, 2015, Remark 5.3) and applying Theorem 13 with $f = r_\Omega\tilde{f} = r_\Omega\tilde{f}_*$, $\Phi_* = \varphi_0$, we obtain that BDIE system (D1) is uniquely solvable and its solution is: $\mathcal{U}_1 = (\mathcal{A}^D)^{-1}(r_\Omega\tilde{f}, \varphi_0)^\top$, $\mathcal{U}_2 = \gamma^+\mathcal{U}_1 - \varphi_0$, where the inverse operator, $(\mathcal{A}^D)^{-1} : H^{-1}(\Omega) \times H^{\frac{1}{2}}(\partial\Omega) \rightarrow H^1(\Omega)$, to the left-hand side operator, $\mathcal{A}^D : H^1(\Omega) \rightarrow H^{-1}(\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$, of the Dirichlet problem (69)–(70), is continuous. Representation (115) and continuity of the operator (116) imply invertibility of (81). To prove the invertibility of operator (82), let us consider the BDIE system (D2) with arbitrary right-hand side

$$\mathcal{F}_*^{D2} = (\mathcal{F}_{*1}^{D2}, \mathcal{F}_{*2}^{D2})^\top \in H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega).$$

Take $\mathcal{F}_1 = \mathcal{F}_{*1}^{D2}$ and $\mathcal{F}_2 = T_a^+(\mathcal{F}_1, u) = \mathcal{F}_{*2}^{D2}$ in Lemma 17 to represent \mathcal{F}_*^{D2} as

$$\mathcal{F}_{*1}^{D2} = \mathcal{F}_1 \quad \mathcal{F}_{*2}^{D2} = T_a^+(\mathcal{F}_1, u) = \mathcal{F}_2$$

and the couple

$$(\tilde{f}_{**}, \Phi_*) = \tilde{\mathcal{C}}_{**}(\mathcal{F}_1, \mathcal{F}_2) \in \tilde{H}^{-1}(\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$$

is unique and the operator

$$\tilde{\mathcal{C}}_{**} : H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega) \rightarrow \tilde{H}^{-1}(\Omega) \times H^{\frac{1}{2}}(\partial\Omega) \quad (117)$$

is linear and continuous. Taking into account (Mikhailov, 2015, Remark 5.3) and applying Theorem 13 with $\tilde{f} = \tilde{f}_{**}$, $\Phi_* = \varphi_0$, we obtain that BDIE system (D2) is uniquely solvable and its solution is: $\mathcal{U}_1 = (\mathcal{A}^D)^{-1}(r_\Omega\tilde{f}, \varphi_0)^\top$, $\mathcal{U}_2 = T_a^+(r_\Omega\tilde{f}, \mathcal{U}_1)$, where the inverse operator, $(\mathcal{A}^D)^{-1} : H^{-1}(\Omega) \times$

$H^{\frac{1}{2}}(\partial\Omega) \rightarrow H^1(\Omega)$, to the left-hand side operator, $\mathcal{A}^D : H^1(\Omega) \rightarrow H^{-1}(\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$, of the Dirichlet problem (69)–(70), is continuous. Representation (115) and continuity of the operator (117) imply invertibility of (82). \square

TWO-OPERATOR BDIE SYSTEMS FOR NEUMANN PROBLEM

Let Ω be a domain in \mathbb{R}^2 bounded by a smooth curve $\partial\Omega$. We shall derive and investigate the two-operator BDIE systems for the following Neumann problem: for $\psi_0 \in H^{-\frac{1}{2}}(\partial\Omega)$ and $\tilde{f} \in \tilde{H}^{-1}(\Omega)$, find a function $u \in H^1(\Omega)$ satisfying

$$Au = r_\Omega \tilde{f} \quad \text{in } \Omega, \quad (118)$$

$$T_a^+(\tilde{f}, u) = \psi_0 \quad \text{on } \partial\Omega. \quad (119)$$

Here Eq. (118) is understood in the distributional sense (4) and the Neumann boundary condition (119) in the weak sense (10). The following assertion is well-known and can be proved e.g. using variational settings and the Lax-Milgram lemma.

Theorem 20. (i) The homogeneous Neumann problem (118)-(119) admits only linearly independent solution $u^0 = 1$ in $H^1(\Omega)$.

(ii) The nonhomogeneous Neumann problem (118)-(119) is solvable if and only if the following solvability condition is satisfied.

$$\langle \tilde{f}, u^0 \rangle_\Omega - \langle \psi_0, \gamma^+ u^0 \rangle_{\partial\Omega} = 0 \quad (120)$$

BDIE system formulation for the Neumann problem. We explore different possibilities of reducing the Neumann problem (118)–(119) with $\tilde{f} \in \tilde{H}^{-1}(\Omega)$, for $u \in H^1(\Omega)$, to two different segregated boundary-domain integral equations (BDIE) systems. Corresponding formulations for the Neumann problem for $u \in H^{1,0}(\Omega, \Delta)$ with $f \in L_2(\Omega)$ in 2D were introduced and analysed in Ayele Tsegaye *et al.* (2017). Let us represent in (48), (66) and (67) the generalised conormal derivative and the trace of the function u as

$$T_a^+(\tilde{f}, u) = \psi_0, \quad \gamma^+ u = \varphi,$$

and will regard the new unknown function $\varphi \in H^{\frac{1}{2}}(\partial\Omega)$ as formally segregated of u . Thus we will look for the couple $(u, \varphi) \in H^1(\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$.

