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Decomposition of BH-Lattices

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ABSTRACT: In this paper we study further properties of BH-lattices which is a subclass of BH-monoids. We furnish certain examples of BH- monoids which are not BH- lattice. We give a characterization of BH-lattices in terms of bounded BH-lattices and commutative l-groups. Also we prove that every BH-lattice is a direct product of Heyting algebra and commutative l-group under certain condition. Further we obtain the decomposition theorem in terms of Boolean algebra and a commutative l-group.

Key words/phrases: **Brouwer-Heyting(BH) monoid, BH-lattice, Brouwerian algebra, Heyting algebra, l-group**

INTRODUCTION

It is Ward and Dilworth [16] initiated the study of residuated lattices and as a result a study on lattice ordered Semigroups with residuation as an operation have been introduced by K.L.N Swamy [10] under the name DRl-Semigroups. Two algebras Brouwerian and Heyting are generalizations to Boolean algebra (lattice) and dual to each other. There is a mismatch in the literature regarding the nomenclature of these two algebras. Brouwerian algebra defined in [7] by Nordhaus, EA and Leolapidus is called as Heyting algebra by Birkhoff G [1]. To avoid this mismatch, we recall the two definitions in the preliminaries and also to clear the confusion between the two algebras Swamy [9] introduced the notion of Brouwer-Heyting monoids (for short BH-monoids) as a general class containing both Brouwerian algebra, its dual Heyting algebra. He also observed that DRl-semigroups and dual DRl-semigroups are examples of BH-monoids. Further he obtained decomposition theorems for both BH-Monoids as well as for BH-lattices.

This paper investigates further properties of BH-lattices and decomposition theorems on BH-lattices. It is divided into 3 sections. The first section is preliminaries in which we recall all the existing literature on BH-monoids and BH-lattices. In section two, we obtain further properties on BH-lattices which we use in the sequel. The last section is devoted for decomposition theorems. In this paper we prove every BH-lattice can be represented as a direct product of Heyting algebra and a commutative l-group with certain conditions.

The following abbreviations are used in this paper. Po-group means partially ordered group, l-group means lattice-ordered groups, BH means Brouwer-Heyting, DRl means Dually residuated lattice ordered.

Preliminaries

In this section, we recall certain definitions and results concerning Heyting algebra [3, 6] and Brouwer-Heyting lattices [9] which will be used in the sequel.

Definition 1.1. *A bounded lattice* $(L, \vee, \wedge)$ *in which to each a, b there is a least x such that* $x \vee a \geq b$ *is called a Brouwerian algebra. The least element is denoted by* $2 - b$.

Definition 1.2. *A bounded lattice* $(L, \vee, \wedge)$ *is called a Heyting algebra if for any given elements* a *and* b *in* L, *there is a greatest* x *such that* $x \wedge a \leq b$.

Remark 1.1. *The greatest element* x *is denoted by* $a \to b$. *Clearly* $a \to b$ *is unique.*

Lemma 1.1. *Every Boolean algebra is a Heyting*

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algebra, with $a \to b$ given by $a' \vee b$.

Theorem 1.1. *Let L be a Heyting algebra and $x, y, z \in L$. Then the following hold*
1. $y \le x \to y$
2. $1 = x \to 1$
3. $x \le y \Leftrightarrow x \to y = 1$.
4. $x \wedge (x \to y) = x \wedge y$
5. $y \wedge (x \to y) = y$ and $((x \wedge y) \to x) \wedge z = z$
6. $y = 1 \to y$
7. If $x \le y$, then $(z \to x) \le (z \to y)$

Theorem 1.2. *Let L be a Heyting algebra and $x, y, z \in L$. Then the following hold*
1. If $x \le y$, then $(y \to z) \le (x \to z)$
2. $x \to (y \to z) = (x \wedge y) \to z = (x \to y) \to (x \to z)$
3. $(x \to y) \wedge (x \to z) = x \to (y \wedge z)$
4. $(x \to y) \wedge z = ((z \wedge x) \to (z \wedge y)) \wedge z$
5. *L is a bounded distributive lattice.*

Note: An equivalent definition of 1.2 is:

Definition 1.3. *A non empty set L with three binary operations $\wedge$, $\vee$ and $\to$ and two distinguished elements 0 and 1 is a Heyting Algebra if the following conditions hold: (H_1) $(L, \vee, \wedge, 0, 1)$ is a lattice with 0,1*
(H_2) $x \wedge (x \to y) = x \wedge y$
(H_3) $x \wedge (y \to z) = x \wedge [(x \wedge y) \to (x \wedge z)]$
(H_4) $(x \wedge y) \to x = 1$.

Definition 1.4. *A system $(G, \circ, e, \rho)$ is called a partially ordered group(po-group) if $(G, \circ, e)$ is a group, (G, ρ) is a poset and for $a, b, x, y \in G$, $x\rho y \Rightarrow (a \circ x \circ b)\rho(a \circ y \circ b)$. And it is called a commutative po-group, if $\circ$ is commutative.*

Definition 1.5. *An l-group is a po-group where (G, ρ) is a lattice.*

Example 1.1. *The additive groups Z - integers, Q - rationals, R - reals are the simplest examples of l-groups.*

Definition 1.6. *A system $(G, \circ, e, \rho, \to)$ is a Brouwer-Heyting (for short BH) monoid if*
1 $(G.\circ, e)$ is a commutative semi group with identity "e"
2 (G, ρ) is a partially ordered set and $\to$ is binary operation on G such that for all x, a, b in G, $(x \circ b)\rho a \Leftrightarrow x\rho(a \to b)$.

The following are examples of Brouwer-Heyting monoids.

Example 1.2. *Let $(G, \circ, e, \rho)$ is a commutative po-group. Define $a \to b = a \circ b^{-1}$*

Example 1.3. *$(B, \vee, \wedge, 0, 1)$ is a Boolean algebra. Let $\circ = \wedge$, $\rho = \le$ defined by $a\rho b$ if*
$a \wedge b = a$, $e = 1$, $a \to b = a \vee b'$.

Example 1.4. *Let $(G, \vee, \wedge, 1)$ be a Heyting algebra. Let $o = \wedge$, $e = 1$, ρ is the lattice order , $a \to b$ is the largest x such that $b \wedge x \le a$. That is, the new arrow operation is defined in terms of the arrow operation in the Heyting algebra by $a \to_N b = b \to a$.*

Example 1.5. *Let $(L, \vee, \wedge, 0)$ be Brouwerian algebra. If we take ρ = (the dual of $\le$*
) $\ge$, $a \to b = a - b$ is the smallest x such that $x \vee b \ge a$, $o = \vee$ and the least 0 as identity element. Thus $(L, \vee, \wedge, 0, -)$ is a BH monoid where ρ is the dual ordering of the lattice $(L, \vee, \wedge)$

Example 1.6. *Let $(G, +, \le, -, 0)$ be a DRl-monoid. We have $x + b \ge a \Leftrightarrow a - b \le x$*
(By the definition of DRl- monoid). Thus the dual of DRl-monoid is a BH monoid .

