# The Study of Interplay of Singlet Superconductivity and Ferromagnetism in Superconducting HoMo$_6$Se$_8$

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**ABSTRACT:** This article reports the effect of magnetic ordering and superconducting order parameter on superconducting and magnetic transition temperatures of interacting singlet superconductivity and ferromagnetism in HoMo$_6$Se$_8$ material. The basic superconducting parameters were calculated starting with the BCS type model Hamiltonian and using the double time temperature dependent Green function formalisms. Results showed that the superconducting critical temperature decreases with enhancement of superconducting order parameter and vice versa. This is perhaps due to the weakening of the binding energy as the temperature approaches its critical value. On the other hand, the superconducting critical temperature demonstrates attenuation with increasing magnetic order parameters $\eta$. It was also observed that the enhancement of magnetic transition temperature with the increase of the magnetic order parameter demonstrates variation with the alteration of the superconducting order parameter. Moreover, the calculations revealed that there is a temperature region where both superconductivity and magnetic order coexist. Furthermore, the present theoretical analysis is broadly in agreement with existing experimental findings.

**Keywords/Phrases:** Ferromagnetism, Green's function, HoMo$_6$Se$_8$, Magnetic ordering, Singlet superconductivity

## INTRODUCTION

The relationship between superconductivity and magnetic ordering has long been a central topic in condensed matter physics (Matthias, 1953). It has been generally accepted that in the context of Bardeen, Cooper, and Schrieffer's (BCS) theory, conduction electrons cannot be both magnetically ordered and superconducting. The phenomena were studied theoretically for the first time by Ginzburg (Ginzburg, 1957) and experimentally by Matthias (Matthias, 1953). According to their study superconductivity and ferromagnetism can coexist due to the effect of dipole field from the moment of a small amount of magnetic impurity dissolved in a nonmagnetic superconductor element. In fact, it has been found experimentally that ferromagnetism is often detrimental to most superconductors (Matthias, 1953; Fertig *et al.*,1977; Chervenak, and Valles, 1995; Garrett et al., 1998; Smith and Ambegaokar, 2000). However, there is no fundamental obstacle to the coexistence of ferromagnetism and superconductivity. When two electrons form a bound state in a Cooper pair, the overall wave function must be anti-symmetric, requiring the spin states to be singlet or triplet. In the singlet state, there is only one

spin state while there are three spin states in a triplet state. Obviously, the two electrons in a singlet Cooper pair always have antiparallel spins while in a triplet superconductor, the two electrons can have parallel spins. The interplay between superconducting and ferromagnetic long-range order has recently renewed interest with the discovery of superconductivity in ferromagnetic compounds such as UGe$_2$ (Kakani et al.,1988), URhGe (Canfield, et al., 1998) and ZrZn$_2$ (Saxena et al., 2000). Such a long-range magnetic order has indeed been found in HoMo$_6$Se$_8$ and in ErRh$_4$B$_4$ (Pfleiderer et al., 2001). In HoMo$_6$S$_8$, ErRh$_4$B$_4$, and HoMo$_6$Se$_8$, superconductivity and the oscillating magnetic order coexist in a narrow temperature range. Clear experimental evidence for the existence of non-uniform magnetic structures was obtained from neutron diffraction measurements in ErRh$_4$B$_4$ (Sinha et al., 1982), HoMo$_6$S$_8$ (Lynn et al.,1981) and HoMo$_6$Se$_8$ (Lynn et al.,1984). BCS theory predicts that a Cooper pair forms two electrons with opposite momentum and also opposite spin in singlet superconductivity. Singlet superconductivity can coexist with ferromagnetism in a simple model with pairing interaction and scattering of copper S-wave pairs by ferromagnetic ordered localized pairs (Desta et al., 2015). In ferromagnetic superconductors, the

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magnetic ordering is due to the indirect exchange interaction going via the conduction electrons. In addition to the exchange mechanism the electromagnetic interaction of superconductivity and ferromagnetism is always present in real compounds. It is related to the Meissner screening of the magnetic induction created by the magnetization. Namely, the magnetization creates a dipolar magnetic field, which on the other hand induces the screening current of conduction electrons; the Meissner effect. This mechanism also favors a non-uniform magnetic structure instead of a ferromagnetic one (Krey, 1973). This article presents the theoretical study of the interplay between singlet superconductivity and ferromagnetism in $HoMo_6Se_8$ using standard model Hamiltonian employing double time temperature-dependent Green's function technique based on quantum field theory.