BDIE system (N1). To reduce BVP (118)–(119) to a BDIE system in this section we will use equation (48) in Ω and equation (67) on $\partial\Omega$.

Then we arrive at the following system, (N1), of two boundary-domain integral equations for the couple of unknowns, (u, φ) ,

$$u + \mathcal{Z}_b u + \mathcal{R}_b u + W_b \varphi = \mathcal{F}_1^{N1}, \quad \text{in } \Omega, \quad (121)$$

$$T_a^+ \mathcal{Z}_b u + T_a^+ \mathcal{R}_b u + \mathcal{L}_{ab}^+ \varphi = \mathcal{F}_2^{N1}, \quad \text{on } \partial\Omega, \quad (122)$$

where

$$\mathcal{F}^{N1} := \begin{bmatrix} \mathcal{F}_1^{N1} \\ \mathcal{F}_2^{N1} \end{bmatrix} = \begin{bmatrix} \mathcal{P}_b \tilde{f} + V_b \psi_0 \\ T_a^+(\tilde{f} + \mathring{E} \mathcal{R}_*^b \tilde{f}, \mathcal{P}_b \tilde{f}) - \psi_0 + \frac{a}{2b} \psi_0 + \mathcal{W}'_{ab} \psi_0 \end{bmatrix} \quad (123)$$

Due to the mapping properties of operators involved in (123) we have $\mathcal{F}^{N1} \in H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega)$ and $\mathcal{F}_0^N := \mathcal{P}_b \tilde{f} + V_b \psi_0 \in H^1(\Omega)$.

Remark 21. Let $\tilde{f} \in \tilde{H}^{-1}(\Omega)$, $\psi_0 \in H^{-\frac{1}{2}}(\partial\Omega)$ and $r_0 > \text{diam}(\Omega)$. Then $\mathcal{F}^{N1} = 0$ if and only if $(\tilde{f}, \psi_0) = 0$.

Proof. The later equality implies the former. Conversely, let $\mathcal{F}^{N1} = 0$, that is, $\mathcal{P}_b \tilde{f} + V_b \psi_0 = 0$ in Ω and $T_a^+(\tilde{f} + \mathring{E} \mathcal{R}_*^b \tilde{f}, \mathcal{P}_b \tilde{f}) - \psi_0 + \frac{a}{2b} \psi_0 + \mathcal{W}'_{ab} \psi_0 = 0$ on $\partial\Omega$. Multiplying the first relation by b gives $\mathcal{P}_\Delta \tilde{f} + V_\Delta \psi_0 = 0$ in Ω . Further, taking into account that $bV_b \psi_0 = V_\Delta \psi_0$ is harmonic and applying Laplace operator we get $\tilde{f} = 0$ in \mathbb{R}^2 and hence $V_b \psi_0 = 0$ in Ω . Then due to Lemma 12(i), we get $\psi_0 = 0$ on $\partial\Omega$. \square

BDIE system (N2). To obtain a segregated BDIE system of the second kind, we will use equation (48) in Ω and equation (66) on $\partial\Omega$. Then we arrive at the following system, (D2), of boundary-domain integral equation systems,

$$u + \mathcal{Z}_b u + \mathcal{R}_b u + W_b \varphi = \mathcal{P}_b \tilde{f} + V_b \psi_0, \quad \text{in } \Omega, \quad (124)$$

$$\frac{1}{2} \varphi + \gamma^+ \mathcal{Z}_b u + \gamma^+ \mathcal{R}_b u + \mathcal{W}_b \varphi = \gamma^+ \mathcal{P}_b \tilde{f} + \mathcal{V}_b \psi_0, \quad \text{on } \partial\Omega, \quad (125)$$

where

$$\mathcal{F}^{N2} := \begin{bmatrix} \mathcal{F}_1^{N2} \\ \mathcal{F}_2^{N2} \end{bmatrix} = \begin{bmatrix} \mathcal{P}_b \tilde{f} + V_b \psi_0 \\ \gamma^+ \mathcal{P}_b \tilde{f} + \mathcal{V}_b \psi_0 \end{bmatrix}. \quad (126)$$

Due to the mapping properties of operators involved in (126), we have the inclusion $\mathcal{F}_1^{N2} = \mathcal{P}_b \tilde{f} + V_b \psi_0 \in H^1(\Omega)$ and $\mathcal{F}^{N2} \in H^1(\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$.

Remark 22. Let $\tilde{f} \in \tilde{H}^{-1}(\Omega)$, $\psi_0 \in H^{-\frac{1}{2}}(\partial\Omega)$ and $r_0 > \text{diam}(\Omega)$. Then $\mathcal{F}^{N2} = 0$ if and only if $(\tilde{f}, \psi_0) = 0$.

