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Zeros of a two-parameter family of harmonic quadrinomials

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ABSTRACT: In this paper, we determine the number of zeros and the zero inclusion regions of a two-parameter family of harmonic quadrinomials. We also determine a curve that separates sense-preserving and sense-reversing regions for these families of quadrinomials. Our work makes practical and effective use of the work of Wilmshurst, Khavinson, Dehmer, and also Bezout's Theorem in the plane.

Keywords/phrases: Analytic polynomials, harmonic polynomials, zero inclusion regions, quadrinomials.

Introduction

A complex-valued harmonic polynomial on a simply connected domain $\mathbb{D} \subset \mathbb{C}$ is a function that can be decomposed as $f(z) = h(z) + \overline{g(z)}$, where h and g are polynomials of degree n and m respectively see [2, Chapter 4]. In this decomposition, $h(z)$ is called analytic part and $g(z)$ is said to be co-analytic part of f.

The location and number of the zeros of analytic polynomials as well as complex-valued harmonic polynomials has been studied by many researchers(see Bshouty [5], Cluine [7], Dehmer [8], Dorff [9], Duren [10,11], Frank [13], Howell [14], Johnson [15], Lerario [20], Marden [22], Neumann [24], and Wilmshurst [27]).

In 1940 Kennedy [16] bounds the roots of analytic trinoimial equations of the form $z^n + az^k + b = 0$, where $ab \neq 0$ by asserting that the roots of this type of trinomials have certain bounds to their respective absolute values.

Also the roots of the trinomial equations can be interpreted as the equilibrium points of unit masses that are located at the vertices of two regular concentric polygons centered at the origin in $\mathbb{C}$ (see [25]). In 2012, Melman [23] determined the regions in which the zeros of analytic trinomials of the form $p(z) = z^n - \alpha z^k - 1$, with integers $n \geq 3$ and $1 \leq k \leq n-1$ with $\gcd(k, n) = 1$, and $\alpha \in \mathbb{C}$, lie. He determines the zero inclusion regions for the following cases: ***a***) for any value of $|\alpha|$; ***b***) $|\alpha| > \sigma(n, k)$; and ***c***) $|\alpha| < \sigma(n, k)$ where a separability threshold, $\sigma(n, k)$, is given by $\sigma(n, k) = \frac{n}{n-k}(\frac{n-k}{k})^{\frac{k}{n}}$. In all these three cases he told useful information on the location of the zeros of p.

Determining the number of zeros of complex-valued harmonic polynomial and locating the zeros are attracting problems in complex analysis. Let $\mathcal{Z}_f$ denote the number of zeros of f, that is, number of points $z \in \mathbb{C}$ satisfying $f(z) = 0$. For $f(z) = h(z) + \overline{g(z)}$ with $\deg h = n > m = \deg g$, we have $n \leq \mathcal{Z}_f \leq n^2$.

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The lower bound is sharp for each m and n due to the generalized argument principle. Note that the zeros of f are isolated. Therefore by applying Bezout's Theorem to the real and imaginary parts of $f(z) = 0$, the upper bound follows. This was shown by Wilmshurst [28]. For instance as illustrated by Wilmshurst in the same paper, $\left(\frac{1}{i}\right)^n Q\left(iz + \frac{1}{2}\right)$ is a polynomial with n^2 zeros where $Q(z) = z^n + (z-1)^n + i\overline{z}^n - i(\overline{z}-1)^n$. For analytic polynomial f of degree n, Fundamental theorem of Algebra stipulates that f has n total number of zeros counting with multiplicity.

Recently, Brilleslyper *et al* [3] studied on the number of zeros of harmonic trinomials of the form $p_c(z) = z^n + c\overline{z}^k - 1$ where $1 \leq k \leq n-1, n \geq 3, c \in \mathbb{R}^+$, and $\gcd(n,k) = 1$. They showed that the number of zeros of $p_c(z)$ changes as c varies and proved that the distinct number of zeros of $p_c(z)$ ranges from n to $n+2k$. Among other things, they used the argument principle for harmonic function that can be formulated as a direct generalization of the classical result for analytic functions. In the same paper,there are a list of open problems and one interesting question was what can be proven on the number of zeros of other families of harmonic polynomials. Due to this, now we move from a one-parameter trinomial to a two-parameter quadrinomials and we are interested in how the number of the zeros and location of the zeros changes as a real parameters $b, c \in \mathbb{R}$ vary.