*Formulation of the problem*

The Model Hamiltonian can be formulated as:

$$H = \sum_{k,\sigma} \varepsilon_k \hat{c}^\dagger_{k,\sigma} \hat{c}_{k,\sigma}$$
$$- \sum_{k,k'} V_{k,k'} \hat{c}^\dagger_{k\uparrow,\sigma} \hat{c}^\dagger_{-k\downarrow} \hat{c}_{-k_\downarrow} \hat{c}_{k'_\downarrow}$$
$$+ \sum_{l,\sigma} \varepsilon_l \hat{b}^\dagger_{l,\sigma} \hat{b}_{l,\sigma} + \sum_{l,m,k} \alpha_{l,m} (\hat{c}^\dagger_{k\uparrow,\sigma} \hat{c}^\dagger_{-k\downarrow} \hat{b}_{l\uparrow} \hat{b}_{m\uparrow}) \quad (1)$$

The energy of free charge carriers and the electron-phonon interaction potentials are represented by $\varepsilon_k$ and $V(k\ k')$ respectively, and the operators of creation (annihilation) of the conduction electron specified by the wave vector k and the spin $\sigma$ are designated by $\hat{c}^\dagger_{k\sigma}(\hat{c}_{k\sigma})$ in the first two terms form the BCS Hamiltonian. In the third term, the energy of the localized electrons is represented by $\varepsilon_l$ and creation (annihilate ion) operators of localized electrons are $\hat{b}^\dagger_{l\uparrow}(\hat{b}_{m\uparrow})$ at the localized sites. The last term stands for the scattering of conduction electrons by localized ferromagnetic electrons with a certain coupling constant $(\alpha_{l,m})$.

In order to find the order parameter and superconducting critical transition temperature $T_c$, we consider the model Hamiltonian and employ the double time temperature dependent Green's function technique (Zubarev,1960), hence, the equation of motion can be written as

$$\omega \left\langle\!\left\langle \hat{c}_{k\sigma}, \hat{c}^\dagger_{-k'\sigma} \right\rangle\!\right\rangle = \left\langle [\hat{c}_{k\sigma}, \hat{c}^\dagger_{-k\sigma}] \right\rangle + \left\langle\!\left\langle [\hat{c}_{k\uparrow}, \hat{H}]; \hat{c}^\dagger_{-k\downarrow} \right\rangle\!\right\rangle$$
$$(2a)$$

Setting $\hat{H}$ and using the RPA to decouple higher-order terms, the following two equations are obtained

$$(\omega - \varepsilon_k) \left\langle\!\left\langle \hat{c}_{k\uparrow}, \hat{c}^+_{k\uparrow} \right\rangle\!\right\rangle = 1 - (\Delta - \eta) \left\langle\!\left\langle \hat{c}^+_{-k\downarrow}, \hat{c}^+_{k\uparrow} \right\rangle\!\right\rangle$$
$$(2b)$$

and

$$(\omega + \varepsilon_k) \langle\!\langle \hat{c}^+_{-k\downarrow} \hat{c}^+_{k\uparrow} \rangle\!\rangle = -(\Delta - \eta) \langle\!\langle \hat{c}_{k\uparrow} \hat{c}^+_{k\uparrow} \rangle\!\rangle$$
$$(3)$$

where the superconducting order parameter is given by $\Delta = V \sum_{p'} \langle \hat{c}_{p'\downarrow}, \hat{c}_{-p'\uparrow} \rangle$ and the magnetic order parameter is given by $\eta = \sum_k \alpha_k^{lm} \langle \hat{b}_{l\uparrow} \hat{b}_{m\uparrow} \rangle$. Substituting Eq. (2) into Eq. (3) for $\langle\!\langle \hat{c}_{k\uparrow}, \hat{c}^+_{k\uparrow} \rangle\!\rangle$ gives:

$$(\omega + \varepsilon_k) \left\langle\!\left\langle \hat{c}^+_{-k\downarrow} \hat{c}^+_{k\uparrow} \right\rangle\!\right\rangle = \frac{-(\Delta - \eta)}{(\omega - \varepsilon_k)} \left[ 1 - (\Delta - \eta) \left\langle\!\left\langle \hat{c}^+_{-k\downarrow} \hat{c}^+_{k\uparrow} \right\rangle\!\right\rangle \right]$$
$$(4)$$