Note: Here after for the sake of convenience, we use $\le$ instead of ρ.

Theorem 1.3. *In BH monoid $(L, o, e, \le, \to)$ the following hold for $x, y, z \in L$*
1. $y \le z \Rightarrow x \circ y \le x \circ z$
2. $x \le (x \circ y) \to y$.
3. $(x \to y) \circ y \le x$
4. $z \to (x \circ y) = (z \to y) \to x = (z \to x) \to y$.
5. $e \to e = e$
6. $x \to e = x$
7. $e \le x \to x$
8. $x \le y \Rightarrow x \to z \le y \to z$
9. $x \le y \Rightarrow z \to y \le z \to x$
10. $y \le x \Leftrightarrow e \le x \to y$
11. $(x \to y) \circ (y \to z) \le x \to z$
12. If $x \vee y$ exists, then $(z \circ x) \vee (z \circ y)$ exists for any z and $z \circ (x \vee y) = (z \circ x) \vee (z \circ y)$
13. If $x \vee y$ exists, then $(z \to x) \wedge (z \to y)$ exist and $z \to (x \vee y) = (z \to x) \wedge (z \to y)$
14. If $x \wedge y$ exists, then $(x \to z) \wedge (y \to z)$ exists and $(x \wedge y) \to z = (x \to z) \wedge (y \to z)$.

Definition 1.7. *A BH monoid $(L, \circ, e, \le, \to)$ is a BH-lattice if*
1. $(L, \le)$ is a lattice with glb and lub denoted $\wedge$

and $\vee$ *respectively*

2. $x \circ (y \wedge z) = (x \circ y) \wedge (x \circ z), \forall x, y, z \in L$
3. $((y \to x) \wedge e) \circ x = x \wedge y, \forall x, y \in L.$

Theorem 1.4. *A lattice* $(L, \vee, \wedge, o, e, \to)$, *where* $(L, \circ, e)$ *is a commutative monoid and* $\to$ *be a binary operation on L,is a BH-lattice if and only if*

1. $(y \to x) \circ x \leq y$
2. $(x \wedge z) \to y \leq x \to y$
3. $x \leq (x \circ y) \to y, \forall x, y \in L$
4. $x \circ (y \wedge z) = (x \circ y) \wedge (x \circ z), \forall x, y, z \in L$
5. $((y \to x) \wedge e) \circ x = x \wedge y, \forall x, y \in L$

Since $\wedge$ is commutative, it follows that $((y \to x) \wedge e) \circ x = ((x \to y) \wedge e) \circ y, \forall x, y \in L.$

Remark 1.2. *Theorem* 1.4 *shows that BH-lattices can be defined by means of identities alone.*

Theorem 1.5. *Let* $(L, \vee, \wedge, o, e, \to)$ *be a BH-lattice and* $a, b, x \in L$, *then the following hold.*

1. $x \to x = e$.
2. L is distributive.
3. $(x \vee y) \circ (x \wedge y) = x \circ y$
4. $x = x \circ e = (x \vee e) \circ (x \wedge e)$.
5. If $e \leq x$, *then* x *is invertible.*
6. $e \vee x$ *is invertible.*
7. If x *is invertible, then* $e \to x$ *is inverse of* x.
8. If y *is invertible, then* $x \to y = x \circ (e \to y)$
9. If x *and* y *are invertible, then* xoy *is invertible and* $(e \to x)o(e \to y)$ *is the inverse of* xoy
10. $e \to x$ *is invertible.*
11. If G is the set of all invertible elements, then G is an l-group.

Theorem 1.6. *(Decomposition Theorem for BH monoids and BH-lattices)*

1. BH monoid L is direct product of po- group and a BH monoid with greatest element if and only if $e \to x$ *is invertible and* $x \to x = e, \forall x \in L$,.
2. BH-lattice L is direct product of a commutative l-group and a BH-lattice with greatest element.

Further Properties of BH-lattices

We begin with the following

Example 2.1. *Boolean algebra, Heyting algebra given in example 1.3 and 1.4 above and l-groups are BH-lattices. Heyting algebra is an example of bounded BH-lattice while the unbounded l-groups are examples of unbounded BH-lattices.*

Now we have the following examples which are BH-monoids but not BH-lattices.

Example 2.2. *The commutative po-group* $(G, \circ, e, \rho)$ *given in the example* 1.2 *above is not a BH-lattice.*

Example 2.3. *Consider the lattice given in Fig* 1. *Clearly it is a Brouwerian algebra and hence a BH-monoid. Since* $[(c \to b) \wedge e]ob \neq c \wedge b$, *it is not a BH-lattice. Thus*

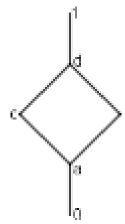

Figure 1. Example of Brouwerian algebra.

Brouwerian algebra is not a BH-lattice.

Example 2.4. *Consider the set A - the multiplicative semigroup of the set of non negative integers ordered by the divisibility relation. Then A is a DRl semigroup with least element 1 and greatest element 0, for* $x, y \in A$, $x \to y = \lfloor \frac{x}{y} \rfloor$, *where* $\lfloor . \rfloor$ *is the floor function,* $x \wedge y = GCF(x, y)$ *and* $x \vee y = LCM(x, y)$. *Then for the BH-monoid induced from the above DRl-monoid,* $[(3 \to 2) \wedge e]o2 = (1 \wedge 1).2 = 1.2 = 2 \neq 3 \wedge 2 = 6$. *Hence it is not a BH-lattice. Hence DRl-monoid is not a BH-lattice.*

Note: Here after L stands for a BH-lattice $(L, \vee, \wedge, o, e, \to)$.

Theorem 2.1. *In BH lattice L, for* $x, y \in L$, *if* x *and* y *are invertible, then* $x \to y$ *is invertible and* $y \to x$ *is the inverse of* $x \to y$.

Proof. Let in BH lattice L, $x, y \in L$ are invertible. From theorem 2.9, $x \to y = xo(e \to y)$ and $y \to x = yo(e \to x)$. Hence $(x \to y)o(y \to x) = [xo(e \to y)]o[yo(e \to x)] = xo[[(e \to y)oy]o(e \to x)]$ (associativity) $= xo[eo(e \to x)] = e$. Hence the result holds. □

Theorem 2.2. *For* $x, y, z \in L$, *the following properties holds.*

1. $x \leq y \to (y \to x)$
2. $z \leq (xoz) \to (x \wedge y)$ *or equivalently* $(x \wedge y) \leq (xoz) \to z$
3. $x \wedge y \leq [(xoy) \to y] \wedge [(xoy) \to x]$

Proof. Proof is a consequence of the definition BH-monoid and 1 of theorem 1.3.