In this paper, we first determine the zero inclusion regions of the zeros of quadrinomials of the form $q(z) = bz^k + \overline{z}^n + c\overline{z}^m + z$ where $k, m, n \in \mathbb{N}$ with $n > m$, and $b, c \in \mathbb{R}$ by considering different cases. Then we determine the possible maximum number of the zeros of these families of harmonic quadrinomials. Finally, we determine a curve that separates sense-preserving and sense-reversing region for this quadrinomial. The main results in this paper are Theorem 8, Theorem 9, Theorem 10, and Theorem 11.

This paper is organized as follows. In the preliminary section, we present some important preliminary results that will be used to proof the main results. In the next section, we state and proof main results. Theorems 8 and 9 determines the zero inclusion regions for the families of quadrinomials $q(z) = bz^k + \overline{z}^n + c\overline{z}^m + z$ where $k, m, n \in \mathbb{N}$ with $n > m$, and $b, c \in \mathbb{R}$ by taking different cases and we are interested in how the number and the location of zeros of $q(z)$ changes as b and c vary. Theorem 10 determines the maximum number of zeros for our quadrinomials. Also we look for some properties of the zeros of $q(z)$ in Theorem 11.

Preliminaries

In this section we review some important concepts and results that we will use later on to prove main results. We begin by stating the well known results and some useful definitions, theorems and lemmas.

Definition 1. A one to one complex-valued harmonic function is said to be univalent harmonic function. A *locally univalent* functions are functions that are one to one locally. That means, a function $f(z)$ is said to be locally univalent in the domain G if there is a small neighborhood around each point $z_0 \in G$, such as a smaller disk centered at z_0 and the function is one to one in that neighborhood. definition

Definition 2. A complex-valued harmonic function $f(z) = h(z) + \overline{g(z)}$ is said to be *sense-preserving* at z_0 if $J_f(z_0) > 0$ and is *sense-reversing* at z_0 if $J_f(z_0) < 0$, where $J_f(z_0)$ is the jacobian of f which is given by

$$J_f = \begin{vmatrix} u_x & v_x \\ u_y & v_y \end{vmatrix} = u_x v_y - u_y v_x.$$

If f is neither sense-preserving nor sense-reversing at z_0, then f is said to be *singular* at z_0.

Definition 3. The *dilatation* of a complex-valued harmonic function $f(z) = h(z) + \overline{g(z)}$ is defined to be

$$\omega(z) = \frac{g'(z)}{h'(z)}.$$

Theorem 1([21, Lewy's Theorem]). If f is a complex-valued harmonic function that is locally univalent in a domain $\mathbb{D} \subset \mathbb{C}$, then its Jacobian , $J_f(z)$, never vanish for all $z \in \mathbb{D}$.

As an immediate consequence of Theorem 1, a complex-valued harmonic function $f(z) = h(z) + \overline{g(z)}$ is locally univalent and sense-preserving if and only if $h'(z) \neq 0$ and $|\omega(z)| < 1$. A harmonic function $f(z) = h(z) + \overline{g(z)}$, is called sense-preserving at z_0 if the Jacobian $J_f(z) > 0$ for every z in some punctured neighborhood of z_0. We also say that f is sense-reversing if $\overline{f}$ is sense-preserving at z_0.

Recall to Argument principle for harmonic functions in [12]: Let f be a complex-valued harmonic function in a Jordan domain $\mathbb{D}$ with boundary Γ. Suppose that f is continuous in $\overline{\mathbb{D}}$ and $f \neq 0$ on Γ. Suppose that f has no singular zeros in $\mathbb{D}$, and let $N = N_+ - N_-$, where N_+ and N_- are the number zeros in sense-preserving region and sense-reversing region of f in $\mathbb{D}$, respectively. Then, $\triangle_\Gamma arg f(z) = 2\pi N$.

Definition 4. Suppose z_0 is a fixed point of a function f, that is, $f(z_0) = z_0$. The number $\lambda = |f'(z_0)|$ is called the *multiplier* of f at z_0. We classify the fixed point according to λ in the following definition.

Definition 5. A point ζ is called a critical point of a polynomial f if $f'(\zeta) = 0$. A fixed point $\zeta \in \mathbb{C}$ is *attractive, repelling* or *neutral* if, respectively, $|f'(\zeta)| < 1, |f'(\zeta)| > 1$ or $|f'(\zeta)| = 1$. A neutral fixed point is rationally neutral if $f'(\zeta)$ is a root of unity. We shall say that a fixed point ζ attracts some point $w \in \mathbb{C}$ provided that the sequence $f^k(w) = \underbrace{f(w) \circ f(w) \circ \cdots \circ f(w)}_{k-copies}$ converges to ζ.