Rearranging Eq. (4) for the expression of the superconducting paired state yields:

$$\left\langle\!\left\langle \hat{c}^+_{-k\downarrow} \hat{c}^+_{k\uparrow} \right\rangle\!\right\rangle = \frac{-(\Delta - \eta)}{(\omega^2 - \varepsilon_k^2 - (\Delta - \eta)^2)}$$
$$(5)$$

Applying Laplace's transform with replacement of $\omega$ by Matsubara's frequency $i\omega_n$, $\omega \to i\omega_n$ where $\omega_n = \frac{(2n+1)\pi}{\beta}$ and $\beta^{-1} = K_B T$, and using in Eq. (5):

$$\langle\!\langle \hat{c}^+_{-k\downarrow} \hat{c}^+_{k\uparrow} \rangle\!\rangle = \frac{\beta^2 (\Delta - \eta)}{(2n+1)^2 \pi^2 + \beta^2 \varepsilon_k^2 + \beta^2 (\Delta - \eta)^2}$$
$$(6)$$

The superconducting order parameter can be expressed as :

$$\Delta = \frac{V}{\beta} \sum_k \langle\!\langle \hat{c}^+_{-k\downarrow} \hat{c}^+_{k\uparrow} \rangle\!\rangle \quad (7)$$

Substituting Eq. (6) into Eq. (7) gives

$$\Delta = V\beta \sum_k \left( \frac{(\Delta - \eta)}{(2n+1)^2 \pi^2 + \beta^2 E^2} \right) \quad (8)$$

Let

$$E = (\varepsilon_k^2 + (\Delta - \eta)^2)^{1/2} \quad (9)$$

Using the trigonometric relations

$$\frac{1}{2x}tanh\left(\frac{x}{2}\right) = \sum_{\infty}^{\infty}\frac{1}{(2n+1)^2\pi^2+x^2} \qquad (10a)$$

where $x = \beta E$, hence, we can rewrite in the following form:

$$\frac{1}{2\beta E}tanh\left(\frac{\beta E}{2}\right) = \sum_{\infty}^{\infty}\frac{1}{(2n+1)^2\pi^2+(\beta E)^2} \qquad (10b)$$

Henceforth, substituting Eq. (10b) in to Eq. (8) one can obtain

$$\Delta = |V|\beta\sum_k(\Delta-\eta)\left(\frac{tanh(\beta E_k/2)}{2\beta E_k}\right) \qquad (11)$$

The equation may have some solution in the Fermi liquid, typically only fermions near the Fermi surface will contribute or an attractive interaction is effective for the region $-\hbar\omega_b < \varepsilon < \hbar\omega_b$, where $\omega_b$ is frequency of wave propagation in the fermions region. The sum of momentum k can be written as an integer of energy. By introducing the density of state that does not vary over this energy interval at the Fermi surface, expressed as:

$$\sum_k \rightarrow \int d^3k = \int_{-\epsilon_F}^{\infty} d\varepsilon N(\varepsilon) \qquad (12)$$

According to Eq. (12), Eq. (11) can be written as

$$\Delta = 2N(0)V\beta\int_0^{\hbar\omega_b}\frac{(\Delta-\eta)}{2\beta(\varepsilon_k^2+(\Delta-\eta)^2)^{\frac{1}{2}}}tanh\left(\beta\frac{(\varepsilon_k^2+(\Delta-\eta)^2)^{1/2}}{2}\right)d\varepsilon \qquad (13)$$

$$\frac{\Delta}{\lambda} = \int_0^{\hbar\omega_b}\frac{(\Delta-\eta)}{(\varepsilon_k^2+(\Delta-\eta)^2)^{\frac{1}{2}}}tanh\left(\beta\frac{(\varepsilon_k^2+(\Delta-\eta)^2)^{1/2}}{2}\right)d\varepsilon \qquad (14)$$

where $\lambda = N(o)V$ is the parameter.

## RESULTS AND DISCUSSION

In this section, the interaction of singlet superconductivity and ferromagnetism and their coexistence in $HoMo_6Se_8$ is critically studied. Expressions to the electron-phonon coupling parameter $\lambda$, the superconducting critical transition temperature $T_c$, the superconducting order parameter $\Delta$, the magnetic ordering parameter $\eta$, and the magnetic transition temperature $T_m$, have been calculated numerically and plotted to describe the required findings.