Proof. The later equality implies the former. Conversely, let $\mathcal{F}^{N2} = 0$, that is, $\mathcal{P}_b\tilde{f} + V_b\psi_0 = 0$ in Ω and $\gamma^+\mathcal{P}_b\tilde{f} + \mathcal{V}_b\psi_0 = 0$ on $\partial\Omega$. Multiplying the first relation by b gives $\mathcal{P}_\Delta\tilde{f} + V_\Delta\psi_0 = 0$ in Ω . Further, taking into account that $bV_b\psi_0 = V_\Delta\psi_0$ is harmonic and applying Laplace operator we get $\tilde{f} = 0$ in \mathbb{R}^2 and hence $V_b\psi_0 = 0$ in Ω . Then due to Lemma 12(i), we get $\psi_0 = 0$ on $\partial\Omega$. \square

BDIE systems equivalence to the Neumann problem.

Theorem 23. Let $\psi_0 \in H^{-\frac{1}{2}}(\partial\Omega)$ and $\tilde{f} \in \tilde{H}^{-1}(\Omega)$.

- (i) If a function $u \in H^1(\Omega)$ solves the BVP (118)-(119), then the couple (u, φ) , where

$$\varphi = \gamma^+u \quad (127)$$

solves the BDIE systems (N1) and (N2).

- (ii) If a couple $(u, \varphi) \in H^1(\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$ solves the BDIE system (N1), then the u solves BDIE system (N2) and u solves the Neumann problem (118)-(119) and φ satisfies (127).

- (iii) If a couple $(u, \varphi) \in H^1(\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$ solves the BDIE system (N2) and $r_0 > \text{diam}(\Omega)$, then the u solves BDIE system (N1) and Neumann problem (118)-(119) and φ satisfies (127).

- (iv) The homogeneous BDIE systems (N1) and (N2) have unique linearly independent solution spanned by $\mathcal{U}_0 = (u^0, \varphi^0)^\top = (1, 1)^\top$ in $H^1(\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$. Condition (120) is necessary and sufficient for solvability of the nonhomogeneous BDIE systems (N1) and, if $r_0 > \text{diam}(\Omega)$, also of the system (N2), in $H^1(\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$.

Proof. (i) Let $u \in H^1(\Omega)$ be a solution to the Neumann BVP (118)-(119). It immediately follows from Theorem 48 and relations (66)-(67) that the couple (u, φ) with $\varphi = \gamma^+u$ satisfies the BDIE systems (N1) and (N2), which proves item (i).

(ii) Let now a couple $(u, \varphi) \in H^1(\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$ solve BDIE system (N1) or (N2). Due to the first equations in the BDIE systems, the hypotheses of Lemma 4 are satisfied implying that u is a solution of equation (118) in Ω , and equations (55)-(58) hold for $\Psi = \psi_0$ and $\Phi = \varphi$.

If a couple $(u, \varphi) \in H^1(\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$ solve the system (N1) then subtracting (58) from (122) gives $T_a^+(\tilde{f}, u) = \psi_0$ on $\partial\Omega$. Thus Neumann (119) is satisfied. Further, from (56) we derive $W_b(\gamma^+u - \varphi) = 0$ in Ω , where $\gamma^+u = \varphi$ on $\partial\Omega$ by Lemma 12 completing the proof of item (ii).

(iii) Let now couple $(u, \varphi) \in H^1(\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$ solve BDIE system (N2). Further, taking the trace of (124) on $\partial\Omega$ and comparing the results with (125), we easily derive that $\gamma^+u = \varphi$ on $\partial\Omega$. Lemma 4 for equation (124) implies that u is a solution of equation (118), while equations (55)-(58) hold for $\Psi = \psi_0$ and $\Phi = \varphi$. Further, from (56) we derive

$$V_b(\psi_0 - T_a(\tilde{f}, u)) = 0 \quad \text{in } \Omega,$$

whence $T_a(\tilde{f}, u) = \psi_0$ on $\partial\Omega$ due to Lemma 12 (i) and u solves Neumann problem (118)-(119) which completes the proof of item (iii).

(iv) Theorem 20 along with items (i) and (ii) imply the claims of item (iv) for BDIE system (N2) and (N1). \square

Properties of BDIE system operators for the Neumann problem. BDIE systems (N1) and (N2) can be written respectively, as

$$\mathfrak{R}^1\mathcal{U}^N = \mathcal{F}^{N1}, \quad \mathfrak{R}^2\mathcal{U}^N = \mathcal{F}^{N2}, \quad (128)$$

where $\mathcal{U}^N = (u, \varphi)^\top \in H^1(\Omega) \times H^{\frac{1}{2}}(\partial\Omega_D)$, while \mathcal{F}^{N1} and \mathcal{F}^{N2} are given by Eqs. (123) and (126) respectively. Due to the mapping properties of potentials in (123) and (126), $\mathcal{F}^{N1} \in H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega)$ and $\mathcal{F}^{N2} \in H^1(\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$.

$$\mathfrak{R}^1 := \begin{bmatrix} I + \mathcal{Z}_b + \mathcal{R}_b & W_b \\ T_a^+\mathcal{Z}_b + T_a^+\mathcal{R}_b & \mathcal{L}_{ab}^+ \end{bmatrix},$$

$$\mathfrak{R}^2 := \begin{bmatrix} I + \mathcal{Z}_b + \mathcal{R}_b & W_b \\ \gamma^+\mathcal{Z}_b + \gamma^+\mathcal{R}_b & \frac{1}{2}I + W_b \end{bmatrix}.$$

Due to the mapping properties of potentials in (123) and (126), the right hand sides of BDIE systems (N1) and (N2) are such that $\mathcal{F}^{N1} \in H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega)$ and $\mathcal{F}^{N2} \in H^1(\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$.