Note that as mentioned in [6], if $deg f > 1$ and ζ is an attracting or rationally neutral fixed point, then ζ attracts some critical point of f.

Definition 6.[18] Let $f(z)$ be a polynomial such that the conditions $|f'(z_0)| = 1$ and $\overline{f(z_0)} = z_0$ are not satisfied simultaneously for any $z_0 \in \mathbb{C}$. Then $f(z)$ is said to be a *regular* polynomial.

Definition 7.[4] The roots that lie on the unit circle are referred to as *uni-modular roots.*

Definition 8. The grand orbit of a point $z \in \hat{\mathbb{C}}$, denoted by $[z]$, under a map f is defined as the set of all points w such that there exists $m, n \geq 0$ with $f^m(z) = f^n(w)$.

Remark 1. A grand orbit under a transformation T is an equivalence class of the relation

$$x \sim y \ iff \ T^p(x) = T^q(y)$$

for some $p, q > 0$.

Theorem 2([26, Descartes' Rule of Signs]). Let $p(x) = a_0x^{b_0} + a_1x^{b_1} + \cdots + a_nx^{b_n}$ denote a polynomial with nonzero real coefficients a_i, where the b_i are integers satisfying $0 \leq b_0 < b_1 < b_2 < \cdots < b_n$. Then the number of positive real zeros of $p(x)$ (counted with multiplicities) is either equal to the number of variations in sign in the sequence $a_0, \cdots, a_n$ of the coefficients or less than that by an even whole number. The number of negative zeros of p(x) (counted with multiplicities) is either equal to the number of variations in sign in the sequence of the coefficients of $p(-x)$ or less than that by an even whole number.

Note that according to Descartes's Rule of Signs, polynomial $p(x)$ has no more positive roots than it has sign changes and has no more negative roots than $p(-x)$ has sign changes.

A polynomial may not achieve the maximum allowable number of roots given by the Fundamental Theorem of Algebra, and likewise it may not achieve the maximum allowable number of positive roots given by the Rule of Signs. The upper bound can always be attained for any sign sequence of coefficients that contains no zeros, and an explicit formula for finding such a polynomial can easily be found and this polynomial can be modified to reduce the number of roots by any even number while maintaining the same sign sequence. For more information, see Anderson et al [1].

Theorem 3 ([19, Bezout's Theorem In the Plane]). Let f and g be relatively prime polynomials in the real variables x and y with real coefficients, and let $\deg h = n$ and $\deg g = m$. Then the two algebraic curves $f(x, y) = 0$ and $g(x, y) = 0$ have at most mn points in common.

Bezout's theorem is one of the most fundamental results about the degrees of polynomial surfaces and it bounds the size of the intersection of polynomial surfaces.

Theorem 4([17, Khavinson]). For a harmonic polynomial $f(z) = h(z) + \overline{g(z)}$ with real coefficients, the equation $f(z) = 0$ has at most $n^2 - n$ solutions that satisfy $(Rez)(Imz) \neq 0$ where $\deg h = n > m = \deg g$.

Theorem 5([28, Wilmshurst]). If $f(z) = h(z)+\overline{g(z)}$ is a harmonic polynomial such that $\deg h = n > m = \deg g$ and

$$\lim_{z\to\infty} f(z) = \infty,$$

then $f(z)$ has at most n^2 zeros.

The proof of Theorem 5 can also readily follows from Bezout's theorem.

Let $f(z) = h(z) + \overline{g(z)} = 0$ with $h = u + iv$ and $g = \alpha + i\beta$. Then

$$f(z) = (u(x, y) + \alpha(x, y)) + i\,(v(x, y) - \beta(x, y))$$
$$= 0 + i0.$$

Now define $\eta(x, y) := u(x, y) + \alpha(x, y)$ and $\zeta(x, y) := v(x, y) - \beta(x, y)$. Then we do have a homogeneous polynomial equations equation

$$\begin{cases} \eta(x, y) = 0 \\ \zeta(x, y) = 0. \end{cases}$$

Here we know that $\deg\eta = n$ and $\deg\zeta = n$. Therefore by Bezout's Theorem the maximum number of zeros of η and ζ in common is n^2.

The following result is stated and peoved in [18, Theorem 1].

Theorem 6. Let $q(z) = z - \overline{p(z)}$, where $p(z)$ is an analytic polynomial with $\deg p = n > 1$. Then $\mathcal{Z}_q \leq 3n - 2$.

