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Spectral problem for the Laplacian and a selfadjoint nonlinear elliptic boundary value problem

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ABSTRACT. In this paper, we present some connections between the spectral problem,

$$\begin{cases} -\Delta u(x) = \lambda_1 u(x) & \text{in } \Omega, \\ u(x) = 0 & \text{on } \partial\Omega \end{cases}$$

and selfadjoint boundary value problem,

$$\begin{cases} \Delta u(x) - \lambda_1 u(x) + g(x, u(x)) = h(x) & \text{in } \Omega, \\ u(x) = 0 & \text{on } \partial\Omega, \end{cases}$$

where λ_1 is the smallest eigenvalue of $-\Delta$, $\Omega \subseteq \mathbb{R}^n$ is a bounded domain, $h \in L^2(\Omega)$ and the nonlinear function g is a Caratheodory function satisfying a growth condition. We initially investigate the existence of solutions for the spectral problem by considering the selfadjoint boundary value problem. The selfadjoint boundary value problem is then considered for both existence and estimation results. We use degree argument in order to show that the selfadjoint boundary value problem has a solution instead of the Landesman-Lazer condition or the monotonocity assumption on the second argument of the function g.

Key words/phrases: Spectral problem, Boundary value problem, Second order elliptic partial differential operator, Selfadjoint operator, Laplace operator.

INTRODUCTION

This paper is inspired by the search of nontrivial solutions in Iannacci et al. [10] and Iannacci and Nkashama [11, 12] for the selfadjoint boundary value problem,

$$\begin{cases} \mathcal{L}u(x) + \lambda_1 u(x) + g(x, u(x)) = h(x) & \text{in } \Omega, \\ u(x) = 0 & \text{on } \partial\Omega, \end{cases}$$

where λ_1 is the smallest eigenvalue of $-\mathcal{L}$, $\Omega \subseteq \mathbb{R}^n$ is a bounded domain, $h \in L^p(\Omega)$ with $p > n$ and the nonlinear function g is a Caratheodory function that grows at most linearly.

The solvability of a boundary value problem involving the Laplace operator was investigated by a number of researchers (see, e.g., [1, 4, 6, 10, 11, 12, 13, 15, 16, 19, 21] and references therein).

Among these, Nica [19] assured that the eigenfunctions for the eigenvalue problem

$$\begin{cases} -\Delta u(x) = \lambda u(x) & \text{in } \Omega, \\ u(x) = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

form an orthonormal basis of $L^2(\Omega)$ and the corresponding eigenvalues grow without bound.

De Figueiredo and Ni [6] studied the existence of solutions for a boundary value problem,

$$\begin{cases} \mathcal{L}u(x) - \lambda_1 u + g(u(x)) = h(x) & \text{in } \Omega, \\ u(x) = 0 & \text{on } \partial\Omega, \end{cases}$$

without using the Landesman-Lazer condition under some prescribed assumptions. Iannacci and Nkashama [12] investigated this same problem by specifying $\mathcal{L}$ and provided sufficient conditions for the solvability of the semilinear two-point boundary value problem,

$$\begin{cases} u''(x) + u(x) + g(x, u(x)) = h(x), & x \in (0, \pi), \\ u(0) = u(\pi) = 0, \end{cases}$$

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in which g is not required to satisfy the Landesman-Lazer condition,

$$\int_0^\pi g_-(x)sinxdx < \int_0^\pi h(x)sinxdx < \int_0^\pi g_+(x)sinxdx,$$

where $g_\pm = \lim_{u\longrightarrow\pm\infty} g(u)$ or the monotonicity assumption. After a while, Iannacci et al. [10] generalized the main results in De Figueiredo and Ni [6], Iannacci and Nkashama[12], Gupta [7, 8] and Ward [20]. In their generalization, it is indicated that there is a solution for the selfadjoint boundary value problem,

$$\begin{cases} \mathcal{L}u(x) + \lambda_1 u(x) + g(x, u(x)) = h(x) \text{ in } \Omega, \\ u(x) = 0 \text{ on } \partial\Omega, \end{cases}$$

where $\mathcal{L}$ is a selfadjoint operator. In order to arrive at this existence result they primarily studied

$$\begin{cases} -\mathcal{L}u(x) = \lambda_1 u(x) \ \ in\ \Omega, \\ u(x) = 0 \text{ on } \partial\Omega, \end{cases}$$

under some assumptions placed on $\mathcal{L}$.

Recently, Xu and Ma [21] investigated the spectrum structure of the eigenvalue problem

$$\begin{cases} u^{(4)}(x) = \lambda u(x), \quad x \in (0, 1), \\ u(0) = u(1) = u'(0) = u'(1) = 0, \end{cases}$$

and for its application, they demonstrated the existence of solutions for the fourth order boundary value problem

$$\begin{cases} -u^{(4)}(x) + \lambda_1 u(x) + g(x, u(x)) = h(x), \\ u(0) = u(1) = u'(0) = u'(1) = 0, \end{cases}$$

at resonance, where λ_1 is the smallest eigenvalue of the corresponding linear eigenvalue problem. They provided sufficient conditions for solving the above problem, in which the nonlinearity of g does not necessarily satisfy the Landesman-Lazer type condition or the monotonicity assumption.