Theorem 24. The operators

$$\mathfrak{R}^1 : H^1(\Omega) \times H^{\frac{1}{2}}(\partial\Omega) \rightarrow H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega), \quad (129)$$

$$\mathfrak{R}^2 : H^1(\Omega) \times H^{\frac{1}{2}}(\partial\Omega) \rightarrow H^1(\Omega) \times H^{\frac{1}{2}}(\partial\Omega), \quad (130)$$

are continuous. They have one-dimensional null spaces, $\ker \mathfrak{R}^1 = \ker \mathfrak{R}^2$, in $H^1(\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$, spanned over the element $(u^0, \varphi^0) = (1, 1)$.

Proof. The mapping properties of the potentials imply continuity of the operators (129) and (130). The claims that $\ker \mathfrak{R}^1$ and $\ker \mathfrak{R}^2$ are

one-dimensional and the couple $(u^0, \varphi^0) = (1, 1)$ belong to $\ker \mathfrak{A}^1 = \ker \mathfrak{A}^2$ directly follows from Theorem 23(iii). \square

To describe in more details the range of operators (129) and (130), i.e., to give more information about the co-kernels of these operators, we will need several auxiliary assertions. First of all, let us remark that for any $v \in H^{s-\frac{3}{2}}(\partial\Omega)$, $s < \frac{3}{2}$, the single layer potential can be defined as follows:

$$\begin{aligned} V_b v(y) &:= -\langle \gamma P_b(\cdot, y), v \rangle_{\partial\Omega} = -\langle P_b(\cdot, y), \gamma^* v \rangle_{\mathbb{R}^3} \\ &= -\mathbf{P}_b \gamma^* v(y), \quad y \in \mathbb{R}^2 \setminus \partial\Omega. \end{aligned} \quad (131)$$

where $\gamma^* : H^{s-\frac{3}{2}}(\partial\Omega) \rightarrow H^{s-2}_{\partial\Omega}$, $s < \frac{3}{2}$, is the operator adjointed to the trace operator $\gamma : H^{2-s}(\mathbb{R}^3) \rightarrow H^{\frac{3}{2}-s}(\partial\Omega)$, and the space $H^s_{\partial\Omega}$ is defined by (2).

Lemma 25. *Let $\tilde{f} \in \tilde{H}^{s-2}(\Omega)$, $s > \frac{1}{2}$ and $r_0 > \text{diam}(\Omega)$. If*

$$r_\Omega \mathbf{P}_b \tilde{f} = 0 \quad \text{in } \Omega, \quad (132)$$

then $\tilde{f} = 0$ in \mathbb{R}^2 .

Proof. Multiplying (132) by b , taking into account the first relation in (34) and applying the Laplace operator, we obtain $r_\Omega \tilde{f} = 0$, which means $\tilde{f} \in H^{s-2}_{\partial\Omega}$. If $s \geq \frac{3}{2}$, then $\tilde{f} = 0$ by (Mikhailov, 2011, Theorem 2.10). If $\frac{1}{2} < s < \frac{3}{2}$, then by the same theorem there exists $v \in H^{s-\frac{3}{2}}(\partial\Omega)$ such that $\tilde{f} = \gamma^* v$. This gives $\mathbf{P}_b \tilde{f} = \mathbf{P}_b \gamma^* v = -V_b v$ in \mathbb{R}^2 . Then (132) reduces to $V_b v = 0$ in Ω , which by Lemma 12(i) (for $s = 1$, which can be generalized to $\frac{1}{2} < s < \frac{3}{2}$) implies $v = 0$ on $\partial\Omega$ and thus $\tilde{f} = 0$ in \mathbb{R}^2 . \square

Theorem 26. *Let $\frac{1}{2} < s < \frac{3}{2}$ and $r_0 > \text{diam}(\Omega)$. The operator*

$$\mathbf{P}_b : \tilde{H}^{s-2}(\Omega) \rightarrow H^s(\Omega) \quad (133)$$

and its inverse

$$(\mathbf{P}_b)^{-1} : H^s(\Omega) \rightarrow \tilde{H}^{s-2}(\Omega) \quad (134)$$

are continuous and

$$\begin{aligned} (\mathbf{P}_b)^{-1} g &= \left[\Delta \mathring{E} (I - r_\Omega V_\Delta \mathcal{V}_\Delta^{-1} \gamma^+) - \gamma^* \mathcal{V}_\Delta^{-1} \gamma^+ \right] \\ &\cdot (bg) \text{ in } \mathbb{R}^2, \quad \forall g \in H^s(\Omega). \end{aligned} \quad (135)$$