*Electron-phonon Coupling parameter $\lambda$ and temperature $T_C$ in HoMo_6Se_8*

In purely non-magnetic superconducting materials the electron-phonon coupling depends on the superconducting order parameter and temperature, as in BCS theory. This can be described in the absence of magnetic ordering parameter following Eq. (14), and reduced to

$$\frac{1}{\lambda} = \int_0^{\hbar\omega_b}\frac{1}{(\varepsilon_k^2+\Delta^2)^{\frac{1}{2}}}tanh(\beta/2)(\varepsilon_k^2+\Delta^2)^{1/2}d\varepsilon_k \qquad (15)$$

The superconducting order parameter $\Delta \rightarrow 0$ as $T \rightarrow T_c$. Applying integration by parts to Eq. (15) we get

$$T_C = 1.14\frac{\hbar\omega_b}{k_B}exp\left[-\frac{1}{\lambda}\right] \qquad (16)$$

Fig. 1 is plotted using the last equation and as can be seen, the superconducting transition temperature $T_c$ increases with the increase of the coupling parameter $\lambda$. This is due to the increase in the electron-phonon interaction parameter, V up to the reciprocal of the experimental value of the density of state ($N^{-1}$ = 1754 K.spin). This enhances effective net attraction between electrons producing the binding energy of pairing interaction to the order of *meV* enough to keep them paired at extremely low temperatures thus, the coupled electrons in the system condensate in the ground state and this raises its transition temperature by a small factor in agreement with the existing experimental results (Lynn et al., 1984; Bennemann et al., 2008).
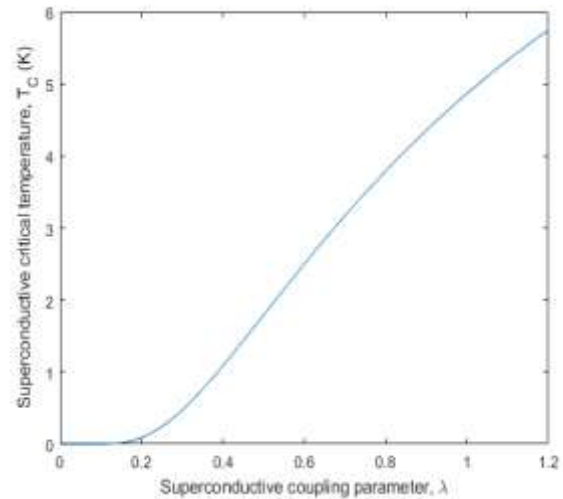


**Figure 1. Superconducting transition temperature $T_c$ versus electron-phonon coupling parameter.**

*The effect of temperature, T on superconducting order parameter in HoMo$_6$Se$_8$ superconductor.*

It has been understood that the variation of temperature affects the current flow in the superconducting material. For example: At zero temperature, current flows when eV = Δ, where V is the voltage across junctions and 2Δ is the energy gap. With the assumption that the Fermi sea below the superconducting gap is filled with cooper pairs, the gap represents the energy required to break one of these pairs and promote an electron into a state lying above the gap.

Starting from Eq. (15) we get:

$$\frac{d\varepsilon_k}{k^2+\Delta^2} tanh \frac{\sqrt{\varepsilon_k^2+\Delta^2}}{2K_BT} = \int_0^{\hbar\omega_b} \frac{2}{\beta} \sum_{n=-\infty}^{\infty} \left(\frac{d\varepsilon_k}{\omega_n^2+E^2}\right)$$

(17)

With the help of Laplace Transform and Matsubara frequency Eq. (17) is changed to:

$$\left(\int_0^{\hbar\omega_b} \frac{d\varepsilon_k}{\sqrt{\varepsilon_k^2+\Delta^2}} tanh \frac{\beta\sqrt{\varepsilon_k^2+\Delta^2}}{2}\right)$$

$$= \int_0^{\hbar\omega_b} d\varepsilon\, tanh \frac{\beta\sqrt{\varepsilon_k^2+\Delta^2}}{2} - \frac{4}{\beta}\int_0^{\hbar\omega_b}\Delta^2 \sum_{n=-\infty}^{\pi} \frac{1}{a^4(1+x^2)^2} d\varepsilon_k = A_1 + A_2 \quad (18)$$

where

$$A_1 = \int_0^{\hbar\omega_b} tanh \frac{\beta\sqrt{\varepsilon_k^2+\Delta^2}}{2}\, d\varepsilon_k = -\ln 1.14\beta\hbar\omega_b$$

(19)

and

$$A_2 = -\frac{4}{\beta}\int_0^{\hbar\omega_b}\Delta^2 \sum_{n=-\infty}^{\infty} \frac{1}{a^4(1+x^2)^2} d\varepsilon_k +\dots \approx -\left(\frac{\beta\Delta}{\pi}\right)^2 1.05$$

(20)

where $x = E\beta$, $E^2 = (\varepsilon_k^2 + \Delta^2)$ and $1/\beta = K_B T_m$.