The main targets of this study are to:

i. investigate the solvability of the spectral problem for the Laplacian,

$$\begin{cases} -\Delta u(x) = \lambda_1 u(x) \text{ in } \Omega, \\ u(x) = 0 \text{ on } \partial\Omega \end{cases} \tag{2}$$

by taking a selfadjoint boundary value problem,

$$\begin{cases} \Delta u(x) - \lambda_1 u(x) + g(x, u(x)) = h(x) \text{ in } \Omega, \\ u(x) = 0 \text{ on } \partial\Omega \end{cases} \tag{3}$$

into consideration.

ii. examine the solvability of problem (3) without using the Landesman-Lazer condition or monotonicity assumption with respect to the second argument of the nonlinear function g, and

iii. find some estimation results by considering problem (3).

In the above boundary value problems (2) and (3):

- $\Omega \subseteq \mathbb{R}^n$ is a bounded domain with boundary $\partial\Omega$ of class $C^{1,\mu}$, $0 < \mu < 1$;
- λ_1 is the smallest eigenvalue of $-\Delta$;
- $h(x) \in L^2(\Omega)$;
- $g : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ is a Caratheodory function that grows at most linearly;
- the second-order partial differential operator Δ is the usual Laplacian.

The paper is organized as follows. In Section 2, some facts concerning the spectral problem for the Laplacian as well as important Lemmas regarding the selfadjoint boundary value problem are discussed.

In Section 3, both existence and estimation results are presented. We have shown that there is a solution for the spectral problem and selfadjoint boundary value problem in an appropriate Sobolev space. For the estimation results we considered the selfadjoint boundary value problem. In section 4, we put a precise conclusion based on the results obtained.

PRELIMINARIES

In this section, we describe some vital facts about a spectral problem for the Laplacian and a selfadjoint boundary value problem involving the Laplace operator. Some auxiliary results are also presented.

Definition 1. Let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain. The spectral problem for the Laplace operator is

$$\begin{cases} -\Delta u = \lambda f(x, u) \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \end{cases} \tag{4}$$

where $f : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ is a given function and λ is a real number. We say that λ is an eigenvalue of (4) if there exists $u \in H_0^1(\Omega)$ such that for any

$\psi \in H_0^1(\Omega)$,

$$\int_\Omega \nabla u \nabla \psi dx - \lambda \int_\Omega f(x,u)\psi = 0.$$

Furthermore, if λ is an eigenvalue of problem (4), then $u \in H_0^1(\Omega)$ given in the above definition is called the eigenfunction corresponding to λ. If $f(x,u) = u$, then (4) reduces to (1). The discreteness of the spectrum of Δ allows one to order the eigenvalues, $0 < \lambda_1 < \lambda_2 < \lambda_3 < \cdots < \lambda_n < \cdots$ monotonically and λ_1 is characterized from a variational point of view as the minimum of the Rayleigh quotient,

$$\lambda_1 = \inf_{u \in H_0^1(\Omega)} \frac{\int_\Omega |\nabla u|^2 dx}{\int_\Omega u^2 dx}.$$

Moreover, it is widely known that λ_1 is simple, all the associated eigenfunctions are multiples of each other[5, 2, 3, 9].

Theorem 1. [18] For the eigenvalue problem (1), all eigenvalues are positive.

Theorem 2. [5, 18, 2] Let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain. There exists a sequence of eigenvalues $\{\lambda_j\}$ and eigenfunctions $\{u_j\}$ for (1) such that

$$0 < \lambda_1 < \lambda_2 < \lambda_3 < \cdots$$

with $\lambda_j \longrightarrow \infty$ as $j \longrightarrow \infty$, $u_j \in H_0^1(\Omega)$ and $-\Delta u_j = \lambda_j u_j$. The family of eigenfunctions is a Hilbert basis of $L^2(\Omega)$ and is orthogonal and complete in $H_0^1(\Omega)$ equipped with the inner product associated with the seminorm.

Theorem 3. [5, 18, 2] There exists u_1 in the kernel of $-\Delta - \lambda_1 I$ such that $u_1(x) > 0$ for all $x \in \Omega$.

In other words, the eigenfunctions associated with the first eigenvalue do not vanish in Ω, and thus we can choose one that is strictly positive in Ω. The result is a consequence of the Krein-Rutman theorem; see [14]. As a rule, the eigenfunctions associated with the other eigenvalues do vanish on nodal sets and change sign.

Define a linear operator $\mathcal{L} : Q \subset L^2(\Omega) \longrightarrow L^2(\Omega)$ by

$$\mathcal{L}u := \Delta u + \lambda_1 u,$$

where $Q := H^2(\Omega) \cap H_0^1(\Omega)$. Then $\mathcal{L}$ is a selfadjoint operator and thus $L^2(\Omega)$ admits the orthogonal direct sum decomposition $L^2(\Omega) = \kappa \oplus \Re$, where κ is the one dimensional null space of $\mathcal{L}$ and $\Re$ is the range space of $\mathcal{L}$, namely,

$$\kappa = \{y(x) \in L^2(\Omega) : y(x) = su_1(x) \text{ for some } s \in \mathbb{R}\}$$
$$\Re = \{y(x) \in L^2(\Omega) : \int_\Omega y(x)u_1(x)dx = 0\}.$$

Therefore, each $u \in H_0^1(\Omega)$ has a unique decomposition:

$$u = su_1(x) + w := \nu(x) + \varphi(x),$$

where $s \in \mathbb{R}$ and $w \in \Re$ so that, with obvious notation

$$H_0^1(\Omega) = \bar{H}_0^1(\Omega) \oplus \tilde{H}_0^1(\Omega),$$

where $\bar{H}_0^1(\Omega)$ and $\tilde{H}_0^1(\Omega)$ are spaces associated with ν and φ, respectively.