Proof. The continuity of equation (133) follows from (Chkadua et al., 2009, Theorem 3.8). By Lemma 25 operator (133) is injective. Let us prove its surjectivity. To this end, for arbitrary $g \in H^s(\Omega)$ let us consider the following equation

with respect to $\tilde{f} \in \tilde{H}^{s-2}(\Omega)$,

$$\mathbf{P}_\Delta \tilde{f} = g \quad \text{in } \Omega. \quad (136)$$

Let $g_1 \in H^s(\Omega)$ be the (unique) solution of the following Dirichlet problem:

$$\Delta g_1 = 0 \text{ in } \Omega, \quad \gamma^+ g_1 = \gamma^+ g,$$

which by (Ayele Tsegaye and Bekele Solomon, 2019, Theorem 2) the single layer potential \mathcal{V}_Δ^{-1} exists and due to Costabel (1988) or (Mikhailov, 2011, Lemma 2.6) can be particularly presented as $g_1 = V_\Delta \mathcal{V}_\Delta^{-1} \gamma^+ g$. Let $g_0 := g - g_1$. Then $g_0 \in H^s(\Omega)$ and $\gamma^+ g_0 = 0$ and thus g_0 can be uniquely extended to $\mathring{E} g_0 \in \tilde{H}^s(\Omega)$, where \mathring{E} is the operator of extension by zero outside Ω . Thus, by (131) equation (136) takes the form

$$r_\Omega \mathbf{P}_\Delta [\tilde{f} + \gamma^* \mathcal{V}_\Delta^{-1} \gamma^+ g] = g_0 \quad \text{in } \Omega. \quad (137)$$

Any solution $\tilde{f} \in \tilde{H}^{s-2}(\Omega)$ of the corresponding equation on \mathbb{R}^2

$$\mathbf{P}_\Delta [\tilde{f} + \gamma^* \mathcal{V}_\Delta^{-1} \gamma^+ g] = \mathring{E} g_0 \quad \text{in } \mathbb{R}^2, \quad (138)$$

solves (137). If \tilde{f} solves (138), then acting with the Laplace operator on (138) we obtain

$$\begin{aligned} \tilde{f} = \tilde{Q} g &:= \Delta \mathring{E} g_0 - \gamma^* \mathcal{V}_\Delta^{-1} \gamma^+ g \\ &= \Delta \mathring{E} (g - r_\Omega V_\Delta \mathcal{V}_\Delta^{-1} \gamma^+ g) \\ &\quad - \gamma^* \mathcal{V}_\Delta^{-1} \gamma^+ g \text{ in } \mathbb{R}^2. \end{aligned} \quad (139)$$

On the other hand, substituting \tilde{f} given by (139) to (138) and taking into account that $\mathbf{P}_\Delta \Delta \tilde{h} = \tilde{h}$ for any $\tilde{h} \in \tilde{H}^s(\Omega)$, $s \in \mathbb{R}$, we obtain that $\tilde{Q} g$ is indeed a solution of equation (138) and thus (137). By Lemma 25 the solution of (138) is unique, which means that the operator \tilde{Q} is inverse to the operator (133), i.e., $\tilde{Q} = (r_\Omega \mathbf{P}_b)^{-1}$. Since Δ is a continuous operator from $\tilde{H}^s(\Omega)$ to $\tilde{H}^{s-2}(\Omega)$, equation (85) implies that operator $(r_\Omega \mathbf{P}_b)^{-1} = \tilde{Q} : H^s(\Omega) \rightarrow \tilde{H}^{s-2}(\Omega)$ is continuous. The relations $\mathbf{P}_b = \frac{1}{b} \mathbf{P}_\Delta$ and $b(x) > c > 0$ then imply the invertibility of the operator (133) and ansatz (135). \square

Theorem 27. *The co-kernel of the operator (129) is spanned over the functional*

$$g^{*1} := ((\gamma^+)^* \partial_n b, 1)^\top \quad (140)$$

in $\tilde{H}^{-1}(\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$, that is, $g^{*1}(\mathcal{F}_1, \mathcal{F}_2) = \langle (\gamma^+ \mathcal{F}_1) \partial_n b + \mathcal{F}_2, \gamma^+ u^0 \rangle_{\partial\Omega}$, where $u^0 = 1$.

Proof. The proof follows from the proof of (Mikhailov, 2015, Theorem 6.7) and Lemma 17. Indeed, let us consider the first equation in (128), i.e. the equation $\mathfrak{A}^1 \mathcal{U} = (\mathcal{F}_1, \mathcal{F}_2)^\top$,

representing the BDIE system (N1)

$$u + \mathcal{Z}_b u + \mathcal{R}_b u + W_b \varphi = \mathcal{F}_1 \quad \text{in } \Omega, \quad (141)$$

$$T_a^+ \mathcal{Z}_b u + T_a^+ \mathcal{R}_b u + \mathcal{L}_{ab}^+ \varphi = \mathcal{F}_2 \quad \text{on } \partial\Omega, \quad (142)$$

with arbitrary right hand side $(\mathcal{F}_1, \mathcal{F}_2)^\top \in H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega)$, for $(u, \varphi) \in H^1(\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$. By Lemma 17 the right-hand side of the system has the form (83)-(84), that is, system (141)-(142) reduces to

$$\begin{aligned} u + \mathcal{Z}_b u + \mathcal{R}_b u + W_b(\varphi + \Phi_*) \\ = \mathcal{P}_b \tilde{f}_{**} \quad \text{in } \Omega, \end{aligned} \quad (143)$$

$$\begin{aligned} T_a^+ \mathcal{Z}_b u + T_a^+ \mathcal{R}_b u + \mathcal{L}_{ab}^+(\varphi + \Phi_*) \\ = T_a^+(\tilde{f}_{**} + \mathring{E} \mathcal{R}_*^b \tilde{f}_{**}, \mathcal{P}_b \tilde{f}_{**}) \quad \text{on } \partial\Omega, \end{aligned} \quad (144)$$

