SINET: Ethiop. J. Sci., 45(1): 97–104, 2022 ISSN: 0379–2897 (PRINT)
© College of Natural and Computational Sciences, Addis Ababa University, 2022
eISSN: 2520–7997

Date received: December 21, 2021; Date revised: March 28, 2022; Date accepted: April 06, 2022
DOI: https://dx.doi.org/10.4314/sinet.v45i1.8

Weak idempotent rings

Dereje Wasihun[1], Tilahun Abebaw[2], Yibeltal Yitayew[2] and Venkateswarlu Kolluru[3*]

[1] Department of Mathematics, Addis Ababa Science and Technology University, Addis Ababa, Ethiopia
[2]Department of Mathematics, Addis Ababa University, Addis Ababa, Ethiopia.
[3] Department of Mathematics, Dr.L.Bullayya College,Visakhapatnam, India.
E-mail: drkvenkateswarlu@gmail.com

ABSTRACT: In this paper is to introduce the notion of weak idempotent rings as a generalization of Boolean like rings. We obtain many formal properties of the class of weak idempotent rings and furnish certain examples of the class of weak idempotent rings. Furthermore, we obtain the properties of completely prime ideal and left and right completely primary ideals of weak idempotent rings.

Key words/phrases: Weak idempotent ring, completely prime ideal, one sided primary ideal, local ring

INTRODUCTION

Many ring theoretic generalizations of Boolean rings have come into light quite for some time; Boolean like rings by Foster (1946), associate rings of Sussman (1958) and p_1 and p_2 rings by Subrahmanyam (1960) are a few of them. Among those generalizations, Boolean like rings arise naturally from general ring duality considerations and preserve many formal properties of the Boolean rings. A Boolean like ring is a commutative ring with unity and is of characteristic 2 in which $ab(1+a)(1+b)=0$ for all elements a and b in the ring. Later Swaminathan (1982) in his dissertation work made an extensive study on Boolean like rings and established many of the new properties that are true in the class of Boolean rings. Swaminathan (1982) also characterized submaximal ideals of Boolean like rings and he has established direct product decomposition theorems, characterization of injective and projective objects in the category of all Boolean like rings. In fact Swaminathan (1982) has constructed a general Boolean like ring by using the method of synthesis which is an improvisation to that of the synthesis adopted by Foster (1946) and Harary (1950).

We consider the fundamental concepts that are studied recently, which are wider classes to that of the class of Boolean Like rings. An element $\boldsymbol{a}$ of a ring $\boldsymbol{R}$ is nil-clean, if $\boldsymbol{a} = \boldsymbol{e} + \boldsymbol{n},$ where $\boldsymbol{e}^2 = \boldsymbol{e} \in \boldsymbol{R}$ and $\boldsymbol{n}$ is a nilpotent element of R; if further $en = ne$, the element $\boldsymbol{a}$ is called strongly nil-clean. A ring $\boldsymbol{R}$ is called nil-clean (resp., strongly nil-clean) if each of its elements are nil-clean (resp., strongly nil-clean), (see Koşan *et al.* (2016)). Calugarean (2015) studied about UU rings. A ring R is called a UU ring if all its units are unipotent that is, 1 +N(R) = U(R), where N(R) is the set of all nilpotent elements of R and U(R) is the set of the units of R. A ring R is called periodic if, for every $\boldsymbol{a}$ ∈ R, there exist distinct positive integers m and n such that $\boldsymbol{a}^m = \boldsymbol{a}^n$, (see Cui and Danchev (2020)).

The square of a nilpotent element need not be zero in Nil Clean rings, UU rings and periodic rings and $1+\boldsymbol{e}$ need not be idempotent element for any idempotent element $\boldsymbol{e}$ in these rings. For any element $\boldsymbol{a}$ in a Nil Clean ring and a UU ring, $\boldsymbol{a}^2$ need not be an idempotent element**.** The above said properties do hold in Boolean like rings. In Boolean like rings, the conditions $a^4 = a^2$ for every a in the ring and being characteristic 2 play major roles. Keeping this in view, it is a quite natural to ask whether the class of Boolean like rings can be extended to a new class, which may be a subclass to the class of Nil clean rings, UU rings and Periodic rings and preserve many of the decent properties of Boolean like rings. In that direction we are motivated to attempt for a larger class of Boolean like rings and the answer is affirmative.

In this work, we introduce the notion of Weak Idempotent rings, furnish certain examples of weak idempotent rings and also we establish the

*Author to whom correspondence should be addressed.

fundamental properties of Weak Idempotent rings. The ideas of completely prime and primary ideals in Weak Idempotent Rings with their basic properties are also considered.