Inserting results of Eq. (19) and Eq. (20) into Eq. (18) for A$_1$ and A$_2$ yields:

$$\left(\int_0^{\hbar\omega} \frac{d\varepsilon}{\sqrt{\varepsilon_k^2+\Delta^2}} tanh \frac{\sqrt{\varepsilon_k^2+\Delta^2}}{2K_BT}\right)$$

$$= -\ln 1.14 \frac{\hbar\omega_b}{K_B T_m} + \left(\frac{\Delta}{\pi K_B T_m}\right)^2 1.05 \quad (21)$$

Considering the condition;

$$T \to 0\,K,\ \beta \to \infty\ \text{and}\ tanh(\beta E/2) \to 1 \quad (22)$$

Using the preceding equation in Eq. (21), we obtain:

$$\frac{1}{\lambda} = \int_0^{\hbar\omega_b} \frac{(\varepsilon_k^2+\Delta^2)^{1/2}}{(\varepsilon_k^2+\Delta)^{\frac{1}{2}}} d\varepsilon = -1.14\left(\ln \frac{\hbar\omega}{K_BT}\right) + (\Delta)^2 \left(\frac{1}{\pi K_BT}\right)^2 1.05$$

Equating Eq. (16) with Eq. (23), one can obtain

$$-1.14\left(\ln \frac{\hbar\omega}{K_BT_C}\right) = -1.14\left(\ln \frac{\hbar\omega}{K_BT}\right) + (\Delta)^2 \left(\frac{1}{\pi K_BT}\right)^2 1.05+\dots$$

(24)

$$\ln\left(\frac{T}{T_C}\right) = -\Delta^2 \left(\frac{1}{\pi K_BT}\right)^2 +\dots$$

Using the Taylor expansion, for very small $\frac{T}{T_c}$, from Eq. (25) yields

$$\ln\left(\frac{T}{T_C}\right) \approx -\left(1 - \frac{T}{T_C}\right)$$

Comparing Eq. (25) with Eq. (26), and substituting experimental value for $T_C$,

$$\Delta(T) = 1.45\ meV \left(1 - \frac{T}{5.5K}\right)^{\frac{1}{2}} \quad (27)$$

Using Eq. (27), we plotted the phase diagram of the superconducting order parameter Δ versus superconducting transition temperature $T_c$ as shown in Fig. 2. We observed that the decrease in superconducting order parameter, Δ enhances the transition temperature up to the value of the critical transition temperature in agreement with experimental findings by (Bennemann et al., 2008). This is perhaps due to the weakening of the binding energy as the temperature approaches its critical value $T_C \approx 5.5K$.
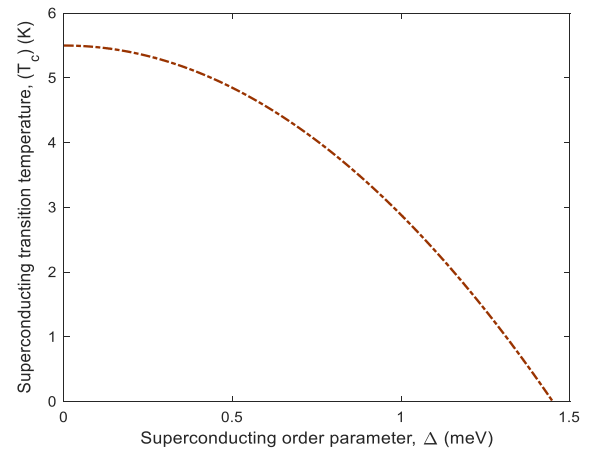


**Figure 2. Superconducting order parameter versus superconducting temperature.**

*Impact of magnetic order parameter $\eta$ on superconducting critical temperature at $\Delta = 0$*