where the couple $(\tilde{f}_{**}, \Phi_*) \in \tilde{H}^{-1}(\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$ is given by (83)-(84). Up to the notations (143)-(144) is the same as in (123) with $\psi_0 = 0$. Then, Theorems 23(iii) and 26 imply that the BDIE system (143)-(144) and hence (141)-(142) is solvable if and only if

$$\begin{aligned} \langle \tilde{f}_{**}, u^0 \rangle_\Omega &= \langle (\mathring{\Delta} b \mathcal{F}_1) + \gamma^*(\mathcal{F}_2 + \\ &+ (\gamma^+ \mathcal{F}_1) \partial_n b), u^0 \rangle_\Omega \\ &= \langle (\nabla \cdot \mathring{E} \nabla (b \mathcal{F}_1) + \gamma^*(\mathcal{F}_2 + \\ &+ (\gamma^+ \mathcal{F}_1) \partial_n b), u^0 \rangle_{\mathbb{R}^2} \\ &= \langle (\nabla \cdot \mathring{E} \nabla (b \mathcal{F}_1, \nabla u^0)_{\mathbb{R}^2} \\ &+ \langle (\mathcal{F}_2 + (\gamma^+ \mathcal{F}_1) \partial_n b), \gamma^+ u^0 \rangle_{\partial\Omega} \\ &= \langle (\mathcal{F}_2 + (\gamma^+ \mathcal{F}_1) \partial_n b), \gamma^+ u^0 \rangle_{\partial\Omega} = 0, \end{aligned} \quad (145)$$

where we took into account that $\nabla u^0 = 0$ in \mathbb{R}^2 . Thus the functional g^{*1} defined by (140) generates the necessary and sufficient solvability condition for the first equation in (128). Hence g^{*1} is basis of the co-kernel of \mathfrak{R}^1 . \square

Theorem 28. *Let $r_0 > \text{diam}(\Omega)$. Then the co-kernel of operator (130) is spanned over*

$$g^{*2} := \begin{pmatrix} -b\gamma^{*+} \left(\frac{1}{2} + \mathcal{W}'_\Delta \right) \mathcal{V}_\Delta^{-1} \gamma^+ u^0 \\ -b \left(\frac{1}{2} - \mathcal{W}'_\Delta \right) \mathcal{V}_\Delta^{-1} \gamma^+ u^0 \end{pmatrix} \quad (146)$$

in $\tilde{H}^{-1}(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega)$, that is,

$$\begin{aligned} g^{*2}(\mathcal{F}_1, \mathcal{F}_2) &= \left\langle -b\gamma^{*+} \left(\frac{1}{2} + \mathcal{W}'_\Delta \right) \mathcal{V}_\Delta^{-1} \gamma^+ u^0, \mathcal{F}_1 \right\rangle_\Omega \\ &+ \left\langle -b \left(\frac{1}{2} - \mathcal{W}'_\Delta \right) \mathcal{V}_\Delta^{-1} \gamma^+ u^0, \mathcal{F}_2 \right\rangle_{\partial\Omega}, \end{aligned}$$

where $u^0 = 1$.

Proof. The proof follows from the proof of (Mikhailov, 2015, Theorem 6.8), (Ayele Tsegaye and Bekele Solomon, 2019, Theorem 2) and Lemma 17. Indeed, let us consider the first

equation in (128), i.e. the equation $\mathfrak{R}^1 \mathcal{U} = (\mathcal{F}_1, \mathcal{F}_2)^\top$, representing the BDIE system (N1)

$$u + \mathcal{Z}_b u + \mathcal{R}_b u + W_b \varphi = \mathcal{F}_1 \quad \text{in } \Omega, \quad (147)$$

$$\begin{aligned} \frac{1}{2} \varphi + \gamma^+ \mathcal{Z}_b u + \gamma^+ \mathcal{R}_b u + \mathcal{W}_b \varphi \\ = \mathcal{F}_2 \quad \text{on } \partial\Omega, \end{aligned} \quad (148)$$

with arbitrary $(\mathcal{F}_1, \mathcal{F}_2)^\top \in H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega)$, for $(u, \varphi) \in H^1(\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$.

Introducing the new variable, $\varphi' = \varphi - (\mathcal{F}_2 - \gamma^+ \mathcal{F}_1)$, BDIE system (147)-(148) takes the form

$$u + \mathcal{Z}_b u + \mathcal{R}_b u + W_b \varphi = \mathcal{F}'_1 \quad \text{in } \Omega, \quad (149)$$

$$\begin{aligned} \frac{1}{2} \varphi' + \gamma^+ \mathcal{Z}_b u + \gamma^+ \mathcal{R}_b u + \mathcal{W}_b \varphi' \\ = \mathcal{F}'_2 \quad \text{on } \partial\Omega, \end{aligned} \quad (150)$$

where

$$\mathcal{F}'_1 = \mathcal{F}_1 - W_b(\mathcal{F}_2 - \gamma^+ \mathcal{F}_1) \in H^1(\Omega).$$