The organization of the paper is as follows. In the second section, definition and examples of Weak Idempotent Rings are given, in the third section basic properties of Weak Idempotent Rings are discussed and in the fourth section the ideas of completely prime and primary ideals in Weak Idempotent Rings with their basic properties are also established.

WEAK IDEMPOTENT RINGS AND EXAMPLES

In this section, we introduce the concept of Weak Idempotent Rings. Throughout this paper the ring R stands for Weak idempotent ring. We begin with the following definition of a Weak Idempotent Ring.

Definition 1. A ring $(R,+,\cdot)$ is said to be Weak idempotent ring if R is a ring of characteristic 2 and $a^4 = a^2$ for each $a \in R$

Remark 1.

1. We call Weak idempotent ring by WIR for short.
2. It is clear that every Boolean ring is a Boolean like ring but not conversely. We substantiate this in the following example.

Example 1. The ring $(H_4,+,\star)$ with $H_4 = \{0,1,p,q\}$ and $+$ and $\star$ are defined by the following tables is a Boolean like ring, but not a Boolean ring.

$+$	0	1	p	q
0	0	1	p	q
1	1	0	q	p
p	p	q	0	1
q	q	p	1	0

Table 1

$\star$	0	1	p	q
0	0	0	0	0
1	0	1	p	q
p	0	p	0	p
q	0	q	p	1

Table 2

Observe that $p \star p = 0 \neq p$. Hence, H_4 is not a Boolean ring.

Lemma 1. Let R be a Weak idempotent ring. Then for all $a \in R$

1. $a^n = a,\ a^2$ or a^3 for any positive integer n.
2. If $0 \neq a$ is a nilpotent element, then $a^2 = 0$.
3. $a = a^2 + (a^2 + a)$, where a^2 is idempotent and $a^2 + a$ is nilpotent.

Proof. 1 and 3 are clear from the definition and for the proof of 2, let $a^3 = 0$.
Then $a^3 a = 0 \Rightarrow a^4 = 0 \Rightarrow a^2 = 0$.

Remark 2. In a WIR R, 0 is the only element that is both nilpotent and idempotent. Every element a of R is a sum of a nilpotent element and an idempotent element, but this representation is not unique as in the case of Boolean like rings. This can be substantiated in the following example.

Example 2. Let $U_2(\mathbb{Z}_2)$ be the ring of 2×2 upper triangular matrices over $\mathbb{Z}_2$ with the usual addition and multiplication of matrices.

Let $a = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \in U_2(\mathbb{Z}_2)$. Then $a^2 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$. Thus, $a = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, which shows that the above representation of an element as a sum of idempotent and nilpotent is not unique.

Remark 3. If the ring is a commutative WIR, then the representation of each element as a sum of an idempotent element and a nilpotent element is unique and we use the notation a_B for the idempotent element a^2 and a_N for the nilpotent element $a + a^2$ of the unique representation, that is, $a = a_B + a_N$.

Example 3. Let $U_2(\mathbb{Z}_2)$ be the ring of 2×2 upper triangular matrices over $\mathbb{Z}_2$ with the usual addition and multiplication of matrices. Then $a^4 = a^2$ and $a + a = 0$ for all $a \in U_2(\mathbb{Z}_2)$.

Clearly $(U_2(\mathbb{Z}_2),+,\cdot)$ is non-commutative ring (hence not a Boolean like ring) with unity $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Thus $U_2(\mathbb{Z}_2)$ is Weak idempotent ring with unity. Furthermore if we let $a = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $b = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, then $ab(a + b + ab) \neq a$

Example 4. Let $R = \mathbb{Z}_2$. Define "$+$" and "$*$" on $\overline{R} = R \times R$ by $(a,b) * (c,d) = (a + c, b + d)$ and $(a,b) * (c,d) = (bc, bd)$ for $(a,b),(c,d) \in \overline{R}$.

Then $(\overline{R},+,*)$ is a Weak idempotent ring with $(a,1)$, for any $a \in R$, as a left unity. But R has no right unity and hence R has no unity.

Furthermore, the ring $\overline{\mathrm{R}}$ is a non-commutative ring since $(1,0)*(a,1)=(0,0)\neq(1,0)=(a,1)*(1,0)$ and we also have $ab(a+b+ab)\neq ab$ for $a=(0,1)$ and $b=(1,1)$.

Thus, $(\overline{\mathrm{R}},+,\cdot)$ is a non-commutative Weak idempotent ring without unity.