Lemma 1. [10] Let $w(x) \in L^\infty(\Omega)$ be such that for a.e. $x \in \Omega$, $0 \leq w(x) \leq \lambda_2 - \lambda_1$ with $w(x)$ in a subset of Ω with positive measure, $\sigma > 0$ and $\delta = \delta(w) > 0$. Then, for all $p \in L^\infty(\Omega)$ satisfying

$$0 \leq p(x) \leq w(x) + \sigma$$

a.e. on Ω and all $u \in Q$ one has

$$\int_\Omega (\Delta u(x) - u(x) + p(x)u(x)) \times (\nu(x) - \varphi(x))dx \geq (\delta - \sigma)\|\varphi\|_{H^1}^2.$$

MAIN RESULTS

From now on

- $(\cdot,\cdot)$ denotes the L^2 inner product in Ω.
- we let
$$u(x) = \nu(x) + \varphi(x),$$
for each $u(x) \in H_0^1(\Omega)$, $\nu(x) = (\int_\Omega u(x)u_1(x)dx)u_1(x)$ with $\|u_1(x)\|_{L^2(\Omega)}^2 = 1$ and $\varphi(x) = u(x) - \nu(x)$. The functions $\nu(x)$ and $\varphi(x)$ as defined above are $L^2(\Omega)$-orthogonal.
- $u : \Omega \longrightarrow \mathbb{R}$ is defined by
$$u(x) = u^+(x) - u^-(x),$$
where
$$u^+(x) = \max\{u(x), 0\}$$
$$u^-(x) = \max\{-u(x), 0\}.$$

- $(-\Delta\varphi(x), \varphi(x)) \geq \lambda_2(\varphi(x), \varphi(x))$, λ_2 is the second eigenvalue of $-\Delta$.

The proof of Theorem 4 will be based on the following assumptions.

$(H1)$. $H_+(x)$, $H_-(x) \in L^2(\Omega)$ in which for a.e. $x \in \Omega$, $H_\pm(x)$ is a nonnegative function and $H_\pm(x) \leq \lambda_1 + \lambda_2$ with

$$\int_{u_2(x)>0} [(\lambda_1+\lambda_2) - H_+(x)](u_2(x))^2 dx + \int_{u_2(x)<0} [(\lambda_1+\lambda_2) - H_-(x)](u_2(x))^2 dx > 0 \tag{5}$$

for all $u_2(x)$ satisfying

$$\begin{cases} -\Delta u_2(x) = \lambda_2 u_2(x) \text{ in } \Omega, \\ u_2(x) = 0 \text{ on } \partial\Omega. \end{cases}$$

$(H2)$. For all $r_\pm(x) \in L^2(\Omega)$, consider

$$0 \leq r_\pm(x) \leq H_\pm(x).$$

$(H3)$. Define $z : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ by

$$z(x, u(x)) = \begin{cases} r_+(x) & \text{if } x \in \Omega \text{ and } u \geq 0, \\ r_-(x) & \text{if } x \in \Omega \text{ and } u < 0 \end{cases}$$

such that

$$0 \leq z(x, u(x)) \leq \lambda_1 + \lambda_2 \text{ with } \frac{z(x,u(x))}{2} \geq \lambda_1.$$

$(H4)$. $u(x) \in H^2(\Omega)$, satisfies

$$\begin{cases} \Delta u(x) - \lambda_1 u(x) + r_+(x)u^+(x) - r_-(x)u^-(x) = 0 \text{ in } \Omega, \\ u(x) = 0 \text{ on } \partial\Omega. \end{cases} \tag{6}$$

Existence result

Theorem 4. If $(H1)$-$(H4)$ hold, then $u(x) \in H^2(\Omega)$ is a solution of (2).

Proof. From problem (3), we have

$$\Delta u(x) - \lambda_1 u(x) + g(x, u(x)) - h(x) = 0, \text{ in } \Omega.$$

Let $z(x, u(x))u(x) = g(x, u(x)) - h(x)$, where $z(x, u(x))$ is defined in $(H3)$. Then

$$\begin{aligned} z(x,u(x))u(x) =& z(x,u(x))(u^+(x) - u^-(x)) \\ =& r_+(x)u^+(x) - r_-(x)u^-(x). \end{aligned}$$

Upon substituting $z(x, u(x))u(x)$ for $r_+(x)u^+(x) - r_-(x)u^-(x)$ in equation (6), we have

$$\begin{cases} \Delta u(x) - \lambda_1 u(x) + z(x, u(x))u(x) = 0 \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega. \end{cases}$$

This in turn leads to:

$$[\Delta u(x) - \lambda_1 u(x) + z(x, u(x))u(x)]\,[\nu(x) - \varphi(x)] = 0.$$

This implies that

$$\begin{aligned} &\int_\Omega [\Delta u(x) - \lambda_1 u(x) + z(x,u(x))u(x)] \\ &\quad \times [\nu(x) - \varphi(x)]\, dx \\ &= \int_\Omega (\nu(x)\Delta\left(\nu(x) + \varphi(x)\right) \\ &\quad - \varphi(x)\Delta\left(\nu(x) + \varphi(x)\right))dx \\ &\quad - \int_\Omega \lambda_1 \left(\nu(x) + \varphi(x)\right)\left(\nu(x) - \varphi(x)\right) dx \\ &\quad + \int_\Omega \left[z(x,u(x))\left(\nu(x) + \varphi(x)\right)\left(\nu(x) - \varphi(x)\right)\right] dx \\ &= \int_\Omega [-\varphi(x)\Delta\varphi(x) + \nu(x)\Delta\nu(x) - \lambda_1(\nu(x))^2 \\ &\quad - (z(x,u(x)) - \lambda_1)(\varphi(x))^2]dx \\ &\quad + \int_\Omega z(x,u(x))(\nu(x))^2 dx. \end{aligned}$$