Remark 2.1. *For a BH-lattice L and* $a, b \in L$, $b \to a = max\{x \in L : aox \leq b\}$

Theorem 2.3. *Let* $x, y, z \in L$. *Then the following properties holds.*
1. $x \leq ((x \vee y)oz) \to z, x \leq ((x \vee y)ox) \to y$
2. $x \to y \leq (z \to y) \to (z \to x)$
3. $x \to y \leq (x \to z) \to (y \to z)$

Proof.

1. By 1 of theorem 1.3, $xoz \leq (x \vee y)oz, xoy \leq (x \vee y)ox \Rightarrow x \leq (x \vee y)oz \to z, x \leq (x \vee y)ox \to y$.

2. By 4 of theorem 1.3, we have $z \to (xoy) = (z \to x) \to y = (z \to y) \to x$. And by 1 of theorem 2.2 above we have $x \leq z \to (z \to x) \Rightarrow x \to y \leq [z \to (z \to x)] \to y$ (by 8 of theorem 1.3). Thus we have $(z \to y) \to (z \to x) = z \to (yo(z \to x)) = (z \to (z \to x)) \to y$. Hence $x \to y \leq (z \to y) \to (z \to x)$.
3. By 4 and 9 of theorem 1.3) we have $(y \to z)oz \leq y \Rightarrow x \to y \leq x \to [(y \to z)oz] = (x \to z) \to (y \to z)$.

Corollary 2.1. *Let* $x, y \in L$. *Then* $x \to y \leq e \to (y \to x)$

Proof. Replace z by x in 3 of theorem 2.3

Notation: For $x \in L$, we shall denote $e \to x$ by the symbol x^-.

Theorem 2.4. *For all* $x, y \in L$ *the following properties hold.*
1. $(xoy)^- = x^- \to y = y^- \to x$
2. $x \leq y \Rightarrow y^- \leq x^-$
3. $x \leq x^{--}$ *and* $x^- = x^{---}$
4. $(x \vee y)^- = x^- \wedge y^-$ *and* $x^- \vee y^- \leq (x \wedge y)^-$
5. $x \to y \leq y^- \to x^-$
6. $x^{--} \to y^{--} = y^- \to x^-$

Proof. (1) and (2) are clear from theorem 1.3. By 2 of theorem 2.2 it follows that $x \leq e \to (e \to x) = x^{--}$. Hence $x \leq x^{--}$. Now replacing x^- in the place of x in the above inequality we have $x^- \leq x^{---}$. But by 2 in the inequality $x \leq x^{--}$, we obtain that $x^{---} = e \to (x^{--}) \leq (e \to x) = x^-$. So $x^{---} = x^-$. By 9 and 13 of theorem 1.3, $e \to (x \vee y) = (e \to x) \wedge (e \to y) = x^- \wedge y^-$. Since $x \wedge y \leq x, y$, it follows that $x^- \leq e \to (x \wedge y) = (x \wedge y)^-$ and similarly $y^- \leq (x \wedge y)^-$. Hence $x^- \vee y^- \leq (x \wedge y)^-$. (5) follows from 2 of theorem 2.3. By 7, 8 and 10 of theorem 1.5, $x^{--} \to y^{--} = x^{--}o(e \to y^{--}) = x^{--}oy^{---} = x^{--}oy^- = y^-o(e \to x^-) = y^- \to x^-$.

Theorem 2.5. *Let* $x, y \in L$, *then the following properties hold.*
1. $x \wedge y \to x \leq e$
2. $e \leq x \to (x \wedge y)$

Proof. Follows from 8 and 9 of theorem 1.3 and 1 of theorem 1.5.

Theorem 2.6. *For* $x, y \in L$, $x \to (y \to x) = x \Rightarrow x \to (x \to y) = y \to (y \to x)$

Proof. Let $x, y \in L$. Then $x \to (x \to y) = (x \to (y \to x)) \to (x \to y) = (x \to (x \to y)) \to (y \to x)$ (by 4 of theorem 1.3). Since from theorem 2.2, $y \leq x \to (x \to y)$ it follow that $y \to (y \to x) \leq (x \to (x \to y)) \to (y \to x) = x \to (x \to y)$. Hence $y \to (y \to x) \leq x \to (x \to y)$. Analogously $x \to (x \to y) \leq y \to (y \to x)$.

Theorem 2.7. *Let* $a, b, x, y \in L$ *such that* $a \leq b$ *and* $x \leq y$, *then the following properties hold.*
1. $aox \leq boy$
2. $(a \to x) \vee (b \to y) \leq b \to x$
3. $a \to y \leq (a \to x) \wedge (b \to y)$
4. $a \to y \leq b \to x$

Proof. Suppose $a \leq b$ and $x \leq y$. By 1 in theorem 1.3 $a \leq b$ implies $aox \leq box$ and $x \leq y$ implies $box \leq boy$. Hence $aox \leq boy$. By 8 and 9 in theorem 1.3, $a \leq b \Rightarrow a \to x \leq b \to x$ and $x \leq y \Rightarrow b \to y \leq b \to x$. Hence $(a \to x) \vee (b \to y) \leq b \to x$. And $a \leq b \Rightarrow a \to y \leq b \to y$ and $x \leq y \Rightarrow a \to y \leq a \to x$. Hence $a \to y \leq (a \to x) \wedge (b \to y)$. (4) follows from 2 and 3.

Theorem 2.8. *Let* $x, y, z, s, t \in L$. *Then the following properties hold.*
1. $x \to y \leq xoz \to yoz$
2. $(x \to y)o(s \to t) \leq (xos) \to (yot)$

Proof. From 2, 4 and 8 of theorem 1.3, $x \to y \leq [(xoz) \to z] \to y = xoz \to yoz$. From (1) we have $x \to y \leq xos \to yos$ and $s \to t \leq yos \to toy$. Now by theorem 2.7 and 11 of theorem 1.3, we have $(x \to y)o(s \to t) \leq (xos \to yos)o(yos \to yot) \leq xos \to yot$. Hence (2) holds.