Let us recall that $\mathcal{P}_b = r_\Omega \mathbf{P}_b : \tilde{H}^{s-2}(\Omega) \rightarrow H^s(\Omega)$ and then by Theorem 26, the operator $\mathcal{P}_b^{-1} = (\mathbf{P}_b)^{-1} : H^s(\Omega) \rightarrow \tilde{H}^{s-2}(\Omega)$ is continuous for $\frac{1}{2} < s < \frac{3}{2}$, while \mathcal{V}_Δ^{-1} exists (Ayele Tsegaye and Bekele Solomon, 2019, Theorem 2). Hence we always represent $\mathcal{F}_1 = \mathcal{P}_b \tilde{f}_*$, with

$$\begin{aligned} \tilde{f}_* &= [\Delta \mathring{E} (I - r_\Omega V_\Delta \mathcal{V}_\Delta^{-1} \gamma^+) - \\ &- \gamma^* \mathcal{V}_\Delta^{-1} \gamma^+] (b \mathcal{F}'_1) \in \tilde{H}^{-1}(\Omega). \end{aligned}$$

For $\mathcal{F}'_1 = \mathcal{P}_b \tilde{f}_*$, the right hand side of BDIE system (149)-(150) is the same as in (126) with $f = \tilde{f}_*$ and $\psi_0 = 0$. Then Theorems 23(iii) implies that the BDIE system (149)-(150) and hence (147)-(148) is solvable if and only if

$$\begin{aligned} \langle \tilde{f}_*, u^0 \rangle_\Omega &= \\ &= \langle [\Delta \mathring{E} (I - r_\Omega V_\Delta \mathcal{V}_\Delta^{-1} \gamma^+) - \gamma^{*+} \mathcal{V}_\Delta^{-1} \gamma^+] (b \mathcal{F}'_1), u^0 \rangle_{\mathbb{R}^2} \\ &= \langle \mathring{E} (I - r_\Omega V_\Delta \mathcal{V}_\Delta^{-1} \gamma^+) (b \mathcal{F}'_1), \Delta u^0 \rangle_{\mathbb{R}^2} \\ &\quad - \langle (\gamma^{*+} \mathcal{V}_\Delta^{-1} \gamma^+) (b \mathcal{F}'_1), u^0 \rangle_{\mathbb{R}^2} \\ &\quad - \langle \gamma^+ (b \mathcal{F}'_1), \mathcal{V}_\Delta^{-1} \gamma^+ u^0 \rangle_{\partial\Omega} = \\ &= - \left\langle \frac{1}{2} [\gamma^+ (b \mathcal{F}_1) + (b \mathcal{F}_2)] - \mathcal{W}_\Delta [b(\mathcal{F}_2 - \gamma^+ \mathcal{F}_1)], \right. \\ &\quad \left. \mathcal{V}_\Delta^{-1} \gamma^+ u^0 \right\rangle_{\partial\Omega} \\ &= \langle -b\gamma^{*+} \left(\frac{1}{2} + \mathcal{W}'_\Delta \right) \mathcal{V}_\Delta^{-1} \gamma^+ u^0, \mathcal{F}_1 \rangle_\Omega \\ &+ \langle -b \left(\frac{1}{2} + \mathcal{W}'_\Delta \right) \mathcal{V}_\Delta^{-1} \gamma^+ u^0, \mathcal{F}_2 \rangle_{\partial\Omega} = 0. \end{aligned} \quad (151)$$

Thus the functional g^{*2} defined by (146) generates the necessary and sufficient solvability condition of the equation $\mathfrak{R}^2\mathcal{U} = (\mathcal{F}_1, \mathcal{F}_2)^\top$. Hence g^{*2} is basis of the cokernel of \mathfrak{R}^2 . \square

Perturbed segregated BDIE systems for Neumann problem. Theorem 23 implies, that even when the solvability condition (120) is satisfied, the solutions of both BDIE systems, (N1) and (N2), are not unique. By Theorem 24, in turn, the BDIE left hand side operators, \mathfrak{R}^1 and \mathfrak{R}^2 , have non-zero kernels and thus are not invertible. To find a solution (u, φ) from uniquely solvable BDIE system with continuously invertible left hand side operators, let us consider, following Mikhailov (1999), some BDIE systems obtained from (N1) and (N2) by finite-dimensional operator perturbations (cf. Ayele Tsegaye (2021b) for the three-dimensional case). Below we use the notations $\mathcal{U} = (u, \varphi)^\top$ and $|\partial\Omega| := \int_{\partial\Omega} dS$.

0.0.1. *Perturbation of BDIE system (N1).* Let us introduce the perturbed counterparts of the BDIE system (N1),

$$\hat{\mathfrak{R}}^1\mathcal{U}^N = \mathcal{F}^{N1}, \tag{152}$$

where

$$\begin{aligned} \hat{\mathfrak{R}}^1 &:= \hat{\mathfrak{R}}^1 + \hat{\mathfrak{R}}^1 \text{ and } \hat{\mathfrak{R}}^1\mathcal{U}^N(y) := g^0(\mathcal{U}^N)\mathcal{G}^1(y) \\ &= \frac{1}{|\partial\Omega|} \int_{|\partial\Omega|} \varphi(x)dS \begin{pmatrix} b^{-1}(y) \\ 0 \end{pmatrix}, \end{aligned}$$

that is,

$$g^0(\mathcal{U}^N) := \frac{1}{|\partial\Omega|} \int_{\partial\Omega} \varphi(x)dS, \quad \mathcal{G}^1(y) := \begin{pmatrix} b^{-1}(y) \\ 0 \end{pmatrix}.$$

For the functional g^{*1} given by (140) in Theorem 27, $g^{*1}(\mathcal{G}^1) = |\partial\Omega|$, while $g^0(\mathcal{U}^0) = 1$. Hence (Mikhailov, 2015, Theorem D.1 in Appendix) and (Ayele Tsegaye *et al.*, 2017) imply the following assertion.