Since

$$-\Delta\nu(x) = \lambda_1\nu(x) \text{ in } \Omega,$$

it follows that

$$\begin{aligned} &\int_\Omega [-\varphi(x)\Delta\varphi(x) - (z(x,u(x)) - \lambda_1)(\varphi(x))^2]dx \\ &\qquad + \int_\Omega (z(x,u(x)) - 2\lambda_1)(\nu(x))^2 dx = 0. \end{aligned}$$

Moreover, we have the information that $(-\Delta\varphi(x), \varphi(x)) \geq \lambda_2(\varphi(x), \varphi(x))$ and $z(x, u(x)) \leq \lambda_1 + \lambda_2$ with $\frac{z(x,u(x))}{2} \geq \lambda_1$ and hence

$$\begin{aligned} &\int_\Omega \left[-\varphi(x)\Delta\varphi(x) - (z(x,u(x)) - \lambda_1)(\varphi(x))^2\right] dx \\ &\quad + \int_\Omega (z(x,u(x)) - 2\lambda_1)(\nu(x))^2 dx \\ &\geq \int_\Omega [-\varphi(x)\Delta\varphi(x) - \lambda_2(\varphi(x))^2]dx \\ &\geq 0. \end{aligned}$$

From the above inequality one can see that

$$\int\limits_{\Omega}[-\varphi(x)\Delta\varphi(x)-\lambda_2(\varphi(x))^2]dx=0.$$

But, this is true if and only if $\varphi(x)=u_2(x)$. i.e.,

$$\begin{aligned}&\int\limits_{\Omega}[-\varphi(x)\Delta\varphi(x)-\lambda_2(\varphi(x))^2]dx\\&=\int\limits_{\Omega}[-u_2(x)\Delta u_2(x)-\lambda_2(u_2(x))^2]dx\\&=\int\limits_{\Omega}[\lambda_2(u_2(x))^2-\lambda_2(u_2(x))^2]dx\\&=0.\end{aligned}$$

Now, for $\varphi(x)=u_2(x)$, we get

$$\begin{aligned}&\int\limits_{\Omega}\left[-u_2\Delta u_2-(z(x,u)-\lambda_1)(u_2(x))^2\right]dx\\&\qquad+\int\limits_{\Omega}(z(x,u)-2\lambda_1)(\nu(x))^2dx=0.\end{aligned}\tag{7}$$

On the other hand,

$$\begin{aligned}&\int\limits_{\Omega}\left[-u_2(x)\Delta u_2(x)-(z(x,u)-\lambda_1)(u_2(x))^2\right]dx\\&=\int\limits_{\Omega}\left[\lambda_2(u_2(x))^2-(z(x,u)-\lambda_1)(u_2(x))^2\right]dx\\&=\int\limits_{\Omega}\left[\lambda_1+\lambda_2-z(x,u(x))\right](u_2(x))^2dx.\end{aligned}$$

Consequently (7) becomes,

$$\begin{aligned}0&=\int\limits_{\Omega}\left[\lambda_1+\lambda_2-z(x,u(x))\right](u_2(x))^2dx\\&+\int\limits_{\Omega}(z(x,u(x))-2\lambda_1)(\nu(x))^2dx\\&\geq\int\limits_{\Omega}\left[\lambda_1+\lambda_2-z(x,u(x))\right](u_2(x))^2dx\\&\geq 0.\end{aligned}$$

Furthermore,

$$\int\limits_{\Omega}\left[\lambda_1+\lambda_2-z(x,u)\right]u_2^2dx=-\int\limits_{\Omega}(z(x,u)-2\lambda_1)\nu^2dx$$

when

$$\int\limits_{\Omega}\left[\lambda_1+\lambda_2-z(x,u(x))\right](u_2(x))^2dx=0\tag{8}$$

and

$$\int\limits_{\Omega}(z(x,u(x))-2\lambda_1)(\nu(x))^2dx=0,\tag{9}$$

where

$$-\Delta\nu(x)=\lambda_1\nu(x)\ \text{ in }\Omega$$

and

$$-\Delta u_2(x)=\lambda_2u_2(x)\ \text{ in }\Omega.$$

If $z=\lambda_1+\lambda_2$ and $z=2\lambda_1$ respectively, then (8) and (9) will be zero.

If we can show that $\nu(x)=u(x)$, then we are done.

Suppose

$$\Omega_1=\{x\in\Omega:\nu(x)\neq 0\}.$$

Assume that $\Omega_1=\emptyset$. Then $u(x)=u_2(x)$. From (8), $(H2)$, $(H4)$ and the fact that $H_{\pm}$ satisfies (5), we have

$$\begin{aligned}0&=\int\limits_{\Omega}[\lambda_1+\lambda_2-z(x,u_2(x))](u_2(x))^2dx\\&=\int\limits_{u_2(x)>0}[\lambda_1+\lambda_2-r_+(x)](u_2(x))^2dx\\&+\int\limits_{u_2(x)<0}[\lambda_1+\lambda_2-r_-(x)](u_2(x))^2dx.\end{aligned}$$

Thus by $(H1)$ and $(H2)$, $u_2(x)=0$ and hence $u(x)=0=\nu(x)$.