Theorem 2.9. *Let* $x, y \in L$. $(x \wedge y) \to (x \vee y) = (x \to y) \wedge (y \to x)$

Proof. By 13 and 14 of theorem 1.3, $x \wedge y \to x \vee y = [(x \wedge y) \to x] \wedge [(x \wedge y) \to y]$
$= [(x \to x)\wedge(y \to x)]\wedge[(x \to y)\wedge(y \to y)] = e\wedge(y \to x)\wedge(x \to y)$ (i)
And also by 1, 3 and 12 of of theorem 1.3, $(x \vee y)o\{(x \to y) \wedge (y \to x)\} = [xo\{(x \to y) \wedge (y \to x)\}] \vee [yo\{(x \to y) \wedge (y \to x)\}] \leq \{xo(y \to x)\} \vee \{yo(x \to y)\} \leq x \vee y$. That is $(x \vee y)o\{(x \to y) \wedge (y \to x)\} \leq (x \vee y) \Rightarrow (x \to y) \wedge (y \to x) \leq (x \vee y) \to (x \vee y) = e$

$\Rightarrow (x \to y)\wedge(y \to x)\wedge e = (x \to y)\wedge(y \to x)$ (ii)
Hence from (i) and (ii) we have $x \wedge y \to x \vee y = (x \to y) \wedge (y \to x)$

Theorem 2.10. *Let* $x, y \in L$. $y \leq x \Rightarrow (y \to x)ox = y$

Proof. By 8 of theorem 1.3 and 3 in the definition of BH-lattices, $y \leq x \Rightarrow (y \to x) \leq e$
$\Rightarrow (y \to x) \wedge e = (y \to x) \Rightarrow ((y \to x) \wedge e)ox = (y \to x)ox \Rightarrow x \wedge y = (y \to x)ox \Rightarrow$
$y = (y \to x)ox$.

Theorem 2.11. *Let* $x, y \in L$. $((x \to y) \wedge e)o(x \vee y) = x$
Proof. By 13 of theorem 1.3 and theorem 2.10, $x = (x \to (x \vee y))o(x \vee y) = ((x \to y) \wedge e)o(x \vee y)$.

Theorem 2.12. *Let* $x, y \in L$. $[(x \to y) \wedge (y \to x)]o(x \vee y) = x \wedge y$

Proof. Follows from theorem 2.9 and theorem 2.10.

Theorem 2.13. *Let* $x, y, z \in L$. $x \leq y \leq z \Rightarrow (x \to y)o(y \to z) = x \to z$

Proof. Let $x \leq y \leq z$. By 4 of theorem 1.3 and theorem 2.10 we have $(y \to z)oz = y$
$\Rightarrow x \to y = x \to \{(y \to z)oz\} = (x \to z) \to (b \to c)$. Hence $(x \to y)o(y \to z) = ((x \to z) \to (y \to z))o(y \to z)$. And by 8 of theorem 1.3 we have $x \to z \leq y \to z$. Hence by theorem 2.10 $(x \to y)o(y \to z) = ((x \to z) \to (y \to z))o(y \to z) = x \to z$.

Theorem 2.14. *For* $x, y, z \in L$ *the following properties hold.*
1. $(x \to y)oz \leq x \to (y \to z)$,
2. $(x \to z)oy \leq (xoy) \to z$
3. $(e \to x)o(e \to y) \leq e \to (xoy)$

Proof. By 3, 4 and 9 of theorem 1.3, we have $(y \to z)oz \leq y$. $\Rightarrow x \to y \leq x \to (y \to z)oz = (x \to (y \to z)) \to z \Rightarrow (x \to y)oz \leq x \to (y \to z)$. By 2, 4 and 8 of

theorem 1.3, $x \leq (xoy) \to y$. $\Rightarrow x \to z \leq ((xoy) \to y) \to z = ((xoy) \to z) \to y$
$\Rightarrow (x \to z)oy \leq (xoy) \to z$. Finally from (1) and 9 of theorem 1.3 we have $(e \to x)o(e \to y) \leq e \to [x \to (e \to y)] \leq e \to (xoy)$.

Theorem 2.15. *For* $x, y \in L$, $(xoy)^{--} \leq (x^{-}oy^{-})^{-} = x^{--}oy^{--}$

Proof. From theorem 2.14, $(e \to x)o(e \to y) \leq e \to xoy$. Now by 9 of theorem 1.3 and theorem 2.4 we have $(xoy)^{--} \leq e \to [(e \to x)o(e \to y)] = (e \to (e \to x)) \to (e \to y) = x^{--} \to y^{-} = (x^{-}oy^{-})^{-}$. Now by 8 and 10 of theorem 1.5, $x^{--} \to y^{-} = x^{--}o(e \to y^{-}) = x^{--}oy^{--}$

Theorem 2.16. *For* $x, y, z \in L$ *the following properties hold.*
1. $x \wedge y = e = x \wedge z \Rightarrow x \wedge (yoz) = e$
2. $x \vee y = e = x \vee z \Rightarrow x \vee (yoz) = e$

Proof. Let $x, y, z \in L$. Let $x \wedge y = e = x \wedge z$. $\Rightarrow e \leq x, y, z$. Hence $e \leq x \wedge y \wedge z$ and by theorem 2.7, $e \leq yoz$. $\Rightarrow e \leq x\wedge(yoz)$. Now by 1 of theorem 1.3, $x = xoe \leq xo(x\wedge y\wedge z)$
$\Rightarrow x \wedge (yoz) \leq xo(x \wedge y \wedge z) \wedge (yoz) = (xox) \wedge (xoz) \wedge (yox) \wedge (yoz) = (x \wedge y)o(x \wedge z) = eoe = e$ i.e, $x \wedge (yoz) \leq e$. So it follows that $x \wedge (yoz) = e$
Let $x \vee y = e = x \vee z$. $\Rightarrow x, y, z \leq e$. Hence $x \vee y \vee z \leq e$ and with theorem 2.7, $yoz \leq e$.
$\Rightarrow x \vee (yoz) \leq e$. By 1 of theorem 1.3 and 12 of theorem 1.3, $xo(x \vee y \vee z) \leq x = xoe$
$\Rightarrow xo(x \vee y \vee z) \vee (yoz) = (xox) \vee (xoz) \vee (yox) \vee (yoz) = (x \vee y)o(x \vee z) = eoe = e$ and $xo(x\vee y\vee z)\vee(yoz) \leq x\vee(yoz)$. i.e, $e \leq x\vee(yoz)$. Hence it follows that $x\vee(yoz) = e$

Notation: For any x and y in L, we shall write $x * y = (x \to y) \wedge (y \to x)$.