Theorem 29. *Let $r_0 > \text{diam}(\Omega)$, then*

- (i) *The operator $\hat{\mathfrak{R}}^1 : H^1(\Omega) \times H^{\frac{1}{2}}(\partial\Omega) \rightarrow H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega)$ is continuous and continuously invertible.*
- (ii) *If condition $g^{*1}(\mathcal{F}^{N1}) = 0$ or condition (120) for \mathcal{F}^{N1} in form (129) is satisfied, then the unique solution of perturbed BDIDE system (152) gives a solution of original BDIE system (N1) such that*

$$g^0(\mathcal{U}^N) = \frac{1}{|\partial\Omega|} \int_{\partial\Omega} \gamma^+ u dS = \frac{1}{|\partial\Omega|} \int_{\partial\Omega} \varphi dS = 0.$$

Perturbation of BDIE system (N2). Let us introduce the perturbed counterparts of the BDIE system (N2)

$$\hat{\mathfrak{R}}^2\mathcal{U}^N = \mathcal{F}^{N2}, \tag{153}$$

where

$$\begin{aligned} \hat{\mathfrak{R}}^2 &:= \mathfrak{R}^2 + \hat{\mathfrak{R}}^2 \text{ and } \hat{\mathfrak{R}}^2\mathcal{U}^N(y) := g^0(\mathcal{U}^N)\mathcal{G}^2(y) \\ &= \frac{1}{|\partial\Omega|} \int_{|\partial\Omega|} \varphi(x)ds \begin{pmatrix} b^{-1}(y) \\ \gamma^+ b^{-1}(y) \end{pmatrix}, \end{aligned}$$

that is,

$$\begin{aligned} g^0(\mathcal{U}^N) &:= \frac{1}{|\partial\Omega|} \int_{|\partial\Omega|} \varphi(x)ds, \\ \mathcal{G}^2(y) &:= \begin{pmatrix} (b^{-1}u^0)(y) \\ \gamma^+(b^{-1}u^0)(y) \end{pmatrix}. \end{aligned}$$

For the functional g^{*2} given by (146) in Theorem 28, since the operator $\mathcal{V}_\Delta^{-1} : H^{\frac{1}{2}}(\partial\Omega) \rightarrow H^{-\frac{1}{2}}(\partial\Omega)$ is positive definite (with additional condition $r_0 > \text{diam}(\Omega)$) and $u^0(x) = 1$, there exists a positive constant C such that

$$\begin{aligned} g^{*2}(\mathcal{G}^2) &= \\ &\langle -b\gamma^{+*}(\frac{1}{2} + \mathcal{W}'_\Delta)\mathcal{V}_\Delta^{-1}\gamma^+u^0, b^{-1}u^0 \rangle_\Omega + \\ &\langle -b(\frac{1}{2} - \mathcal{W}'_\Delta)\mathcal{V}_\Delta^{-1}\gamma^+u^0, \gamma^+(b^{-1}u^0) \rangle_{\partial\Omega} \\ &= -\langle (\frac{1}{2} + \mathcal{W}'_\Delta)\mathcal{V}_\Delta^{-1}\gamma^+u^0 + \\ &+ (\frac{1}{2} - \mathcal{W}'_\Delta)\mathcal{V}_\Delta^{-1}\gamma^+u^0, \gamma^+u^0 \rangle_{\partial\Omega} \\ &= -\langle \mathcal{V}_\Delta^{-1}\gamma^+u^0, \gamma^+u^0 \rangle_{\partial\Omega} \\ &\leq -C\|\gamma^+u^0\|_{H^{\frac{1}{2}}(\partial\Omega)}^2 \\ &\leq -C\|\gamma^+u^0\|_{H^{\frac{1}{2}}(\partial\Omega)}^2 = -C|\partial\Omega|^2 < 0. \end{aligned} \tag{154}$$

Due to (154) and $g^0(\mathcal{U}^0) = 1$, (Mikhailov, 2015, Theorem D.1) and (Ayele Tsegaye *et al.*, 2017, Theorem 7) imply the following assertion.

Theorem 30. *Let $r_0 > \text{diam}(\Omega)$, then*

- (i) *The operator $\hat{\mathfrak{R}}^2 : H^1(\Omega) \times H^{\frac{1}{2}}(\partial\Omega) \rightarrow H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega)$ is continuous and continuously invertible.*
- (ii) *If condition $g^{*2}(\mathcal{F}^{N2}) = 0$ or condition (120) for \mathcal{F}^{N2} in form (130) is satisfied, then the unique solution of perturbed BDIDE system (153) gives a solution of original BDIE system (N2) such that*

$$g^0(\mathcal{U}^N) = \frac{1}{|\partial\Omega|} \int_{\partial\Omega} \gamma^+ u dS = \frac{1}{|\partial\Omega|} \int_{\partial\Omega} \varphi dS = 0.$$

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