Assume again that $\Omega_1\neq\emptyset$. Then $\nu(x)\neq 0$ and hence in (9)

$$z(x,u(x))=2\lambda_1$$

for a.e $x\in\Omega$.

In (8), $\lambda_1+\lambda_2-z(x,u(x))\neq 0$ and hence $u_2(x)=0$. This implies that

$$u(x)=\nu(x)+u_2(x)=\nu(x).$$

From the fact that

$$\begin{cases}-\Delta\nu(x)=\lambda_1\nu(x)\ \text{ in }\Omega,\\ \nu(x)=0\ \text{ on }\partial\Omega,\end{cases}$$

and $u(x)=\nu(x)$ (from the computation above), one can conclude that

$$\begin{cases}-\Delta u(x)=\lambda_1u(x)\ \text{ in }\Omega,\\ u(x)=0\ \text{ on }\partial\Omega.\end{cases}$$

$\square$

The existence of solutions for problem (3) will be guaranteed by taking the following assumptions into account.

$(H5)$. $g(x, u(x))u(x) \geq 0$ for a.e. $x \in \Omega$ and all $u \in \mathbb{R}$.

$(H6)$. For any constant $\sigma > 0$, there exist a constant $R = R(\sigma) > 0$ and a function $b = b(\sigma) \in L^{\infty}(\Omega)$ such that

$$|g(x, u(x))| \leq (w(x) + \sigma)|u(x)| + b(x),$$

for a.e. $x \in \Omega$ and all $u \in \mathbb{R}$ with $|u| \geq R$, where $w \in L^{\infty}(\Omega)$ is such that for a.e. $x \in \Omega$

$$0 \leq w \leq \lambda_2 - \lambda_1.$$

$(H7)$. $g : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ is a Caratheodory function, for any constant $r > 0$ there exists a function $q_r \in L^2(\Omega)$ such that

$$|g(x, u(x))| \leq q_r(x),$$

for a.e. $x \in \Omega$ and all $u \in \mathbb{R}$ with $|u| \leq r$.

$(H8)$. $\Delta u(x) - \lambda_1 u(x) > 0$, for all $u \in Q$.

Theorem 5. Assume that $(H5)$-$(H8)$ hold. Then (3) has at least one solution $u \in Q$ for any $h \in L^2(\Omega)$ with

$$\int_{\Omega} h(x)u_1(x)dx = 0. \tag{10}$$

Proof. Let $\delta > 0$ be associated with function w and $p \in (0, \lambda_2 - \lambda_1)$ be a fixed constant with $p < \frac{\delta}{2}$. According to the Leray-Scahuder continuation approach [17], proving that (3) has at least one solution is all about demonstrating that the possible solutions of the homotopy

$$\begin{cases} \Delta u(x) - \lambda_1 u(x) + (1-s)pu(x) + sg(x, u(x)) = sh(x) & \text{in } \Omega, \\ u(x) = 0 & \text{on } \partial\Omega \end{cases} \tag{11}$$

has a priori bound in Q.

Our claim is to show that if $u \in Q$ is a solution of the homotopy (11), then there exists a constant $\rho > 0$ independently of $s \in [0, 1)$ such that

$$\|u\|_Q < \rho. \tag{12}$$

If we assume that our claim is false, then there exist sequences $\{s_n\} \subset (0, 1)$ and $\{u_n\} \subset Q$ with $\|u_n\|_Q \geq n$ for all $n \in \mathbb{N}$ such that

$$\begin{cases} \Delta u_n(x) - \lambda_1 u_n(x) + pu_n(x) = s_n[-pu_n(x) - g(x, u_n(x)) + h(x)] & \text{in } \Omega, \\ u_n(x) = 0 & \text{on } \partial\Omega. \end{cases} \tag{13}$$

Setting $v_n = \frac{u_n}{\|u_n\|_Q}$, we have

$$\begin{cases} \Delta v_n(x) - \lambda_1 v_n(x) + pv_n(x) = s_n[-pv_n(x) - \frac{g(x,u_n(x))}{\|u_n\|_Q} + \frac{h(x)}{\|u_n\|_Q}] & \text{in } \Omega, \\ v_n(x) = 0 & \text{on } \partial\Omega. \end{cases} \tag{14}$$

Define an operator $E : Q \longrightarrow L^2(\Omega)$ by $Ev(x) = \Delta v(x) - \lambda_1 v(x) + pv(x) = s[-pv(x) - \frac{g(x,u(x))}{\|u\|_Q} + \frac{h(x)}{\|u\|_Q}]$, where E is invertible and its inverse is compact from $L^2(\Omega)$ into Q.

It follows from $(H6)$ and $(H7)$, there exists a function $c \in L^2(\Omega)$ depending only on $R = R(\delta) > 0$ such that

$$|g(x, u(x))| \leq (q(x) + \frac{\delta}{4})|u(x)| + b(x) + c(x)$$

for a.e. $x \in \Omega$ and all $u \in \mathbb{R}$. Therefore, $\frac{g(x,u_n(x))}{\|u_n\|_Q}$ is bounded in $L^2(\Omega)$. Similarly, from $(H6)$, we have

$$|g(x, u(x))| \leq (q(x) + \frac{\delta}{4})|u(x)| + b(x) \tag{15}$$

for a.e. $x \in \Omega$ and all $u \in \mathbb{R}$ with $|u| \geq R$, where R is chosen such that $\frac{b(x)}{|u|} < \frac{\delta}{4}$.