Theorem 7([8, Dehmer, M.]). Let $f(z) = a_n z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$, $a_n \neq 0$ be a complex polynomial. All zeros of $f(z)$ lie in the closed disc $K(0, \max(1, \delta))$, where $M := \max\{|\frac{a_j}{a_n}|\}$ $\forall j = 0, 1, 2, \cdots , n - 1$ and $\delta \neq 1$ denotes the positive root of the equation $z^{n+1} - (1 + M)z^n + M = 0$.

Note that this theorem is also another classical result for the location of the zeros of analytic complex polynomial which depends on algebraic equation's positive root. Descartes' Rule of Signs plays a great role in the proof as shown by Dehmer.

Lemma 1. A complex valued harmonic function $f(z) = h(z) + \overline{g(z)}$ is analytic if and only if $f_{\overline{z}} = 0$.

Main results

Under this section, we determine the possible maximum number of zeros and bound all the zeros of complex valued harmonic quadrinomials in a closed disk and come up with a certain conclusion. We find the maximum number of the zeros of of a two-parameter family of harmonic quadrinomials $q(z) = bz^k + \overline{z}^n + c\overline{z}^m + z$, we derive a locations for the zeros and we will say that location of zeros changes as a parameter in co-analytic part varies without any restriction to non-zero coefficients in analytic part. For this families of quadrinomials, also we determine the curve $\Gamma_{b,c}$ which separates zeros in sense-preserving region from the zeros in sense-reversing region.

On the location of the zeros of $q(z)$

In this section we find an upper bound on the moduli of all the zeros of complex-valued harmonic quadrinomial $q(z) = bz^k + \overline{z}^n + c\overline{z}^m + z$, where $k, m, n \in \mathbb{N}$ with $n > m$, and $b, c \in \mathbb{R}$. The following theorem derives a closed disk in which all zeros are included and hence it bounds the moduli of the zeros of this families of the complex-valued harmonic quadrinomials.

Theorem 8. Let $q(z) = bz^k + \overline{z}^n + c\overline{z}^m + z$ where $k, m, n \in \mathbb{N}$ with $n > m$ and $b, c \in \mathbb{R} \setminus \{0\}$. Let $|c| > 1$ and $k > n$. If $\delta_1 \neq 1$ is the positive real root of the equation

$$|b|x^{k+1} - (|b| + |c|)x^k + |c| = 0,$$

then all the zeros of $q(z)$ lie in the closed disk $D(0; R)$ where $R = \max\{1, \delta_1\}$.

Proof. Consider $q(z) = bz^k + \overline{z}^n + c\overline{z}^m + z$ with given conditions. This quadrinomial can be rewritten as

$$q(z) = \left(bz^k + z\right) + \overline{(z^n + cz^m)}.$$

Since both $bz^k + z$ and $z^n + cz^m$ are analytic complex-valued polynomials, $q(z)$ is a harmonic polynomial. Note that b is a parameter in analytic part and c is a parameter in co-analytic part of this quadrinomial. Therefore using the triangle inequality, we have

$$|q(z)| = |\left(bz^k + z\right) + \overline{(z^n + cz^m)}| \geq \left| |bz^k| - \left[\left| z + \overline{(z^n + cz^m)} \right| \right] \right|.$$

Since

$$\left| z + \overline{(z^n + cz^m)} \right| \leq |z| + \left| \overline{(z^n + cz^m)} \right|$$

and the modulus of the conjugate of any complex number equals with its modulus, the direct forward calculation gives us:

$$|q(z)| \geq |b| \left[|z|^k - \left(\frac{1}{|b|}|z| + \frac{1}{|b|}|z|^n + \frac{|c|}{|b|}|z|^m \right) \right].$$

Then we have

$$|q(z)| \geq |b| \left[|z|^k - \frac{|c|}{|b|} (|z| + |z|^n + |z|^m) \right].$$

Since also we have assumed that $k > n > m$, we have

$$\begin{aligned} |q(z)| &\geq |b| \left[|z|^k - \frac{|c|}{|b|} \left(|z|^k + |z|^{k-1} + \cdots + 1 \right) \right] \\ &= |b| \left[|z^k| - \frac{\frac{|c|}{|b|} \left(|z|^k - 1 \right)}{|z| - 1} \right] \\ &= |b| \left[\frac{|z|^{k+1} - \left(1 + \frac{|c|}{|b|} \right) |z|^k + \frac{|c|}{|b|}}{|z| - 1} \right]. \end{aligned}$$

Descartes' Rule of Signs gives us that the function $x^{n+1} - \left(1 + \frac{|c|}{|b|}\right) x^n + \frac{|c|}{|b|}$ has exactly two distinct positive zeros, since $x = 1$ is a root with multiplicity one [26]. We then conclude that $|q(z)| \neq 0 \ \forall z \in \mathbb{C} \setminus D(0, R)$. Hence all the zeros of $q(z)$ lie in the closed disk $D(0, R)$.