Example 5.. Let $\mathrm{R}=\mathbb{Z}_2$. Define " + " and " * " on $\overline{\mathrm{R}}=\mathrm{R}\times\mathrm{R}$ by $(a,b)*(c,d)=(a+c,b+d)$ and $(a,b)*(c,d)=(ac,ad)$ for $(a,b),(c,d)\in\overline{\mathrm{R}}$.

Then $(\overline{R},+,*)$ is a Weak idempotent ring with right unity $(1,a)$, for any $a\in R$, and has no left unity.

Example 6. The quaternion ring Q over the field $\mathbb{Z}_2$ is a commutative ring with unity satisfies that $a^4=a^2$ and $a+a=0$ for all a $\in$ Q. Hence Q is a commutative WIR with unity.

The following two examples are examples of non-commutative Weak idempotent rings. The first one, Example 7, is with unity and the second one, Example 8, is without unity.

Example 7. Let

R=$\begin{bmatrix}0&0\\0&0\end{bmatrix},\begin{bmatrix}1&0\\0&1\end{bmatrix},\begin{bmatrix}0&1\\1&0\end{bmatrix},\begin{bmatrix}1&1\\0&0\end{bmatrix},\begin{bmatrix}0&0\\1&1\end{bmatrix},\begin{bmatrix}1&0\\1&0\end{bmatrix},$ $\begin{bmatrix}0&1\\0&1\end{bmatrix},\begin{bmatrix}1&1\\1&1\end{bmatrix}\}$ be a subset of a set of a 2×2 matrices over $\mathbb{Z}_2$ with the usual addition and multiplication of matrices. Then R is a non-commutative Weak idempotent ring with unity that satisfies $ab(a+b+ab)=ab$ and $a+a=0$ for all $a,b\in R$.

Example 8. Let R= $\left\{\begin{bmatrix}0&0\\0&0\end{bmatrix},\begin{bmatrix}1&0\\1&0\end{bmatrix},\begin{bmatrix}0&1\\0&1\end{bmatrix},\begin{bmatrix}1&1\\1&1\end{bmatrix}\right\}$, a subset of the ring of a 2×2 matrices over $\mathbb{Z}_2$ with the usual addition and multiplication of matrices.

Then R is a non-commutative Weak idempotent ring without unity that satisfies $ab(a+b+ab)=ab$ and $a+a=0$ for all $a,b\in R$.

BASIC PROPERTIES OF WEAK IDEMPOTENT RINGS

The following is a an immediate consequence of the definition of a Weak idempotent ring R.

Theorem 1. Every non-zero and non-unit element in a Weak idempotent ring R with unity is a zero divisor.

Proof. Let R be WIR. Then for a nonzero and non-unit element $a\in R$, $a^4=a^2$ implies $a(a^3+a)=0$.

If $(a^3+a)\neq0$, then a is a zero divisor. Otherwise, $a^3+a=0$, that is, $a(a^2+1)=0$. Since a is non-unit, $a^2+1\neq0$.

Hence, a is a zero divisor in R.

Notation.
Let R be a WIR. We denote the set of all idempotent elements of R by R_B and the set of all nilpotent elements of R by N.

Theorem 2. The set of all unit elements of a WIR R with unity is precisely $\{\,1+n: n\in N\}$.

Proof. Let R be a WIR and a be a unit element of R. Then $(1+a)^2=1+a+a+a^2=0$, as $a^2+1=0$.

Hence, $1+a$ is nilpotent and $a=1+(1+a)$. On the other hand, for any nilpotent element n in R, $(1+n)^2=1+n+n+n^2=1$.

Thus, $1+n$ is a unit element in R.

Lemma 2. If R is a local ring with unity, then the only idempotent elements of R are 0 and 1.

Proof. Let R be a local ring and $\mathrm{R}=\mathrm{A}\cup\mathrm{M}$, where A is the set of all unit elements in R and M is the maximal ideal of R and.

For every $x\in R$ such that $x^2=x$, either $x\in A$ or $x\in M$. If $x\in A$, then $x=1$. Otherwise, $x\in M$ and hence $x-1\in A$ and $x-1$ is an idempotent. so $x-1=1$ implies that $x=0$.

Hence 0 and 1 are the only idempotents of the ring R.

Remark 4. Let R be a Weak idempotent ring.

1. If R is a non-commutative, then the set of all idempotent elements $\mathrm{R_B}$ need not be a subring of R. (See Example 3).
2. Commutativity is sufficient condition for $\mathrm{R_B}$ to be a subring of R and N to be an ideal of R.
3. If I is an ideal of R, then R/I is also Weak idempotent ring.