This implies that

$$s_n[pv_n(x) - \frac{g(x, u_n(x))}{\|u_n\|_Q} + \frac{h(x)}{\|u_n\|_Q}]$$

is bounded in $L^2(\Omega)$.

One can rewrite the equation in (14) as

$$v_n = E^{-1}\left(s_n[pv_n(x) - \frac{g(x, u_n(x))}{\|u_n\|_Q} + \frac{h(x)}{\|u_n\|_Q}]\right).$$

Since $E^{-1} : L^2(\Omega) \longrightarrow Q$ is compact, there exists $v \in Q$ such that

$$\begin{cases} \lim_{n \to \infty} v_n = v & \text{in } Q, \\ \|v\|_Q = 1. \end{cases}$$

Let us define $\tilde{q} : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ by

$$\tilde{q}(x, u) = \begin{cases} u^{-1}g(x, u) & \text{for } |u| \geq R \\ R^{-1}g(x, R)(\frac{u}{R}) + (1 - \frac{u}{R})w(x) & \text{for } 0 \leq u < R \\ R^{-1}g(x, -R)(\frac{u}{R}) + (1 + \frac{u}{R})w(x) & \text{for } -R < u \leq 0 \end{cases} \tag{16}$$

Then (16) together with (15) and $(H5)$ gives

$$0 \leq \tilde{q}(x, u) \leq w(x) + \frac{\delta}{2} \tag{17}$$

for a.e. $x \in \Omega$ and all $u \in \mathbb{R}$.

In addition, $\tilde{q}(x,u)u$ satisfies the Caratheodory condition and define $f : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ by

$$f(x,u) = g(x,u) - \tilde{q}(x,u)u, \tag{18}$$

for a.e. $x \in \Omega$ and all $u \in \mathbb{R}$ such that by $(H7)$

$$|f(x,u)| \leq v(x) \tag{19}$$

for some $v \in L^2(\Omega)$.

Upon substituting (18) in (11), we have

$$\begin{cases} \Delta u(x) - \lambda_1 u(x) + (1-s)pu(x) + s\tilde{q}(x,u(x))u + sf(x,u) = sh(x) \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega. \end{cases} \tag{20}$$

From the fact that $p \in (0, \lambda_2 - \lambda_1)$ with $p < \frac{\delta}{2}$ and (17), we get

$$0 \leq (1-s)p + s\tilde{q}(x,u) \leq q(x) + \frac{\delta}{2} \tag{21}$$

for a.e. $x \in \Omega$ and all $u \in \mathbb{R}$.

If $u \in Q$ is a solution of (20) for some $s \in (0,1)$ (for $s = 0$, we have a trivial solution), then by Lemma 1 and the Cauchy-Schwartz inequality, we have

$$\begin{aligned} &\int_\Omega (\nu(x) - \varphi(x))[\Delta u(x) - \lambda_1 u(x) + (1-s)pu(x) \\ &\quad + s\tilde{q}(x,u)u]dx + \int_\Omega (\nu(x) - \varphi(x)) \\ &\quad \times [sf(x,u) - sh(x)]dx \\ &\geq \frac{\delta}{2}\|\varphi(x)\|^2_{H^1} - (\|\varphi(x)\|_{L^2} + \|\nu(x)\|_{L^2}) \\ &\quad \times (\|f\|_{L^2} + \|h\|_{L^2}). \end{aligned}$$

This implies from the compact embedding of $H^1(\Omega)$ into $L^2(\Omega)$ and (19) that

$$0 \geq \frac{\delta}{2}\|\varphi(x)\|^2_{H^1} - \alpha(\|\varphi(x)\|_{H^1} + \|\nu(x)\|_{H^1}) \tag{22}$$

for some constant $\alpha > 0$.

From the inequality (22), one deduces immediately that $\tilde{\varphi}_n \longrightarrow 0$ in $H^1(\Omega)$ for $n \longrightarrow \infty$, where $v_n = \bar{\nu}_n + \tilde{\varphi}_n$. Therefore, $v_n = \bar{\nu}_n$. Since $\|v\|_Q = 1$, one is required to take

$$v = eu_1 \text{ for some } e > 0.$$

Now, using the fact that $v = 0$ on $\partial\Omega$, $v_n \longrightarrow v$ in Q for $n \longrightarrow \infty$ with $v > 0$ in Ω, we have that there exists $n_0 \in \mathbb{N}$ such that $n \geq n_0$, $v_n(x) > 0$ in Ω. So that, for $n \geq n_0$,

$$u_n(x) > 0 \text{ for all } x \in \Omega. \tag{23}$$

For $s_n \in (0,1)$, taking the inner product in $L^2(\Omega)$ of the equation in (14) with $\bar{\nu}_n$ and considering (10), we get

$$\begin{aligned} &\int_\Omega (\Delta\bar{\nu}_n(x) - \lambda_1\bar{\nu}_n(x))\,\bar{\nu}_n(x)dx + p(1+s_n) \\ &\times \int_\Omega (\bar{\nu}_n(x))^2 dx + \frac{s_n}{\|u_n(x)\|_Q}\int_\Omega g(x,u_n(x))\bar{\nu}_n(x)dx \\ &= 0 \end{aligned} \tag{24}$$

for all n sufficiently large. From $(H8)$ and $\bar{\nu}_n(x) > 0$, we have

$$\int_\Omega (\Delta\bar{\nu}_n(x) - \lambda_1\bar{\nu}_n(x))\,\bar{\nu}_n(x)dx > 0.$$

Moreover, $p(1+s_n)\int_\Omega \bar{\nu}_n(x)^2 dx > 0$.