Theorem 9. Let $q(z) = bz^k + \overline{z}^n + c\overline{z}^m + z$ where $k, m, n \in \mathbb{N}$ with $n > m$ and $b, c \in \mathbb{R} \setminus \{0\}$. Let $|c| \leq 1$ and $k > n$. If $\delta_2 \neq 1$ is the positive real root of the equation

$$|b|x^{k+1} - (|b| + 1)x^k + 1 = 0,$$

then all the zeros of $q(z)$ lie in the closed disk $D(0; R)$ where $R = \max\{1, \delta_2\}$.

Proof. If $|c| \leq 1$, then $\max(\frac{1}{|b|}, \frac{|c|}{|b|}) = \frac{1}{|b|}$. Since $\delta_2 \neq 1$ is a positive real root of

$$|b|x^{k+1} - (|b| + 1)x^k + 1 = 0,$$

it is also a positive real root of $x^{k+1} - (1 + \frac{1}{|b|})x^k + \frac{1}{|b|} = 0$. Thus we have for a complex-valued harmonic polynomial $q(z) = bz^k + \overline{z}^n + c\overline{z}^m + z$ with $k > n$, $\delta_2 \neq 1$ is a positive real root of

$x^{k+1}-(1+\frac{1}{|b|})x^k+\frac{1}{|b|}=0$. Then we have

$$\begin{aligned}|q(z)| &= |bz^k+\overline{z}^n+c\overline{z}^m+z|\\ &\geq \left||bz^k|-|z+\overline{z}^n+c\overline{z}^m|\right|\end{aligned}$$

By triangle inequality, we then have

$$\begin{aligned}|q(z)| &\geq |b|\left[|z|^k-\left(\frac{1}{|b|}|z|+\frac{1}{|b|}|z|^n+\frac{|c|}{|b|}|z|^m\right)\right]\\ &\geq |b|\left[|z|^k-\frac{1}{|b|}\left(|z|+|z|^n+|z|^m\right)\right]\end{aligned}$$

Since also we have assumed that $k>n>m$, we have

$$\begin{aligned}|q(z)| &\geq |b|\left[|z|^k-\frac{1}{|b|}\left(|z|^k+|z|^{k-1}+\cdots+1\right)\right]\\ &= |b|\left[|z^k|-\frac{\frac{1}{|b|}\left(|z|^k-1\right)}{|z|-1}\right]\\ &= |b|\left[\frac{|z|^{k+1}-\left(1+\frac{1}{|b|}\right)|z|^k+\frac{1}{|b|}}{|z|-1}\right].\end{aligned}$$

Descartes' Rule of Signs gives us that the function $x^{n+1}-(1+\frac{1}{|b|})x^n+\frac{1}{|b|}$ has exactly two distinct positive zeros, since $x=1$ is a root with multiplicity one [26]. We then conclude that $|q(z)|\neq 0\ \forall z\in\mathbb{C}\setminus D(0,R)$. Hence all the zeros of $q(z)$ lie in the closed disk $D(0,R)$.

Remark 2. For a non-zero real numbers b and c, the location of the zeros of quadrinomial

$$q(z)=bz^k+\overline{z}^n+c\overline{z}^m+z$$

varies as the coefficient of co-analytic part varies.

On the number of the zeros of $q(z)$

In this section we find an upper bound for the possible number of the zeros of quadrinomial $q(z)=bz^k+\overline{z}^n+c\overline{z}^m+z$, where $k,m,n\in\mathbb{N}$ with $n>m$, and $b,c\in\mathbb{R}$.

Now consider the harmonic quadrinomial $q(z)=bz^k+\overline{z}^n+c\overline{z}^m+z$.

This function can be rewritten as

$$q(z)=z-\overline{(-b\overline{z}^k-z^n-cz^m)}.$$

Then Theorem 5 and Theorem 6 play a great role to arrive at the following theorem.