Theorem 2.17. *For any* $x, y, z \in L$, *the following properties hold.*
1. $x * y \leq e$ *with equality iff* $x = y$.
2. $x * y = y * x$
3. $(x \vee y) * (x \wedge y) = x * y$
4. $(x * y)o(y * z) \leq x * z$

5. $x * y \le (x \to y) * e$
6. $x \le e \Rightarrow x * e = x$

Proof. Using 8 of theorem 1.3 and theorem 2.9, $x * y = (x \to y) \wedge (y \to x) = (x \wedge y) \to (x \vee y) \le e$. If $x = y$, then by 1 of theorem 1.5, $x * y = e$. Conversely if $x * y = e$, then $x * y = (x \wedge y) \to (x \vee y) = e \Rightarrow e \le (x \wedge y) \to (x \vee y)$. $\Rightarrow (x \vee y) \le (x \wedge y)$ $\Rightarrow x \vee y = x \wedge y$ and consequently it follows that $x = y$. Hence (1) holds. Evidently (2) is trivial. Since by theorem 2.9, $(x \vee y)*(x \wedge y) = [(x \vee y)\wedge(x \wedge y)] \to [(x \vee y)\vee(x \wedge y)] = (x \wedge y) \to (x \vee y) = x * y$. Hence (3) holds. Again $(x \to y) \wedge (y \to x) \le x \to y, y \to x$ and $(y \to z) \wedge (z \to y) \le y \to z, z \to y$. $(x * y)o(y * z) = [(x \to y) \wedge (y \to x)]o[(y \to z) \wedge (z \to y)] \le (x \to y)o(y \to z)$ (by theorem 2.7) $\le x \to z$ (by 11 of theorem 1.3). That is $(x * y)o(y * z) \le x \to z$. By a similar argument $(x * y)o(y * z) \le z \to x$. So it follows that $(x * y)o(y * z) \le (x \to z) \wedge (z \to x) = x * z$. Hence (4) holds. By theorem 2.3 we have $y \to x \le (x \to x) \to (x \to y) = e \to (x \to y) \Rightarrow x * y = (x \to y) \wedge (y \to x) \le (x \to y) \wedge (e \to (x \to y)) = (x \to y) * e$. So $x * y \le (x \to y) * e$. Finally let $x \le e \Rightarrow e \to e = e \le e \to x$ (by 9 of theorem 1.3). $\Rightarrow x = e \wedge x \le x \wedge (e \to x) = x * e \le x$. Hence $x * e = x$

Definition 2.1. *For $n \in \mathrm{N}$ and $x \in L$, define $x^n = xoxo...ox$ (n times).*

Theorem 2.18. *If there exists an element $x \in L$ such that $e < x$, then the set L is an infinite and not bounded above.*

Proof. Let $e < x$ and $x \neq e$. Consider the sequence $\{x^{2^n}\}_{n\in \mathrm{N}}$, where N is the set of non-negative integers. With 1 of theorem 1.3, $x \le x^2$. If $x^2 = x$, then by 2 of theorem 1.3, $x \le x^2 \to x = x \to x = e$. i.e; $x \le e$. Since $e < x$, it follows that $x = e$ which is a contradiction. So $x^2 \neq x$. If $x^2 = e$, then $x = x \to e = x \to x^2 = (x \to x) \to x = e \to x$. Since $e < x$, by 9 of theorem 1.3, $e \to x \le e \to e = e$. This implies that $x = e \to x \le e$. Hence $x = e$. In both cases there is a contradiction. So $e < x < x^2$. This implies that $x^2 \le x^4$. If $x^2 = x^4$, then by 2 of theorem 1.3, $x^2 \le [x^2ox^2] \to x^2 = e$. i.e; $x^2 \le e$. Hence $x^2 = e$ which is a contradiction. Hence $e < x < x^2 < x^4$. Suppose that $e < x < x^2 < x^4 < ... < x^{2^n}$ for some $n \in \mathrm{N}$. This implies that $x^{2^n} \le x^{2^n}ox^{2^n}$. If $x^{2^n} = x^{2^n}ox^{2^n}$, then by 2 of theorem 1.3, $x^{2^n} \le [x^{2^n}ox^{2^n}] \to x^{2^n} = e$. i.e; $x^{2^n} \le e$. Hence $x^{2^n} = e$ which is a contradiction. Hence by the principle of mathematical induction, the chain $e < x < x^2 < x^4 < ... < x^{2^n} ...$ does not terminate at some point. So, the sequence of elements x^{2^n} are all distinct. Hence the set L is infinite and unbounded above.

Corollary 2.2. *If L is bounded above by the element t, then t = e.*

Theorem 2.19. *For x, y $\in L$ the following properties hold.*
1. $x \le e$ *and* $y \le e \Leftrightarrow xoy \le x \wedge y$
2. $e \le x$ *and* $e \le y \Rightarrow x \wedge y \le xoy$
3. $e \le x$ *and* $y \le e \Rightarrow y \le xoy \le x$

Proof. Follows from 1 of theorem 1.3.

Theorem 2.20. *For $x \in L$ and any $n \in \mathrm{N}$, $x^n \le e \Leftrightarrow x \le e$.*

Proof. Let $x^n \le e$. Repeatedly applying the associative and distributive properties of o over the operation $\vee$, we have $(x\vee e)^n = x^n\vee x^{n-1}\vee...\vee x\vee e = (x^n\vee e)\vee(x^{n-1}\vee...\vee x\vee e) = x^{n-1}\vee...\vee x\vee e$ (as $x^n \le e$) $= (x\vee e)^{n-1}$. That is $(x\vee e)o(x\vee e)^{n-1} = (x\vee e)^{n-1} \Rightarrow (x\vee e) \le [(x \vee e)o(x \vee e)^{n-1}] \to (x \vee e)^{n-1} = (x \vee e)^{n-1} \to (x \vee e)^{n-1} = e \Rightarrow x \vee e = e \Rightarrow x \le e$. Hence $x \le e$ (by 2 of theorem 1.3).
The converse follows from theorem 2.7 by induction.

Corollary 2.3. *For $x \in L$, $x * e = x \Leftrightarrow x \le e$. Proof.* Follows from theorem 2.17 and theorem 2.20.

Corollary 2.4. *Let x, y $\in L$ and y is invertible element. Then for any $n \in \mathrm{N}$, $x^n \le y^n \Leftrightarrow x \le y$.*

Proof. Let $x^n \le y^n$. Then $(xo(e \to y))^n = x^no(e \to y)^n \le y^no(e \to y)^n = (yo(e \to y))^n = e$. Thus by theorem 2.20, $xo(e \to y) \le e$ and hence $x \le y$. The converse follows from 1 of theorem 2.7.