For (24) to be true,

$$\frac{s_n}{\|u_n\|_Q}\int_\Omega g(x,u_n(x))\bar{\nu}_n(x)dx < 0$$

and hence

$$\int_\Omega g(x,u_n)\bar{\nu}_n dx < 0.$$

From (23) and $(H5)$, one can conclude a contradiction and the proof is complete. $\square$

Some estimation results

For the next theorems, we consider problem (3) with $s(x) \in L^\infty(\Omega)$.

Theorem 6. Suppose $j(x) \in L^\infty(\Omega)$ such that $0 \leq j(x) \leq \lambda_1$ and $\mu > 0$. Let

$$0 \leq s(x) \leq j(x) - \mu.$$

Then there exists a constant $\mu = \mu(j) > 0$ such that for all $u \in Q$, we have

$$\int_\Omega [\Delta u - \lambda_1 u + s(x)u(x)]\,[\nu(x) - \varphi(x)]\,dx \geq 2\mu\|\varphi\|^2_{H^1(\Omega)}.$$

Proof. Let $g(x,u) - h(x) = s(x)u(x)$ in (3). Then, in similar fashion as in Theorem 4, we have

$$\int_\Omega [\Delta u(x) - \lambda_1 u(x) + s(x)u(x)]\,[\nu(x) - \varphi(x)]\,dx$$
$$= \int_\Omega s(x)(\nu(x))^2 dx - \int_\Omega \varphi(x)\Delta\varphi(x)dx + \int_\Omega (\lambda_1 - s(x))(\varphi(x))^2 dx.$$

From the facts that $s(x) \geq 0$ and Green's first identity, we get

$$\int_\Omega [\Delta u(x) - \lambda_1 u(x) + s(x)u(x)]\,[\nu(x) - \varphi(x)]\,dx$$
$$\geq \int_\Omega -\varphi(x)\Delta\varphi(x)dx + \int_\Omega (\lambda_1 - s(x))(\varphi(x))^2 dx$$
$$= \int_\Omega |\nabla\varphi(x)|^2 dx + \int_\Omega (\lambda_1 - s(x))(\varphi(x))^2 dx.$$

But, $s(x) \leq j(x) - \mu$ implies $\lambda_1 - s(x) \geq \mu - j(x) + \lambda_1$ and hence

$$\int_\Omega [\Delta u(x) - \lambda_1 u(x) + s(x)u(x)]\,[\nu(x) - \varphi(x)]\,dx$$
$$\geq \int_\Omega |\nabla\varphi(x)|^2 dx + \int_\Omega (\lambda_1 - j(x) + \mu)(\varphi(x))^2 dx$$
$$= \int_\Omega |\nabla\varphi(x)|^2 dx + \int_\Omega (\lambda_1 - j(x))(\varphi(x))^2 dx + \int_\Omega \mu(\varphi(x))^2 dx.$$

Here one can choose a constant $\lambda_1 - j(x) \geq 0$ as μ and hence

$$\int_\Omega [\Delta u(x) - \lambda_1 u(x) + s(x)u(x)]\,[\nu(x) - \varphi(x)]\,dx$$
$$\geq \int_\Omega (\lambda_1 - j(x))(\varphi(x))^2 dx + \int_\Omega \mu(\varphi(x))^2 dx$$
$$= \int_\Omega \mu(\varphi(x))^2 dx + \int_\Omega \mu(\varphi(x))^2 dx$$
$$= 2\mu\|\varphi\|^2_{H^1(\Omega)}.$$

□

Theorem 7. If $s(x) \leq \lambda_1$ and $g(x,u) = s(x)u(x)$ in (3), then $\int_\Omega h(x)u(x)dx \leq 0$.

Proof. Rewriting the equation in (3), we have

$$0 = \Delta u(x) - \lambda_1 u(x) + s(x)u(x) - h(x) \text{ in } \Omega. \quad (25)$$

From (25), we get

$$\int_\Omega [u(x)\Delta u(x) - \lambda_1(u(x))^2 + s(x)(u(x))^2]dx - \int_\Omega h(x)u(x)dx$$
$$= -\int_\Omega |\nabla u(x)|^2 dx + \int_{\partial\Omega} u(x)\frac{\partial u(x)}{\partial\eta}dx - \int_\Omega \lambda_1(u(x))^2 dx + \int_\Omega s(x)(u(x))^2 dx - \int_\Omega h(x)u(x)dx$$
$$= \quad -\int_\Omega |\nabla u(x)|^2 dx - \int_\Omega \lambda_1(u(x))^2 dx + \int_\Omega s(x)(u(x))^2 dx - \int_\Omega h(x)u(x)dx$$
$$= -\int_\Omega |\nabla u(x)|^2 dx + \int_\Omega (s(x) - \lambda_1)(u(x))^2 dx - \int_\Omega h(x)u(x)dx$$
$$\leq -\int_\Omega h(x)u(x)dx.$$

This in turn gives $\int_\Omega h(x)u(x)dx \leq 0$ if $s(x) \leq \lambda_1$. □

For the next estimation result we will use the following assumptions.