Theorem 10. Let $q(z)=bz^k+\overline{z}^n+c\overline{z}^m+z$ where $k,m,n\in\mathbb{N}$ with $k>n>m$, $b,c\in\mathbb{R}$ and $\mathcal{Z}_q$ denotes the number of the zeros of $q(z)$. Then

$$\mathcal{Z}_q\leq\begin{cases}3n-2,\ \mathit{if}\ b=0\\ k^2,\ \mathit{if}\ b\neq 0\ \mathit{and}\ n=k-1\\ n(n-1)+3k-2,\ \mathit{if}\ b\neq 0\\ \mathit{and}\ n<k-1\end{cases}.$$

We prove this theorem in two different cases and our prove bounds the number of zeros of quadrinomial from above. Let us consider the following lemmas in order to achieve the goal:

The case for $b=0$

Lemma 2. A complex valued harmonic function $b\overline{z}^k+z^n+cz^m$ is analytic if and only if $b=0$.

Proof. By Lemma 1 , the function $b\overline{z}^k+z^n+cz^m$ is analytic if and only if

$$\frac{\partial}{\partial\overline{z}}(b\overline{z}^k+z^n+cz^m)=0.$$

But this is true if and only if $bk\overline{z}^{k-1}=0$ which directly implies that $b=0$.

Note that if $b=0$, then we have $q(z)=z+\overline{z^n+cz^m}=z-\overline{p(z)}$ where $p(z)=-z^n-cz^m$. Since $degp=n$, we have the following lemmas and we can prove them directly by the same procedure as in [18]. Also under this case, note that a harmonic function $\overline{z}^n+c\overline{z}^m+z$ where $m,n\in\mathbb{N}$ with $n>m$, and $c\in\mathbb{R}$ at a point z_0, is sense-preserving if and only if

$$2\mathrm{Re}z^{n-m}<\frac{|z|^{2(1-m)}}{cmn}-\frac{n|z|^{2(n-1)}}{cm}-\frac{cm}{n},$$

sense-reversing if and only if

$$2\mathrm{Re}z^{n-m}>\frac{|z|^{2(1-m)}}{cmn}-\frac{n|z|^{2(n-1)}}{cm}-\frac{cm}{n},$$

and singular if and only if

$$2\mathrm{Re}z^{n-m}=\frac{|z|^{2(1-m)}}{cmn}-\frac{n|z|^{2(n-1)}}{cm}-\frac{cm}{n}.$$

Lemma 3. The cardinality of the set of points

$$\{z \in \mathbb{C} : z = \overline{-z^n - cz^m}\} \bigcap$$
$$\{z \in \mathbb{C} : |nz^{n-1} + cmz^{m-1}| \leq 1\}$$

is at most $n-1$ where $m, n \in \mathbb{N}$ with $n > m$, and $c \in \mathbb{R}$.

Proof. Let $G(z) := \overline{-\overline{z}^n - c\overline{z}^m}$; where $g(z) = -z^n - cz^m$. Here $G(z)$ is analytic polynomial of degree n^2. If $|g'(z_0)| = 1$ and $g(z_0) = z_0$, then $G'(z_o) = 1$. All points in the set

$$\{z \in \mathbb{C} : z = -\overline{z}^n - c\overline{z}^m\} \bigcap$$
$$\{z \in \mathbb{C} : |nz^{n-1} + cmz^{m-1}| \leq 1\}$$

are fixed points of $G(z)$, which are either attracting or rationally neutral. So each of them attracts a critical point of G. As it was proved by Khavinson [18], if $-\overline{z_0}^n - c\overline{z_0}^m$ and $c \in \mathbb{C}$, then $(-\overline{z}^n - c\overline{z}^m)^k \to z_0$ if and only if $G^k(z) \to z_0$. If $G'(c) = 0$, then there are atleast $n+1$ critical points of G, counted with multiplicities, which all belongs to the same grand orbit under $\overline{g(z)}$. Furthermore, if $G'(c) = 0, g(z_0) = \overline{z_0}$ and $G^k(c) \to z_0$, then there are $n+1$ critical points of G, counted with multiplicities, all attracted to z_0 under the iteration of G. Each point in the set

$$\{z \in \mathbb{C} : z = -\overline{z}^n - c\overline{z}^m\} \bigcap$$
$$\{z \in \mathbb{C} : |nz^{n-1} + cmz^{m-1}| \leq 1\}$$

attracts a critical point of G, but then it attracts $n+1$ of them. Since different fixed points attract a disjoint set of critical points and the degree of $G(z)$ is n^2, the total number of its critical points counted with multiplicities is $n^2 - 1$.