Using Eq. (22) in Eq. (14), we first get the expression of BCS energy gap:

$$\frac{\Delta}{\lambda} = \int_0^{\hbar\omega_b} \frac{(\Delta-\eta)}{(\varepsilon_k^2+(\Delta-\eta)^2)^{\frac{1}{2}}} \, d\varepsilon_k \qquad (28)$$

Let $a = (\Delta - \eta)$ and $x = \varepsilon_k$, one can apply the method of solving inverse hyperbolic function in Eq. (28) can be simplified into:

$$\frac{\Delta}{\lambda} = \int_0^{\hbar\omega_b} \frac{(\Delta-\eta)}{(\varepsilon_k^2+(\Delta-\eta)^2)^{\frac{1}{2}}} \varepsilon_k = \int_0^{\hbar\omega_b} \frac{a}{(a^2+x^2)} \, dx = a \sin^{-1}\left(\frac{x}{a}\right)$$
(29)

$$\frac{1}{\lambda} = \left(1 - \frac{\eta}{\Delta}\right) \sin^{-1}\left(\frac{x}{a}\right) = \left(1 - \frac{\eta}{\Delta}\right) \ln\left[\left(\frac{x}{a}\right) + \sqrt{\left(\frac{x}{a}\right)^2 + 1}\right]$$

Evaluated between zero (0) and $\hbar\omega_b$, gives:

$$\frac{1}{\lambda} \approx \left(1 - \frac{\eta}{\Delta}\right) \ln\left\{\left(\frac{\hbar\omega_b}{\Delta-\eta}\right) + \left(\frac{\hbar\omega_b}{\Delta-\eta} + 1\right)\right\}$$
(30)

$$\frac{1}{\lambda} \approx \left(1 - \frac{\eta}{\Delta}\right) \ln\left(\frac{2\hbar\omega_b}{\Delta - \eta}\right)$$

$$(\Delta - \eta) = 2\hbar\omega_b \exp\left(-\frac{1}{\lambda(1-\eta/\Delta)}\right)$$
(31)

In the absence of the magnetic order parameter, Eq. (31) gives the expression of BCS energy gap.

$$\Delta = 2\hbar\omega_b \exp\left(-\frac{1}{\lambda}\right)$$
(32)

This equation leads to the zero-temperature value of the energy gap where $\hbar\omega_b = \hbar\omega_D \cong 10^{-3} eV$ for BCS model. Next, by using the experimental value, $T_C = 5.5K$, the zero-temperature value of the energy gap for HoMo$_6$Se$_8$ is given by $\frac{2\Delta(0)}{(K_B T_C)} = 3.5$.

This leads to

$$\Delta(0) = 0.833 \ meV/K \qquad (33)$$

Finally, the interdependence of superconducting transition temperature Tc and the magnetic order parameter $\eta$ is shown as follows by combining Eqs. (31-33):

$$\Delta(0) - \eta = \left(\frac{\Delta}{\frac{\Delta(o)-\eta}{\Delta(o)}}\right) \qquad (34)$$

From which one can obtain the next equation by making simple rearrangements by using the preceding equations and conditions.

$$T_C = \frac{(0.863 - \eta)^2}{0.13} \qquad (35)$$

This equation is used to analyze the effect of magnetic order parameter $\eta$ on superconducting transition temperature T$_c$, as shown in Fig. 3.

The effect of the magnetic order parameter, $\eta$ on the superconducting transition temperature T$_c$ in HoMo$_6$Se$_8$ superconductor is studied using Eq. (35). As can be seen, the superconducting transition temperature T$_c$ decreases to zero when the magnetic order parameter $\eta$ increases. This could be due to the formation of the interaction between spins of localized electrons and the magnetic moments. The coupling of the localized electrons in the specimen became strong and act as a magnetic impurity interacting with the electric charge and, hence, scatter both electrons of a cooper pair, perhaps, causing cooper pair breaking in the superconductor depending on the magnetic structure and electronic bands.
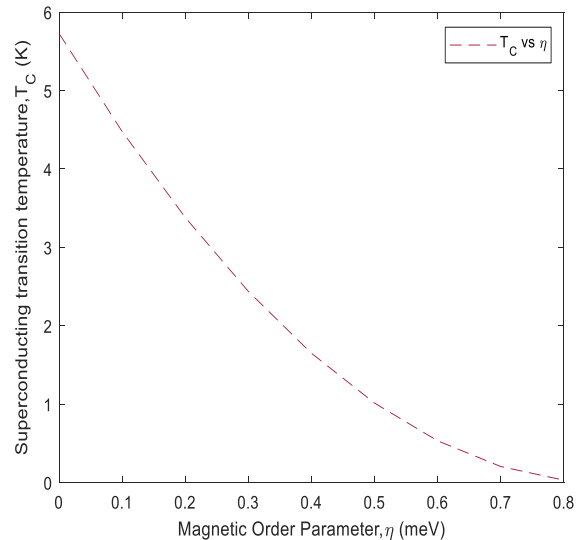
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**Figure 3. Transition temperature Tc versus magnetic order parameter $\eta$ for HoMo$_6$Se$_8$.**