Theorem 2.21. *For $x \in L$ and any $n \in \mathrm{N}$, $x^n = e \Leftrightarrow x = e$.*

Proof. From theorem 2.20, $x \le e$. Observe that $x = x \to e = x \to x^n = x \to (xox^{n-1}) = (x \to x) \to x^{n-1} = e \to x^{n-1} = x^n \to x^{n-1}$ (by 4 of theorem 1.3). Since $x \le e$, $x \wedge e = x$. So by the distributive property of o over $\wedge$ repeatedly, we have $x^n = (x \wedge e)^n = x^n \wedge x^{n-1} \wedge ... \wedge x \wedge e = x^{n-1} \wedge ... \wedge x \wedge e = (x \wedge e)^{n-1} = x^{n-1}$. Hence $x = x^n \to x^{n-1} = e$. The converse is trivial.

Corollary 2.5. *In an l-group, every element other than the identity element has an infinite order.*

Definition 2.2. *An element $l \in L$ is called unity if and only if $xo(l \to x) = lol, \forall x \in L$.*

Lemma 2.1. *In L with unity l, the following properties hold.*
1. $lol = l$
2. $l \leq e$
3. $e = e \to l = (e \to l) \to l$

Proof. Let $x = e$ in definition 2.2. Then it is immediate that $lol = l$. Thus by 2 and 4 of theorem 1.3 it follows that $l \leq (lol) \to l = e$. Let $t = e \to l$. Then $t = e \to (lol) = (e \to l) \to l = t \to l$ i.e $t = t \to l$. By 4 of theorem 1.3, $e = t \to t = (t \to l) \to t = t \to (lot) = (t \to t) \to l = e \to l = t$. Thus $e \to l = e$.

Theorem 2.22. *Unity element is unique, if it exists in L.*

Proof. Let l and l' be unities. This implies $e \to l = e = e \to l'$ and $l \leq e \Rightarrow l \to l' \leq e \to l' = e$. Now by 3 in the definition of BH-lattices and definition of unity, $l \wedge l' = [(l \to l') \wedge e]ol' = (l \to l')ol' = l$. By a similar argument $l' \wedge l = l'$. Hence $l = l'$.

Corollary 2.6. *If BH-lattice L has unity, then L is an l-group if and only if $l = e$.*

Theorem 2.23. *If L with unity contains a least element a, then $a = l$*

Proof. Since a is least element of L, $a \leq e \Rightarrow (aoa) \leq eoa = a$ (by 1 of theorem 1.3). This implies $aoa = a$ (as a is least). Thus $l = (l \to a)oa = (l \to a)oaoa = loa \leq eoa = a$ (since $l \leq e$ and by 1 of theorem 1.3). Since a is least, $a = l$.

Theorem 2.24. *If L with unity l, contains an element x such that $x < l$, then the set L is an infinite set and unbounded below.*

Proof. Let $x < l$ and $x \neq l$. Since by lemma 2.1 $l \leq e$, it follows that $x < l \leq e \Rightarrow x^2 \leq lox \leq x$ (by 1 of theorem 1.3). If $x^2 = x$, then $x = x^2 \leq lox \leq x \Rightarrow x = lox$. Then the above argument together with l and 4 of theorem 1.3, $e = e \to l = (x \to x) \to l = x \to (xol) = x \to [xo(l \to x)]ox = x \to xo(l \to x) = x \to l$. $\Rightarrow e \leq x \to l \Rightarrow l < x$ which is a contradiction. Thus $x^2 \neq x$
If $x^2 = l$, then $x \to l = x \to x^2 = (x \to x) \to x = e \to x$. Since $x \leq e$, by 9 of theorem 1.3, $e \leq e \to x$. So $e \leq e \to x = x \to l \Rightarrow l < x$ which is a contradiction. Hence $x^2 \neq l$. In both cases there is a contradiction, hence $x^2 < x < l$. $\Rightarrow x^2ox^2 \leq x^2$. If $x^2 = x^4$, then $l = x^2o[x^4 \to l] = x^2o(x^2 \to l) = xol$. Hence by the same argument as above $l < x$, which is a contradiction. Hence $x^4 < x^2 < l \leq e$. In the similar fashion, the sequence of elements x^{2^n} are all distinct. Hence the set L is infinite and unbounded
below.

Corollary 2.7. *If L with unity is bounded below by the element t, then $t = l$.*

Theorem 2.25. *Let L be a BH-lattice with unity l. Then $L_l = \{x \in L : x \to l = e\}$ is a BH-lattice with least and greatest element. And L_l is subset of the BH-lattice $L^e = \{x \in L : e \to x = e\}$.*

Proof. By lemma 2.1, $lol = l$ and $e \to l = e$. Hence both l and e belong to L_l. For $x \in L_l$, $e \to x = (x \to l) \to x = (x \to x) \to l = e \to l = e$. Hence e is the greatest element and l is the least element in L_l.

Let $x, y \in L_l$. Then $x \leq e$ and $y \leq e$. This implies that $xoy \leq e$. Hence $(xoy) \to l \leq e \to l = e$. Furthermore by theorem 2.14 and 8 of theorem 1.3, $(x \to l)oy \leq (xoy) \to l$. This implies $y \leq (xoy) \to l$. Hence $e = y \to l \leq \{(xoy) \to l\} \to l = (xoy) \to l$. Thus $(xoy) \to l = e$. By 14 of theorem 1.3, $(x \wedge y) \to l = (x \to l) \wedge (y \to l)$. Hence L_l is closed under both o and $\wedge$. Since $x, y \leq x \vee y \leq e$, by 8 of theorem 1.3, $e = (x \to l) \wedge (y \to l) \leq (x \vee y) \to l \leq e \to l = e$. Hence $x \vee y \in L_l$. Moreover $(x \to y) \to l = (x \to l) \to y = e \to y = (y \to l) \to y = (y \to y) \to l = e \to l = e$. So $x \to y \in L_l$. Hence L_l is a BH-lattice with l as least element and e as greatest element. Clearly $L_l \subseteq L^e$. Finally by the proof of theorem 4.2 of [9], L^e is a BH-lattice with greatest element.

Theorem 2.26. (L, o) *is a group iff* $\forall x, y, z \in L, x \to y = x \to z \Rightarrow y = z$.
Proof. Let (L, o) be a group and let for any $x, y, z \in L$, $x \to y = x \to z$. Hence by 9 of theorem 1.5 it follows that $(x \to y)^{-1} = y \to x = z \to x = (x \to z)^{-1}$. $\Rightarrow (y \to x) \to (e \to x) = (z \to x) \to (e \to x)$. Using 4 of theorem 1.3 and 8 of theorem 1.5, this

implies that $(y \to (xo(e \to x)) = (z \to (xo(e \to x))$. So $y = y \to e = z \to e = z$.
Conversely let $x, y, z \in L, x \to y = x \to z \Rightarrow y = z$. Then $e \to \{yo(e \to y)\} = (e \to y) \to (e \to y) = e = e \to e$. So that $yo(e \to y) = e$. Hence y is invertible element. Hence (L, o) is a group.