Lemma 4. If $\overline{z}^n + c\overline{z}^m$ is regular polynomial, then the cardinality of the set

$$\{z \in \mathbb{C} : z = \overline{-z^n - cz^m}\} \bigcap$$
$$\{z \in \mathbb{C} : |nz^{n-1} + cmz^{m-1}| > 1\}$$

is at most $2n-1$ where $m, n \in \mathbb{N}$ with $n > m$, and $c \in \mathbb{R}$.

Proof. Rewrite $q(z) := z - \overline{g(z)}$, where $g(z) = -z^n - cz^m$. Let Γ_+ be a region where $z - \overline{g(z)}$ is sense-preserving and Γ_- be a region where $z - \overline{g(z)}$ is sense-reversing. Then Γ_+ and Γ_- are separated by $\partial\Gamma_+$. Moreover, make Γ_-^0 compact by intersecting Γ_- with a large disk $D(0, R)$ chosen so that $\partial\Gamma_+ \subset D(0, R)$, all zeros of $z - \overline{g(z)}$ are in $D(0, R)$ and the argument change of $z - \overline{g(z)}$ along the circle $C(0, R)$ is $-n$. Then by argument principle and lemma 3,

$$\triangle_{\partial\Gamma_+}\left(z - \overline{g(z)}\right) \leq 2\pi(n-1).$$

Hence,

$$\triangle_{C(0,R)} - \triangle_{\partial\Gamma_+} \geq -2\pi(2n-1).$$

Since $C(0, R) - \partial\Gamma_+$ is the oriented boundary of Γ_-^0, argument principle means that $\frac{-1}{2\pi}\left[\triangle_{C(0,R)} - \triangle_{\partial\Gamma_+}\right]$ is the number of zeros of $z - \overline{g(z)}$ in Γ_-. This proves our lemma.

Now by using Lemmas 3 and 4, we have

$$\mathscr{Z}_q \leq (n-1) + (2n-1) = 3n-2$$

for $b = 0$.

The case for $b \neq 0$

If we consider our quadrinomial $q(z) = bz^k + \overline{z}^n + c\overline{z}^m + z$, our aim was bounding the possible maximum number of roots of this aforementioned complex-valued harmonic polynomial. Considering first the case $n = k-1$. We have the following:

$$q(z) = bz^k + \overline{z}^n + c\overline{z}^m + z = 0$$
$$\Rightarrow bz^k + \overline{z}^{k-1} + c\overline{z}^m + z = 0.$$

Also we know that $k > n > m$. Therefore, the possible maximum power of analytic part of this complex-valued harmonic polynomial is k and that of co-analytic part is $k-1$. We are about equating this quadrinomial with $0 = 0+0i$. Again if we observe the real and imaginary part of the quadrinomial, their degree is not more than k. Let us have the real part of the quadrinomial to be $\alpha(x,y)$ and its imaginary part is $\beta(x,y)$. Then we have

$$\alpha(x,y)=0 \textit{ and } \beta(x,y)=0.$$

By Bezout's Theorem $\alpha(x,y)$ and $\beta(x,y)$ have at most k^2 solutions in common. Furthermore,

$$\lim_{z\to\infty}(bz^k+\overline{z}^n+c\overline{z}^m+z)=\infty.$$

Therefore, if $n=k-1$ and $b\neq 0$, the possible maximum number of the roots of quadrinomial $q(z)$ is k^2. That means, $\mathcal{Z}_q\leq k^2$.
On the last, if we consider the case for $n<k-1$ and $b\neq 0$, then q has at most $m(m-1)$ distinct zeros whenever $Re(z)Im(z)\neq 0$ and it has at most $3n-2$ distinct zeros whenever $Re(z)Im(z)=0$. Hence it follows that

$$\mathcal{Z}_q\leq n(n-1)+3k-2.$$

This finishes the proof of Theorem 10.

Further analysis for $n=k>m=1$

Under this section, we determine a curve that separates sense-preserving and sense-reversing region for the family of quadrinomials of the form $q(z)=bz^k+\overline{z}^n+c\overline{z}^m+z$, where $k,m,n\in\mathbb{N}$ with $n=k>m=1$, and $b,c\in\mathbb{R}$.