*Correlation between Conduction and localized electrons*

Considering the equation of motion for the localized electrons in addition and employing double time temperature-dependent Green's function technique, we get the following two equations:

$$(\omega - \varepsilon_l)\langle\langle \hat{b}_{l\uparrow}, \hat{b}^+_{l\uparrow}\rangle\rangle = 1 + \sum_{k,l,m} \alpha_{l,m} \langle\langle \hat{c}^+_{k\downarrow} \hat{c}_{k\uparrow} \hat{b}_{l\uparrow} \hat{b}_{m\uparrow}\rangle\rangle$$
(36)

and using the random phase approximation RPA higher order terms are decoupled to give

$$(\omega + \varepsilon_l)\langle\langle \hat{b}^+_{m\uparrow}, \hat{b}^+_{l\uparrow}\rangle\rangle = \frac{\Delta_l}{(\omega - \varepsilon_l)}\langle\langle \hat{b}^+_{m\uparrow}, \hat{b}^+_{l\uparrow}\rangle$$
(37)

Combining Eq. (36) and (37) and rearranging gives:

$$\langle\langle \hat{b}^+_{m\uparrow}, \hat{b}^+_{l\uparrow}\rangle\rangle = \frac{\Delta_l}{(\omega^2 - \varepsilon_l{}^2 - \Delta_l{}^2)}$$
(38)

Changing $\omega \to i\omega_n$, where $\omega_n = \frac{(2n+1)\pi}{\beta}$, and $\beta^{-1} = K_B T$ we obtain,

$$\langle\langle \hat{b}^+_{m\uparrow}, \hat{b}^+_{l\uparrow}\rangle\rangle = \frac{-\beta^2 \Delta_l}{((2n+1)^2\pi^2 + \beta^2 E_l^2)}$$
(39)

We see that the ferromagnetic order parameter $\eta$ for localized electrons is given by

$$\eta = -\frac{\alpha}{\beta}\sum_{l,m} \langle\langle \hat{b}^+_{m\uparrow}\hat{b}^+_{l\uparrow}\rangle\rangle$$
(40)

$$\eta = \alpha \sum_{l,m} \frac{\beta \Delta_l}{((2n+1)^2\pi^2 + \beta^2 E_l^2)}$$
(41)

The magnetic ordering $\eta$ is related to the correlation between local electrons. Using the trigonometric relations in Eq. (10) into Eq. (41), yields:

$$\eta = \Delta_l N(0)\alpha \sum_{l,m} \frac{\tanh(\beta/2)E_l}{E_l} d\varepsilon = \Delta_l \lambda_l \sum_{l,m} \frac{\tanh(\beta/2)E_l}{E_l} d\varepsilon$$
(42)

where $E_l{}^2 = \varepsilon_l{}^2 - \Delta_l{}^2$.

By introducing the density of states $N(0)$ assumed constant in the effective region and replacing summation by integration we get

$$\eta = \lambda_l \int_0^{\hbar\omega_b} \left(\frac{\Delta_l}{(\varepsilon_l{}^2 + \Delta_l{}^2)^{1/2}}\right) \tanh\left(\frac{\beta(\varepsilon_l{}^2 + \Delta_l{}^2)^{1/2}}{2}\right) d\varepsilon$$
(43)

which can be rewritten as

$$\frac{\eta}{\lambda_l \Delta_l} = \left(\int_0^{\hbar\omega_b} \frac{d\varepsilon}{\sqrt{\varepsilon^2 + \Delta_l{}^2}} \tan \frac{\sqrt{\varepsilon^2 + \Delta_l{}^2}}{2K_B T}\right)$$
(44)

where $\lambda = N(0)\alpha$.
Integrating Eq. (44), the magnetic transition temperature, $T_m$:

$$T_m = 1.14 \frac{\hbar\omega_b}{K_B} exp\left(\frac{\eta}{\lambda_l \Delta_l}\right)$$
(45)