$(H9)$ Assume that $v(x) \in Q$ solves

$$\begin{cases} \Delta u(x) - \lambda_1 u(x) + s(x)u(x) = 0 \text{ in } \Omega, \\ u(x) = 0 \text{ on } \partial\Omega \end{cases}$$

and $w(x) \in Q$ is a solution for

$$\begin{cases} \Delta u(x) + \lambda_1 u(x) + s(x)u(x) = 0 \text{ in } \Omega, \\ u(x) = 0 \text{ on } \partial\Omega. \end{cases}$$

$(H10)$ $w(x) - v(x) \geq 0$ in Ω and $h(x) \geq 0$ in Ω

Theorem 8. Let $(H9)$-$(H10)$ hold. Then $\int_\Omega v(x)w(x) \leq 0$.

Proof. By $(H9)$, we have

$$\Delta v(x) - \lambda_1 v(x) + s(x)v(x) = h(x) \tag{26}$$

and

$$\Delta w(x) + \lambda_1 w(x) + s(x)w(x) = h(x). \tag{27}$$

Multiplying (26) by $w(x)$ and (27) by $v(x)$, we get

$$w(x)\Delta v(x) - \lambda_1 v(x)w(x) + s(x)v(x)w(x) = h(x)w(x) \tag{28}$$

and

$$v(x)\Delta w(x) + \lambda_1 w(x)v(x) + s(x)v(x)w(x) = h(x)v(x). \tag{29}$$

Consequently, we obtain

$$\int_\Omega (w(x)\Delta v(x) - v(x)\Delta w(x))dx - 2\lambda_1 \int_\Omega v(x) \times w(x)dx = \int_\Omega h(x)(w(x) - v(x))dx. \tag{30}$$

However, we know from Green's second identity that

$$\int_\Omega (w(x)\Delta v(x) - v(x)\Delta w(x))dx = \int_{\partial\Omega} w(x)\frac{dv(x)}{d\eta}dx - \int_{\partial\Omega} v(x)\frac{dw(x)}{d\eta}dx = 0.$$

As a result, by $(H10)$ equation (30) becomes $\int_\Omega v(x)w(x)dx = -\frac{1}{2\lambda_1}\int_\Omega h(x)(w(x) - v(x))dx \leq 0.$ □

Theorem 9. Let $(v(x) + w(x)) \in Q$ be a solution of $\Delta u(x) - \lambda_1 u(x) + s(x)u(x) = h(x)$ in Ω. If $h(x) \geq 0$, $\lambda_1 \geq s(x)$ and $0 \leq w(x) \leq v(x)$ in Ω, then $\int_\Omega (v(x)\Delta v(x) - w(x)\Delta w(x))dx \geq 0$.

Proof. Since $v(x)+w(x)$ solves $\Delta u(x) - \lambda_1 u(x) + s(x)u(x) = h(x)$ in Ω, we have

$$\Delta(v(x) + w(x)) - \lambda_1(v(x) + w(x)) + s(x)(v(x) + w(x)) = h(x).$$

This suggests that

$$\Delta v(x) + \Delta w(x) - \lambda_1(v(x) + w(x)) + s(x)(v(x) + w(x)) = h(x). \tag{31}$$

Multiplying (31) by $v(x)$ and $w(x)$ give respectively

$$v(x)\Delta v(x) + v(x)\Delta w(x) - \lambda_1(v(x) + w(x))v(x) + s(x)(v(x) + w(x))v(x) = h(x)v(x) \tag{32}$$

and

$$w(x)\Delta v(x) + w(x)\Delta w(x) - \lambda_1(v(x) + w(x))w(x) + s(x)(v(x) + w(x))w(x) = h(x)w(x). \tag{33}$$

The above equations in (32) and (33) leads to,

$$\int_\Omega (v\Delta v - w\Delta w)dx - \int_\Omega \lambda_1((v(x))^2 - (w(x))^2)dx + \int_\Omega s(x)((v(x))^2 - (w(x))^2)dx = \int_\Omega h(v(x) - w(x))dx.$$

This is the same as

$$\int_\Omega (v\Delta v - w\Delta w)dx = \int_\Omega (\lambda_1 - s(x))((v(x))^2 - (w(x))^2)dx + \int_\Omega h(x)(v(x) - w(x))dx$$
$$= \int_\Omega (v(x) - w(x))[h(x) + (\lambda_1 - s(x)) (w(x) + v(x))]dx.$$

After imposing the assumptions $h(x) \geq 0$, $\lambda_1 \geq s(x)$ and $0 \leq w(x) \leq v(x)$ in Ω, we have

$$\int_\Omega (v(x)\Delta v(x) - w(x)\Delta w(x))dx \geq 0.$$

□

CONCLUSIONS

In this study, we took a closer look on the existence of solutions for problem (2) and (3) and some estimation results involving problem (3). The existence of solutions for problem (2) was guaranteed by taking problem (3) into account. In our existence result for problem (2), one can observe that φ must be equal to u_2. If $\varphi = au_2$ for any $a \in \mathbb{R}$, then the obtained existence result fails to be true. This is from the fact that if u satisfying $(H4)$, then $u = \nu + \varphi$, cannot be any function in $H^2(\Omega)$ with $p > n$, where ν is the function defined in Section 3 and $\varphi = au_2$ for any $a \in \mathbb{R}$. The existence of solutions for problem (3) is based on the Leray-Scahuder continuation approach. Besides this, Theorem 6 is valid whenever the newly added assumptions, namely $\lambda_1 - j(x) > 0$ and $0 \le s(x) \le j(x) - \mu$ hold. The remaining estimation results are also valid after employing their respective assumptions correctly.

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