Theorem 2.27. *L is an l-group if (L, o) is a group and further $a \to b$ is the solution of the equation $box = a$.*

Proof. If (L, o) is a group, then by the definition of BH-lattice and 1 of theorem 1.3 $L = (L, o, \leq, \to)$ is an l-group. Again by 7 and 8 theorem 1.5, $bo(a \to b) = bo[ao(e \to b)] = a$. Hence $a \to b$ is the solution of the equation $box = a$.

Theorem 2.28. *BH-lattice L bounded below is a Heyting Algebra if $xoy = x \wedge y, \forall x, y \in L$.*
Also, in L, if $(L, \leq, \to)$ is a Heyting Algebra then, $xoy = x \wedge y, \forall x, y \in L$.

Proof. The first part of the theorem is trivial. For the second part let $(L, \leq, \to)$ be a Heyting Algebra. For $a, b \in L, a \to b$ is the largest x such that $x \wedge b \leq a$. $a \wedge b \leq a \Rightarrow a \leq a \to b \Rightarrow a \wedge b \leq (a \to b) \wedge b = a \wedge b$ (by 4 of theorem 1.1, as L is Heyting algebra). Hence $a \wedge (a \to a) = a \wedge a \Rightarrow a \wedge e = a \Rightarrow a \leq e$. Hence e is the largest element of the lattice. Hence by 1 of theorem 1.3, $aob \leq a, b \Rightarrow aob \leq a \wedge b. \Rightarrow aob = (aob) \wedge (a \wedge b) = (a \wedge b) \wedge [aob \to a \wedge b]$ (by 4 of theorem 1.1 in a Heyting Algebr $(L, \leq, \to)$) $= a \wedge b$ (by 3 of theorem 1.1).

Theorem 2.29. *Let L be with unity l, if $(L, \vee, \wedge)$ is a Boolean algebra, then $o = \wedge$ and $x' = l \to x$.*

Proof. By theorem 2.23 and corollary 2.2, l is the least element and e is the greatest element of the Boolean algebra. Let $x \in L$. Then there exists an element $x' \in L$ such that $x \vee x' = e$ and $x \wedge x' = l$. Hence by theorem 2.28, $o = \wedge$ and $xox' = x \wedge x' = l$. Hence $x \wedge (l \to x) = xo(l \to x) = l$ and $xox' = l \Rightarrow x' \leq l \to x$. $\Rightarrow e = x \vee x' \leq x \vee (l \to x) \Rightarrow x \vee (l \to x) = e$. So $l \to x$ is the complement of x in the Boolean algebra. Hence by uniqueness of complement $x' = l \to x$

Decomposition Theorems of BH-lattices

Lemma 3.1. *Let L be a BH-lattice and for $x, y, z \in L$, $x \to (zoz) \leq (x \to z)o(e \to z)$ holds. Then the following are equivalent.*

1. $H = \{x \in L : xox \to x = e\}$
2. $H' = \{x \in L : xox = x\}$
3. $B = \{x \in L : e \to x = e\}$

Proof. Let $x \to (zoz) \leq (x \to z)o(e \to z)$. Since by theorem 2.8, $(x \to z)o(e \to z) \leq (xoe) \to (zoz)$, it follows that $(xoe) \to (zoz) = (x \to z)o(e \to z)$.
Let $x \in H \Rightarrow xox \to x = e \Rightarrow x \leq xox$. As $x \leq xox \to x = e$, it follows that $xox \leq x$(by 1 and 2 of theorem 1.3). So that $xox = x$. Thus H is the set of all idempotent elements with respect to the operation o. Also if $xox = x$, then clearly $xox \to x = x \to x = e$. Let $x \in H \Rightarrow (xox) \to x = e$. Thus by 4 of theorem 1.3,
$e \to x = \{(xox) \to x\} \to x = (xox) \to (xox) = e$. Hence $x \in B$. Furthermore let
$y \in B \Rightarrow e \to y = e$. Then by 1 and 3 of theorem 1.3, $y = yo(e \to y) \leq e \Rightarrow y^2 \leq y$. And $e = y^2 \to y^2 = (y^2 \to y)o(e \to y) = y^2 \to y \Rightarrow y \leq y^2$. Hence $y^2 = y$.

Theorem 3.1. *A BH-lattice L is direct product of Heyting algebra and a commutative l-group if*
1. $x \to (zoz) \leq (x \to z)o(e \to z)$

2. there exists an idempotent element $0 \in L$ such that $0 \leq x$, for any idempotent element $x \in L$.

Furthermore if L is the direct product of a Heyting algebra and a commutative l-group, then condition (1) holds.

Proof. Let the conditions (1) and (2) hold. Since by theorem 2.8, $(x \to z)o(e \to z) \leq (xoe) \to (zoz)$, it follows that $(xoe) \to (zoz) = (x \to z)o(e \to z)$.

BH-lattice with greatest element e is in similar line to the proof 2 of theorem 1.6 [9]. Further for $x, y \in H$, as $xoy \leq x$ and $xoy \leq y$ it follows that $xoy \leq x \wedge y$. By theorem 2.7, $x \wedge y \leq x, y$ implies that $x \wedge y = (x \wedge y)o(x \wedge y) \leq xoy$. Hence $x \wedge y = xoy$. Thus by theorem 2.28 H is a Heyting algebra. Moreover, as $0, e \in H$, H is non-trivial.
Now consider the set $G = \{a \in L : (aoa) \to a = a\}$. Since $e \in G, G \neq \emptyset$. Let $a \in G$. Then $ao(e \to a) = \{(aoa) \to a\}o(e \to a) = (aoa) \to (aoa) = e$ (by (a) above). Hence a is an invirtible element. Let $a \in L$ be an invirtible element. Then $ao(e \to a) = e = (aoaoe) \to (aoa) = [(aoa) \to a]o(e \to a)$(by (a) above and 7 of theorem 1.5)$\Rightarrow a = (aoa) \to a$. Hence G is the set of all invertible elements of L. By 11 of theorem 1.5, G is a commutative l-

group. As $e \in G$ and using 10 of theorem 1.5, $e \to x \in G, \forall x \in L$, G is non-trivial.
Hence L is the direct product of G and H, by a proof in a comparable strip to the proof 2 of theorem 1.6 [9].
Furthermore if L is the direct product of a Heyting algebra and a commutative l-group, then trivially condition (1) holds.