Note. For a ring R, if R/I and I are Weak idempotent rings, then R need not be a Weak idempotent ring. For instance, consider $\mathrm{R}=\mathbb{Z}_4$, the set of all integers modulo 4, and $I=\{0,2\}$. Then R/I and I are Weak idempotent rings but R is not a WIR as the characteristic of R is4, not 2.

Theorem 3. Let R be a local WIR with unity. Then,

1. the set N of all nilpotent elements of R is the unique maximal ideal of R;
2. R is a commutative ring.

Proof. Let R be a local WIR with unity.

1. Let M be the unique maximal ideal of R. Then by Lemma 2, $R_B=\{0,1\}$.
   Let $a\notin N$. Then $a=a_N+1$. This implies $a^2=a_N^2+a_N+a_N+1=1$. Hence a is a unit element of R. That is, $a\notin\mathrm{M}$ and then $\mathrm{M}\subseteq\mathrm{N}$. Let $a\in N$. Then a is a non-unit element of R. Thus, $a\in\mathrm{M}$, since M contains all non-unit elements of R and hence $\mathrm{M}=\mathrm{N}$.

2. As N is an ideal of R, for all $a, b \in$ N, $a + b \in$ N. This implies $(a+b)^2 = 0 \Longrightarrow a^2 + ab + ba + b^2 = 0$. This implies $ab + ba = 0$ since $a^2 = 0$ and $b^2 = 0$.
   Therefore, $ab = ba$.
   Now take any two elements $a, b \in$ R. Then $ab = (a_B + a_N)(b_B + b_N) = a_B b_B + a_B b_N + a_N b_B + a_N b_N = b_B a_B + b_N a_B + b_B a_N + b_N a_N = (b_B + b_N)(a_B + a_N) = ba$, since R_B is in the center of the ring.
   Hence, R is a commutative ring.

Theorem 4. Every non-commutative Weak idempotent ring R with unity is not local.

Proof. Let R be a non-commutative ring with unity. Suppose R is a local ring. Then by Theorem 3, N is an ideal of R and for all $a, b \in N$ and $ab = ba$.

Let $c, d \in R$, where $c = c_B + c_N$ and $d = d_B + d_N$. Then $cd = (c_B + c_N)(d_B + d_N) = c_B d_B + c_B d_N + c_N d_B + c_N d_N = d_B c_B + d_B c_N + d_N c_B + d_N c_N = dc$ since c_B and d_B are in the center of the ring (either 0 or 1). Thus R is commutative and it is a contradiction. Hence R is not local.

Theorem 5. Let R be a commutative Weak idempotent ring. Then R_B is isomorphic to R/N.

Proof. Define $f: R_B \rightarrow R/N$ by $f(a) = a + N$. Clearly f is a well-defined ring homomorphism. Suppose for $, b \in \mathrm{R_B}$, $f(a) = f(b)$. Then $a + N = b + N \Rightarrow a + b \in N \Rightarrow (a+b)^2 = 0 \Rightarrow a^2 + ab + ba + b^2 = 0 \Rightarrow a + b = 0 \Rightarrow a = b$.

Thus, f is monomorphism. For $a + \mathrm{N} \in R/N$, $a + N = a_B + a_N + N = a_B + N = f(a_B)$, where $a_B \in R_B$. Thus f is an epimorphism and hence it is an isomorphism.

Remark 5. The product of any two nilpotent elements of a Boolean like ring is zero. However this is not true in the case of Weak idempotent rings.

For instance, consider Example 6.

The elements $1 + i$ and $1 + \mathrm{j}$ of Q are nilpotent elements, but $(1+i)(1+j) \neq 0$.

Theorem 6. If R is a commutative WIR with unity and the product of any two elements of N (the nilradical of R) is zero, then R is a Boolean like ring.

Proof. Let R be a WIR and a, $b \in R$. Then $a + a^2$ and $b + b^2$ are both nilpotent elements of R. So, $(a + a^2)(b + b^2) = 0 \Rightarrow a(1+a)b(1+b) = 0$. Hence, R is Boolean like ring.

Foster (1946) proved that a ring R is Boolean like ring if and only if the following are satisfied:

i. R is a commutative ring with unity;
ii. R is a ring of characteristic 2;
iii. each element of R can be expressed as the sum of an idempotent and a nilpotent element and
iv. the product of any two nilpotent elements in is zero.

We have the following equivalent result for a commutative WIR with unity.

Theorem 7. A commutative ring R with unity is a Weak idempotent ring if and only if the following are satisfied.