The phase diagram of the magnetic transition temperature, $T_m$ versus magnetic ordering $\eta$ is plotted using Eq. (45). As seen from Fig. 4, the magnetic transition temperature $T_m$ increases with the magnetic order parameter $\eta$. On the contrary, the magnetic transition temperature $T_m$ decreases from top to dawn with the increase of the superconducting order parameter $\Delta_l$ from $\Delta_1$ to $\Delta_3$. The experimental value of the magnetic transition temperature $T_m \approx$ 0.53 K, describes that the magnetic order parameter, $\Delta_l$ in $HoMo_6Se_8$ lies below the third among the three.
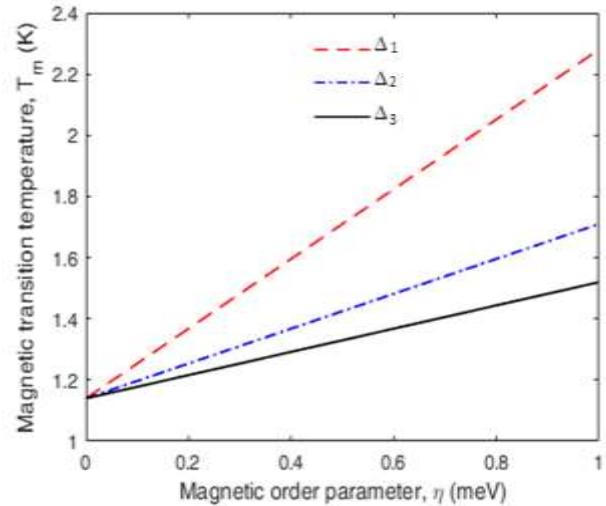


**Figure 4. The magnetic transition temperature, $T_m$ versus magnetic order parameter, $\eta$.**

Finally, by merging Figs. 3 and 4, we observed a region where both superconductivity and ferromagnetism coexist below the critical temperature, $T_c$ in $HoMo_6Se_8$ superconductor, in agreement with experimental observations (Lynn et al.,1984) and (Bennemann et al., 2008).
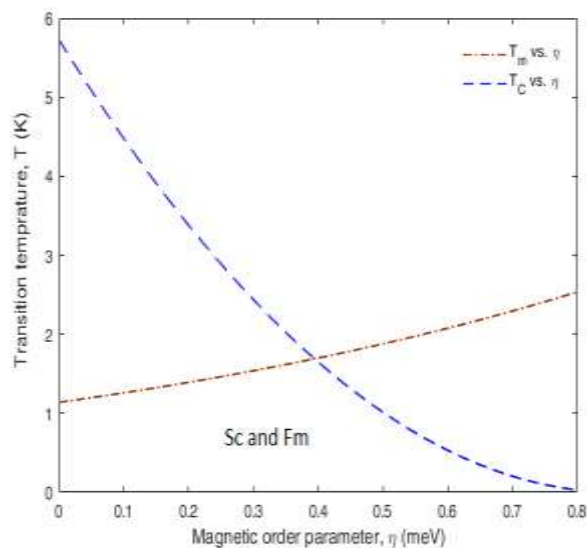
**Figure 5. Coexistence of superconductivity and ferromagnetism in HoMo$_6$Se$_8$.**

## CONCLUSIONS

In the present work, we have demonstrated the interplay between singlet superconductivity and ferromagnetism in HoMo$_6$Se$_8$ using the model Hamiltonian with pairing interaction and scattering of cooper pairs by ferromagnetically ordered localized pairs. Numerical calculations were done for the expressions of interdependent portions and parameters. Our results show that the superconducting critical transition temperature T$_C$ decreases with the increase of the magnetic order parameter, $\Delta$. On the other hand, the superconducting order parameter, shows decrement with the enhancement of T$_C$. This is perhaps due to the weakening of the binding energy as temperature approaches its critical value 5 K. We have also observed the enhancement of magnetic transition temperature T$_m$ and decrement in superconducting transition temperature, T$_C$ with the increase of magnetic order parameter $\eta$. As shown in Fig. 5, there is a lower temperature region where both superconductivity and magnetic ordering coexist. Finally, our theoretical predictions demonstrate broad agreement with existing experimental results (Bulaevskii, et al. 1985).

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