Note that a quadrinomial $q(z)=bz^k+\overline{z}^n+c\overline{z}^m+z$ where $k,m,n\in\mathbb{N}$ with $k,n>m$ and $b,c\in\mathbb{R}^+$ is a locally univalent and sense-preserving if and only if

$$|z|\neq\left(\frac{1}{kb}\right)^{\frac{1}{k-1}}$$

and its dilatation function satisfies

$$|\omega(z)|=\frac{|nz^{n-1}+cmz^{m-1}|}{|bkz^{k-1}+1|}<1.$$

Therefore a curve $\omega(z)=1$ separates the zeros in a sense-preserving region from the zeros in sense-reversing region and this region is determined by

$$|nz^{n-1}+cmz^{m-1}|=|bkz^{k-1}+1|.$$

Theorem 11. Let $q(z)=bz^k+\overline{z}^n+c\overline{z}^m+z$ where $k,m,n\in\mathbb{N}$ such that $k,n>m=1$ and $b,c\in\mathbb{R}$ with $b\neq\pm1$. Suppose $n=k$ and $z^k\overline{z}$ be a pure imaginary number. Then $|\omega(z)|=1$ if and only if

$$|z|=\left(\frac{1}{k}\right)^{\frac{1}{k-1}}\left(\frac{c^2-1}{b^2-1}\right)^{\frac{1}{2k-2}}$$

where $\omega(z)$ is a dilatation function of a quadrinomial $q(z)$.

Proof. By definition, $|\omega(z)|=1$ iff

$$\frac{|(z^n+cz^m)'|}{|(bz^k+z)'|}=1.$$

After a forward calculations we arrive at $|\omega(z)|=1$ iff

$$n^2|z|^{2n}+c^2m^2|z|^{2m}-b^2k^2|z|^{2k}-|z|^2$$
$$=2bk\mathrm{Re}(z^k\overline{z})-2cmn\mathrm{Re}(z^n\overline{z}^m).$$

But here we have assumed that $z^k\overline{z}$ is a pure imaginary number. As a result we do have the following.
$|\omega(z)|=1$

$$\Leftrightarrow n^2|z|^{2n}+c^2m^2|z|^{2m}-b^2k^2|z|^{2k}-|z|^2=0.$$

$$\Leftrightarrow k^2|z|^{2k-2}+c^2-b^2k^2|z|^{2k-2}=1.$$

$$\Leftrightarrow |z|^{2k-2}=\left[\frac{1}{k^2}\left(\frac{c^2-1}{b^2-1}\right)\right].$$

$$\Leftrightarrow |z|=\left[\frac{1}{k^2}\left(\frac{c^2-1}{b^2-1}\right)\right]^{\frac{1}{2k-2}}.$$

Now let us denote

$$\mathcal{M}_{b,c}:=\left(\frac{c^2-1}{k^2(b^2-1)}\right)^{\frac{1}{2k-2}}$$

and let us introduce the following as definition.

Definition 9. The roots of the quadrinomial $q(z)=bz^k+\overline{z}^n+c\overline{z}^m+z$ that lie on the circle of radius $\mathcal{M}_{b,c}$ are said to be the $\mathcal{M}_{b,c}$-*modular* roots.

Conclusion and Discussion

In this paper we have found the maximum number of the zeros of of a two-parameter family of harmonic quadrinomials $q(z) = bz^k + \overline{z}^n + c\overline{z}^m + z$ where $k, m, n \in \mathbb{N}$ with $k > n > m$, and $b, c \in \mathbb{R}$ to be $3n - 2$ if $b = 0$ and is $n^2 - n + 3k - 2$ if $b \neq 0$. Also we have derived location for the zeros and the result shows that the location of the zeros also changes as a parameter in co-analytic part varies without any restriction to non-zero coefficients in analytic part. For this families of quadrinomials, also we have determined the curve

$$\Gamma_{b,c} = \left\{ z \in \mathbb{C} : |z| = \mathcal{M}_{b,c} = \left(\frac{c^2 - 1}{k^2(b^2 - 1)} \right)^{\frac{1}{2k-2}} \right\}$$

which separates zeros in sense-preserving region from the zeros in sense-reversing region by considering the relation $k = n > m = 1$.

For further investigation we have the following to be considered:

(1) How many roots of $q(z)$ are $\mathcal{M}_{b,c}$-*modular*? What can be said on the number of zeros inside and outside of the circle $\Gamma_{b,c} = \{z \in \mathbb{C} : |z| = \mathcal{M}_{b,c}\}$?

(2) What is image of the circle

$$\Gamma_{b,c} = \{z \in \mathbb{C} : |z| = \mathcal{M}_{b,c}\}$$

under the quadrinomial $q(z) = bz^k + \overline{z}^n + c\overline{z}^m + z$?

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Declaration of Interest of Statement

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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