1. It is a ring of characteristic 2;
2. Each element can be expressed as the sum of an idempotent and a nilpotent elements and
3. $n^2 = 0$ for all $n \in N$ (N is the nilradical of R)

Proof. Let R be a commutative with unity.

Suppose R is a Weak idempotent ring. Then the three conditions are clear.

To prove the converse, let $a \in$ R. Then $a = a_B + a_N$. Now $a^2 = a_B$ and hence $a^4 = a_B^2 = a_B = a^2$.

Therefore, R is a Weak idempotent ring.

For a commutative Weak idempotent ring R, the following theorem is easy to prove.

Theorem 8. Let R be a commutative weak idempotent ring with unity**.** For any two elements a and b of R , the following are satisfied.

1. $(a+b)_B = a_B + b_B$ and $(a+b)_N = a_N + b_N$
2. $(ab)_B = a_B b_B$ and $(ab)_N = a_B b_N + a_N b_B + a_N b_N$
3. $(ab)_B = 0$ and $(ab)_N = ab$, if b is nilpotent.

COMPLETELY PRIME AND PRIMARY IDEALS IN WEAK IDEMPOTENT RINGS

In any ring R, an ideal P of R is a prime ideal if and only if for two ideals A and B of R, $\mathrm{AB} \subseteq P$ implies $A \subseteq P$ or $B \subseteq P$. An ideal P of R is called completely prime if and only if for $a, b \in R$, $ab \in P$ implies $a \in P$ or $b \in P$. An ideal which is completely prime is prime but the converse is not generally true. These concepts coincides in the case of commutative ring.

Example 9. Let $R = \{0,1,2,3,4,5,6,7\}$ and define the operations $'+'$ and $'*'$ by the following tables, '+' in Table 3 and 'Table 4'.

+	0	1	2	3	4	5	6	7
0	0	1	2	3	4	5	6	7
1	1	0	4	7	2	6	5	3
2	2	4	0	5	1	3	7	6
3	3	7	5	0	6	2	4	1
4	4	2	1	6	0	7	3	5
5	5	6	3	2	7	0	1	4
6	6	5	7	4	3	1	0	2
7	7	3	6	1	5	4	2	0

Table 3

*	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7
2	0	2	2	0	0	5	3	5
3	0	3	0	0	3	0	3	3
4	0	4	0	0	4	0	4	4
5	0	5	2	3	3	5	0	2
6	0	6	0	0	6	0	6	6
7	0	7	2	3	6	5	4	1

Table 4

Then $(R,+,*)$ is non-commutative ring with unity and the conditions $a + a = 0$ and $a^4 = a^2$ are satisfied and the ideals $P_1 = \{0,2,3,5\}$ and $P_2 = \{0,3,4,6\}$ are completely prime ideals in R.

The ideal $\{0\}$ is prime, but not completely prime ideal of R and we also have that $Q = \{0,3\}$ is not a prime ideal.

Theorem 9. If P is a completely prime ideal of a Weak idempotent ring R with unity, then R/P is isomorphic to the 2- element field.

Proof. Let P be a completely prime ideal of a Weak idempotent ring R with unity and let $a + P$ be idempotent element of R/P.

Then $(a + P)^2 = a + P$ implies $a(a + 1) \in P$. Since P is completely prime ideal, either $a \in P$ or $a + 1 \in P$ that is $a + P = P$ or $a + P = 1 + P$. Thus, P and $1 + P$ are the only idempotent elements of R/P. Suppose $a + P \in N$. Then $(a + P)^2 = P$. So, $a^2 \in P$. Since P is completely prime, $\in P$. This implies $a + P = P$ and hence P is the only nilpotent element of R/P. Since every element of R/P is a sum of an idempotent element and a nilpotent element of R/P, we have $R/P = \{P, 1 + P\}$.

Hence R/P is isomorphic to the 2- element field.

Example 10. In Example 6, it is given that
$R = \{0,1, i, j, k, 1 + i, 1 + j, 1 + k, i + j, i + k, j + k, 1 + i + j, 1 + j + k, 1 + i + k, i + j + k, 1 + i + j + k\}$, $R_B = \{0,1\}$ and $N = \{0,1 + I, 1 + j, 1 + k, i + j, i + k, j + k, 1 + i + j + k\}$. Thus $R = R_B \oplus N$. Clearly N is a prime ideal in R and R is a local ring. Furthermore, the ideals $Q_1 = \{0,1 + i, j + k, 1 + i + j + k\}$, $Q_2 = \{0,1 + j, i + k, 1 + i + j + k\}$ and $Q_3 = \{0,1 + k, i + j, 1 + i + j + k\}$ are all primary ideals, but they are not prime ideals.