Theorem 3.2. *BH-lattice L is direct product of a BH-lattice with least and greatest elements and a commutative l-group if and only if*
1. $(xoy) \to (zoz) \leq (x \to z)o(y \to z)$
2. *There exists an element l such that* $xo(l \to x) = lol$

Proof. Let the conditions given in (1) and (2) hold. Then by lemma 2.1, $lol = l$ and $e \to l = e$, And from (1) by 2 of theorem 2.8 it follows that $(xoy) \to (zoz) = (x \to z)o(y \to z)$. Consider $G = \{x \in L : x \to l = x\}$ and $L_l = \{x \in L : x \to l = e\}$. Then by theorem 2.25, L_l is a BH-lattice with least element l and greatest element e. Since both e and l are in L_1, it is non-trivial.
Fore $x, y \in G$, $(xoy) \to l = (xoy) \to (lol) = (x \to l)o(y \to l)$ and $(x \wedge y) \to l = (x \to l) \wedge (y \to l)$. Hence G is closed under o and $\wedge$. Furthermore $(x \to y) \to l = (x \to l) \to y = x \to y$ and consequently $x \to y \in G$. And $xo(e \to x) = (x \to l)o\{(l \to l) \to x\} = (x \to l)o\{(l \to x) \to l\} = \{xo(l \to x)\} \to (lol) = l \to l = e$. Hence x is invertible element. Thus G is a $\wedge$ semi lattice and hence G is a commutative l-group. By lemma 2.1, $e \in G$ and using 10 of theorem 1.5, $e \to x \in G, \forall x \in L$. So G is non-trivial.
Now for $a \in L$, let $t = a \to l$ and $s = a \to t$. Then $t \to l = (a \to l) \to l = a \to (lol) = a \to l = t$ and $s \to l = (a \to (a \to l)) \to l = (a \to l) \to (a \to l) = e$. Thus $t \in G$ and $s \in L_l$. Since $l \leq e$ by 9 of theorem 1.3, $a = a \to e \leq a \to l$. Thus by theorem 2.10, $tos = (a \to l)o(a \to (a \to l)) = a$.
Now let $a = t'os'$, where $t' \in G$ and $s' \in L_l$. Then $a \to l = (t'os') \to lol = (t' \to l)o(s' \to l) = eo(t' \to l) = t'$. Hence $t = t'$ and consequently $tos = t'os' = tos' \Rightarrow (e \to t)o(tos) = (e \to t)o(tos') \Rightarrow s = s'$. Clearly $\{e\} = G \cap B$. Thus L is the direct product of B and G.
Conversely, if L is the direct product of BH-lattice with least and greatest element B and commutative l-group G, then trivially conditions (1) and (2) in the theorem hold.

Corollary 3.1. *For a BH-lattice L with unity and bounded below the following are equivalent*
1. $(xoy) \to (zoz) \leq (x \to z)o(y \to z), \forall x, y, z \in L.$
2. $x \to (zoz) \leq (x \to z)o(e \to z), \forall x, z \in L.$
3. *L is the direct product of Heyting algebra and a commutative l-group.*

Theorem 3.3. *BH-lattice L is direct product of a Boolean algebra and a commutative l-group if and only if*
1. $x \to (yoy) \leq (x \to y)o(e \to y)$ *for all* $x, y \in L$
2. *there exists an element l in L such that* $(l \to x)ox = lol$ *and* $l \to (l \to x) = x$ *for all* $x \in L$.

Proof. Suppose that the condition in (1) and (2) hold. Let G be the set of all invertible elements of L and H be the set of all idempotent elements of L. By lemme 3.1 and the same argument as in the proof of theorem 3.1, H is a BH-lattice with greatest element e, $o = \wedge$ and L is direct product of G and H.

For any $x \in H$, by 9 of theorem 1.3) $x \leq e \Rightarrow e \leq e \to x \leq e \Rightarrow e = e \to x$. Hence as $l \in H$, it follows that $e = e \to (l \to x)$. So by the condition given in 2 and 4 of theorem 1.3, $l \to (l \to x) = x \Rightarrow [l \to (l \to x)] \to l = x \to l \Rightarrow e = e \to (l \to x) = [l \to (l \to x)] \to l = x \to l \Rightarrow l \leq x$. Hence H is bounded below by the element l. Thus by theorem 2.28, H is a Heyting algebra.
Now for $x \in H$, $x \wedge (l \to x) = [l \to (l \to x)] \wedge (l \to x) = [l \to (l \to x)]o(l \to x) = l$.
Moreover, $l = [l \to \{x \vee (l \to x)\}]o[x \vee (l \to x)] = [[l \to \{x \vee (l \to x)\}] \wedge x] \vee [[l \to \{x \vee (l \to x)\}] \wedge (l \to x)]$. This implies $[l \to \{x \vee (l \to x)\}] \wedge x = l$ and $[l \to \{x \vee (l \to x)\}] \wedge (l \to x)] = l$. Hence $l \to \{x \vee (l \to x)\} \leq l \to x$ and $l \to \{x \vee (l \to x)\} \leq l \to (l \to x) = x$. Hence $l \to \{x \vee (l \to x)\} \leq (l \to x) \wedge x = l$. Hence $l \to \{x \vee (l \to x)\} = l$. Hence $e = l \to l = l \to [l \to \{x \vee (l \to x)\}] = x \vee (l \to x)$. Hence H is a Boolean algebra. Thus L is the direct product of Boolean algebra and a commutative l-group.
Conversely if L is the direct product of a Boolean algebra H and a commutative l-group G, then trivially condition (1) and (2) hold.

Definition 3.1. *A BH-lattice L is called idempotent if* $x^2 = x, \forall x \in L.$

Theorem 3.4. *An idempotent BH-lattice L with unity l is a direct product of Boolean algebra and commutative l-group iff* $l \to (l \to x) = x, \forall x \in L.$

Proof. Suppose that $l \to (l \to x) = x, \forall x \in L$. Let $H = \{x \in L : e \to x = e\}$ and G be the set of all invertible elements of L. The proof of H is a BH-lattice with greatest element e is analogous to the proof 2 of theorem 1.6[9] and further L is the direct product of H and G can be obtained. Furthermore for $x, y \in L$, $xoy \leq x$ and $xoy \leq y$. Hence $xoy \leq x \wedge y$. By theorem 2.7, $x \wedge y = (x \wedge y)o(x \wedge y) \leq xoy$. Thus $xoy = x \wedge y$. Finally by the same argument as in the proof of theorem 3.3, $x' = l \to x, \forall x \in H$. Thus H is a Boolean algebra.

Conversely if L is the direct product of Boolean algebra and commutative l-group, then the condition $l \to (l \to x) = x, \forall x \in L$ is trivial.

Open problem

1. Which group of BH-lattice can be decomposeble in to irreducible non-trivial sub algebras of BH-lattices?

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