Definition 2. Let S be an arbitrary ring with unity and Q be an ideal of a ring S. Then Q is said to be:

1. left completely primary ideal of S if, for $a, b \in S$, $ab \in Q$ implies $a \in Q$ or $b^n \in Q$ for some positive integer n;
2. right completely primary ideal of S if, for $a, b \in S$, $ab \in Q$ implies $a^n \in Q$ or $b \in Q$ for some positive integer n.

Theorem 10. An ideal $I$of a WIR R with unity and $I \neq R$ is left completely primary ideal if and only if R/I has only two idempotents.

Proof. Let I be a left completely primary ideal and $x + I$ be a zero divisor. Then $(y + I)(x + I) = I$ for some $y \notin I \Rightarrow yx + I = I \Rightarrow yx \in I \Rightarrow x^n \in I$ for some positive integer n.

Thus, $x^n + I = I \Rightarrow (x + I)^n = I$. Hence $x + I$ is nilpotent.

Suppose, for $a \in R$, $(a + I)^2 = a + I$. This implies $a^2 + a + I = I \Rightarrow (a + 1 + I)(a + I) = I$.

If $a + 1 + I \neq I$, then $a + I$ is the zero divisor. Hence $a + I$ is nilpotent. If $a + 1 + I = I$, then $a + I = 1 + I$. Hence, I and $1 + I$ are the only idempotents.

Conversely, suppose I and $1 + I$ are the only idempotent elements of R/I. Then $a + I$ is not a nilpotent element. If $a_B = 0$, then $a + I = a_N + I$ is nilpotent. This contradicts the assumption. Hence, $a_B \neq 0$.

Thus By our assumption $a_B = 1$ and hence $a + I = 1 + a_N + I$. That is,, $a^2 + I = 1 + I$. This implies $a + I$ is a unit and hence $a + I$ is not zero divisor. Thus, every zero divisor is nilpotent element.

Let $ab \in I \Rightarrow ab + I = I \Rightarrow (a + I)(b + I) = I$. If $a + I = I$ or $b + I = I$, then $a \in I$ or $b \in I$. If $a + I \neq I$ and $b + I \neq I$, then $b + I$ is zero divisor. Thus, $b + I$ is nilpotent. That is $(b + I)^2 = I \Rightarrow b^2 + I = I \Rightarrow b^2 \in I$.

Hence, I is a left completely primary ideal of R.

Theorem 11. An ideal of a WIR R with unity is left completely primary if and only if for any idempotent element $b \in R$, either $b \in I$ or $1 + b \in I$.

Proof. For $a \in R$ it is clear that $a + I = a_B + a_N + I$. If $a_N \in I$, then $a + I = a_B + I$. Thus, $a + I$ is idempotent. If $a_B \in I$, then $a + I = a_N + I$. Thus, $a + I$ is a nilpotent element.

Let I be left completely primary ideal of R and b be an idempotent in R. Then $b^2 = b \Rightarrow b(b + 1) = 0 \in I$. Since I is left completely primary, $b \in I$ or $(1 + b)^n \in I$ for some positive integer n.

Hence, $b \in I$ or $(1 + b) \in I$.

Conversely, let I be an ideal of R and for any idempotent element $b \in R$, either $b \in I$ or $1 + b \in I$. Now let $a + I$ be an idempotent of R/I. Then $a_N \in I$. But $a_B \in I$ or $1 + a_B \in I$ by our assumption. Thus, $a_B + a_N \in I$ or $1 + a_B + a_N \in I \Rightarrow a \in I$ or $1 + a \in I$. Hence, the only idempotent elements of R/I are I and $1 + I$. Hence, By Theorem 10, I is left completely primary ideal of R.

Theorem 12. In a WIR with unity, a left completely primary ideal I is completely prime if and only if the nilradical of R is a subset of I.

Proof. (⇐) Suppose I is a left completely primary ideal of R such that the nilradical N of R is a subset of I. Let $ab \in I$ and $a \notin I$. Then $b^n \in I$ for some positive integer n. That is b or b^2 or $b^3 \in I$. If $b^2 \in I$, then $b = b_B + b_N = b^2 + b_N \in I$.

If $b^3 \in I$, then $b^3 = b^2 b = b_B(b_B + b_N) = b_B b_B + b_B b_N = b_B + b_B b_N \in I$. This implies $b_B \in I$, that is, $b^2 \in I$. By the above case, $b \in I$. Hence, I is completely prime.

(⇒) Suppose an ideal I of R is completely prime. Then for $a \in N$, $a^2 = 0 \in I$ implies that $a \in I$. Hence nilradical of R is a subset of I.

Corollary 1. An ideal of a WIR R with unity is completely prime (and hence maximal) if and only if nilradical of R is a subset of I and $b \in R_B$ implies that $b \in I$ or $1 + b \in I$.

Proof. (⇒) Let R be a WIR with unity and I be an ideal of R. Suppose I is a left completely prime ideal of R. Since every completely prime ideal is left completely primary, I is left completely primary. Then, by Theorem 11 and Theorem 12, $N \subseteq I$ and $b \in R_B$ implies that $b \in I$ or $1 + b \in I$.

(⇐) Suppose that $N \subseteq I$ and $b \in R_B$. Then $b \in I$ or $1 + b \in I$.

Thus, by Theorem 11, I is left completely primary and hence, by Theorem 12, I is a completely prime ideal of R.

Corollary 2. An ideal I of a WIR R with unity is left completely primary if and only if $I \cap R_B$ is a completely prime ideal of R_B provided that R_B is the subring of R.

Proof.(⇒) Let R be a WIR with unity and I be an ideal of R. Assume that I is left completely primary ideal.

The nilradical of R_B is $\mathrm{N} = \{0\} \subseteq I \cap R_B$. By Corollary 1, $b \in R_B \Rightarrow b \in I$ or $1 + b \in I$ and hence, $b \in I \cap R_B$ or $1 + b \in I \cap R_B$ since $b \in R_B$ and $1 + b \in R_B$.

Hence, $I \cap R_B$ is left completely primary ideal of R_B. By Corollary 1, $\mathrm{I} \cap \mathrm{R_B}$ is completely prime ideal of R_B.

(⇐) Assume that $I \cap R_B$ is a completely prime ideal of R_B. Let $b \in R_B$. Then $b \in I \cap R_B$ or $1 + b \in I \cap R_B$.

Thus, $b \in I$ or $1 + b \in I$ and hence, I is left completely primary.

Note.

1. The theory of right completely primary ideals is analogues to that of left completely primary ideals. If the weak idempotent ring is commutative, then the notions of left completely primary ideal, right completely primary ideal and primary ideal coincide.
2. Primary ring is a commutative ring with unity in which {0} is primary ideal. Equivalently, a ring is primary if and only if every zero divisor of the ring is nilpotent. In a primary ring, the intersection of all primary ideals is obviously {0}.

Theorem 13. If a WIR R with unity is local, then it is a primary ring.

Proof. Let R be a WIR with unity and $a \in R$. Suppose R is local. Then by Lemma 2, $R_B = \{0,1\}$ and by Theorem 3, R is commutative.

Let $a \in R$ be neither idempotent nor nilpotent. Then $a = 1 + n$, where n is non-zero nilpotent element. By Theorem 2, a is a unit.

Let $x \in R$ be a zero divisor. Then $x \notin R_B$ and x is not a unit. Thus, x is a nilpotent element of R. Therefore, R is a primary ring.

Theorem 14. Let R be a commutative WIR with unity. If R_B has no zero divisor, then R is a primary ring.

Proof. Let R be a commutative WIR with unity and $a \in R$ be a zero divisor. Then $ab = 0$ for some $b \in R$ and $b \neq 0$.

Suppose $a_B \neq 0$. Then $ab = (a_B + a_N)b = 0$ implies $(a_B + a_N)^2 b = 0$. Since $(a_B + a_N)^2 = a_B$, we have $a_B b = 0$. That is, a_B is a zero divisor which is a contradiction. Thus $a_B = 0$ and hence a is a nilpotent.

Therefore, R is a primary ring.

Remark 6. In the following discussion, we need to show that there exists a commutative WIR with unity which is not primary ring. We substantiate this by the following example.

Consider Example 4, R is a commutative WIR with unity. Define $\bar{R} = R \times R$ with the usual cross product. Then $\bar{R}$ is a commutative WIR with unity but not primary since for (1,0) and (0,1) in $\bar{R}$, $(1,0)(0,1) = (0,0)$ but (1,0) is not nilpotent.

Theorem 15. In a commutative WIR R with unity, the intersection of all primary ideals is $\{0\}$.

Proof. Let R be a commutative WIR with unity and n be non-zero nilpotent element of R. Now we claim that there exists a primary ideal of R that is not containing n.

Let M be the set of all ideals of R which do not contain n. Then M is non-empty, since $\{0\} \in$ M. By Zorn's lemma, M ordered by inclusion, has a maximal element.

Let I be a maximal element of M. Let $xy \in I$ and $x \notin I$. Clearly, $n \notin I$. Since $x \notin I$, $I + Rx \notin M$ and hence $n \in I + Rx$. Then $n = i + rx$ for $i \in I$ and $r \in R$. Hence $ny = iy + rxy \in I$. Assume no positive power of y belongs to I, that is, $y^3 \notin I$. Hence, $n \in I + Ry^3$. Let $n = j + sy^3 = j + sy^2 + sy^3 + sy^4$ where $j \in I$ and $s \in R$ which implies $n = j + sy_B(1 + y_N)$. Multiplying both sides by $1 + y_N$, we get $n + ny_N = k + sy_B$ where $k \in I$. Then $n = j + sy^3 \Rightarrow ny = jy + sy_B$. Hence , $n + ny_N + ny = k + jy \in I$. But $ny \in I$ and $ny_N \in I$. Therefore, $n \in I$ which is a contradiction. Thus, $y^m \in I$ for some positive integer m. Hence, I is primary and also $n \notin I$.

Thus, the intersection of all primary ideals of R is $\{0\}$.

Theorem 16. In a commutative WIR R with unity, every primary ideal is prime if and only if R is a Boolean ring.

Proof. Let R be a WIR with unity.

(⇒) Assume that every primary ideal of R is a prime ideal.

Then, the intersection of all prime ideals of R is $N(\text{nilradical of } R) = \{0\}$ =The intersection of all primary ideals = $\{0\}$ (by Theorem 15).

Hence, R is a Boolean ring.

(⇐) Let R be a Boolean ring and I be an ideal of R. Suppose I is primary ideal of R.

Let $xy \in$ I and $x \notin$ I. Then $y^n \in$ I, for some positive integer n. Then, $y^n = $ y $\in$ I since R is a Boolean ring. Hence, I is a prime ideal of R.

CONCLUSIONS

In this work, we have defined some special type of rings, Weak Idempotent Rings (WIRs), provided examples of WIRs and studied some basic properties of these rings. Completely Prime and Primary Ideals in weak idempotent rings were also studied in this work. These concepts may motivate to study further on the ideal structures of weak idempotent rings.

ACKNOWLEDGMENTS

The authors thank the anonymous reviewers for their valuable suggestions and comments that helped to improve the manuscript. The first author would like to thank the authorities of Addis Ababa Science and Technology University for supporting to do research.

The second and the third authors gratefully acknowledge ISP (International Science Program, Uppsala University, Sweden).

REFERENCES

1. Anderson, D.D. (1986). Generalization of Boolean, Boolean like and Von Neumann regular rings. *Commentarii mathematici Universitatis Sancti Pauli*, **35**(6):69-76.
2. Calugareanu, G. (2015). UU Rings. CARPATHIAN J. *MATH,* **31**(2):157-163.
3. Cui, J. and Danchev, P. (2020). Some new characterizations of periodic rings. *Journal of Algebra and Its Applications*, **19**(12).
4. Foster, .A.L. (1946). The theory of Boolean like rings. *Trans.Amer.Math.Soc*..**59(1)**: 166-187
5. Givant, S. and Halmos, P. (2009). Introduction to Boolean Algebras. *Springer*.
6. Harary, F. (1950). Atomic Boolean like rings with finite radicals. *Duke Math. Journal*, **17**(1):273-276.
7. Khuzam, H. A. and Yaqub, A. (2013). Generalized Boolean and Boolean like rings. *Int. Journal of Alg*.,**7**(9): 429 - 438.
8. Koşan, T., Wang, Z. and Zhou, Y. (2016). Nil-clean and strongly nil-clean rings. *Journal of Pure and Applied Algebra*, **220**(2)**:**.633-646.
9. Lam, T.Y. (2001). A First Course in Noncommutative Rings. *Graduate Texts in Math., Springer-Verlag,* **13.**
10. McCoy, N.H. (1949). Prime ideals in general rings. *Amer. J. Math*.,.**71**(4):823-833.
11. Subrahmanyam, N.V.(1960). Lattice theory for certain classes of rings. *Math Ann*. **139**: .275-286.
12. Sussman, I. (1958). A generalization of Boolean rings *Math. Ann.* **136**: 326-328.
13. Swaminathan, V. (1982). Boolean like rings. *PhD dissertation, Andhra University, India.*
14. Yaqub, A. (2007). A Generalization of Boolean Rings. *Int. Journal of Alg*., **1**(8):353-362.