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Spectrum of a regular weighted Sturm-Liouville problem and certain infinite sets of integers

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ABSTRACT: Certain infinite subsets of the set of positive integers are investigated as possible spectra of Regular Weighted Sturm-Liouville eigenvalue problem with separated homogeneous boundary conditions. With the (conditional) exception of the set of square integers, it is shown that all the sets considered herein are not spectra of such a problem. Concepts adopted from the area of study in Mathematical analysis known as Asymptotic analysis will figure prominently in the proofs of the main results.

Keywords/phrases: Asymptotic Density, Asymptotic Estimate, Large/Small sets, Spectrum, Weighted Sturm-Liouville problem.

INTRODUCTION

In a 2011 research article Mingarelli answered a question posed in Zettl (2005). He showed that there does not exist a regular Sturm-Liouville problem whose spectrum consists of infinitely many prime numbers (Mingarelli, 2011). Zettl pointed out that given any finite set of distinct real numbers, a Sturm-Liouville Problem of Atkinson-type (Atkinson, 1964) can be found whose spectrum is precisely that set (Zettl, 2005). In proving this result, Mingarelli used a tool from Analytic Number Theory which provides an asymptotic estimate for the n^{th} prime number. Such an estimate is a consequence of the celebrated Prime Number Theorem. Here, we recall that the asymptotic estimate for the n^{th} prime is oftentimes expressed as:

$$p_n \sim nlog(n),$$

where p_n denotes the n^{th} prime number.

On top of this, Mingarelli (2011) also used results on asymptotic estimates for the n^{th} eigenvalue of a regular Sturm-Liouville eigenvalue problem from his previous paper written in collaboration with Atkinson concerning spectral asymptotics (Atkinson and Mingarelli, 1987). Namely:

$$\lambda_n^+ \sim \frac{n^2\pi^2}{(\int_a^b \sqrt{(\frac{r(x)}{p(x)})_\pm}dx)^2}.$$

Evidently,

$$\lim_{n\to\infty} \frac{p_n}{\lambda_n^+} \sim \lim_{n\to\infty} \frac{K^2nlog(n)}{n^2\pi^2}$$

Nevertheless,

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$$\begin{aligned}\lim_{n\to\infty}\frac{K^2 nlog(n)}{n^2\pi^2} &= \lim_{n\to\infty}\frac{K^2 log(n)}{n\pi^2}\\ &= \frac{K^2}{\pi^2}\lim_{n\to\infty}\frac{log(n)}{n}\end{aligned}$$

where $K = \int_a^b \sqrt{(\frac{r(x)}{p(x)})_\pm}dx$.

Observe that $\lim_{n\to\infty}\frac{log(n)}{n} = 0.$

This leads to $\lim_{n\to\infty}\frac{p_n}{\lambda_n^+} = 0.$

By the definition of "$\sim$",

$$p_n \sim \lambda_n^+ \Leftrightarrow \lim_{n\to\infty}\frac{p_n}{\lambda_n^+} = 1.$$

Hence, $p_n \nsim \lambda_n^+$.

This proves that the set of primes is not the spectrum of such a problem.
In fact, the above argument, adopted from Mingarelli (2011); proves more than the aforementioned assertion. Specifically, it proves that any infinite subsequence of distinct prime numbers is not the spectrum of (1).

In the subsequent discussions we consider the regular weighted Sturm-Liouville problem with separated homogeneous boundary conditions given by the following system (Billingham *et al.*, 2003):

$$\begin{aligned}-\frac{d}{dx}[p(x)\frac{dy}{dx}] + q(x)y = \lambda r(x)y,\\ c_1y(a) + d_1y'(a) = 0,\ c_1^2 + d_1^2 \neq 0, \qquad (1)\\ c_2y(b) + d_2y'(b) = 0,\ c_2^2 + d_2^2 \neq 0\end{aligned}$$

where p; p'; q and r are continuous functions over the finite interval [a, b], λ is generally a complex parameter, p and r are strictly positive on [a, b] and $\frac{1}{p} \in$ L [a, b].

Also conjectured in Mingarelli (2011) was that there may exist a Sturm-Liouville problem with Dirichlet data whose Dirichlet spectrum agrees with the set of all rational primes if the parameter dependence is nonlinear.

In a subsequent work Adalar and Amirov (2017) showed that there is no function q(x) $\in L_2(0,1)$ which is the potential of a Sturm-Liouville problem with Dirichlet data whose spectrum is the set of prime numbers. In their proof they adopted the following estimate for the eigenvalues N(λ) = λ;

$$\pi\lambda_n = n\pi + \frac{\int_0^1 q(x)}{2n\pi} + O(n^{-\frac{1}{2}}).$$

Using this estimate, Adalar and Amirov (2017) proved the following two theorems:

Given:

$$\begin{aligned}-y'' + q(x)y = (\pi N(\lambda))^2 y\\ y(0) = y(1) = 0 \qquad (2)\end{aligned}$$

1. if N(λ) = $\frac{\lambda}{ln\lambda}$, then there is no function q $\in L_2$[0, 1] such that the spectrum of the BVP (2) is the set of prime numbers.

2. if N(λ) = li(λ), then there is no function q $\in L_2$[0, 1] such that the spectrum of the BVP (2) is the set of prime numbers.

Although the consideration of regular weighted Sturm-Liouville problems with separated homogeneous boundary conditions whose parameter dependence is non-linear is potentially a tempting endeavor, the generalization of the result in Mingarelli (2011) to a wider class of infinite sets of integers has been deemed by the authors a worthwhile undertaking. Consequently, in what follows we will focus on the BVPs defined in (1) whose parametric dependence is linear.

More on the important role the sign of the weight (density) function r(x) plays in the study of the nature of the spectrum of a general weighted Sturm-Liouville problem can be found in the joint work of Kikonko and Mingarelli (2016).

PRELIMINARIES

Definition 1 (Regular Sturm-Liouville Problems). *A Sturm-Liouville problem is said to be regular on a finite interval [a, b] whenever p(x) and r(x) are strictly positive on [a, b], $p \in C^1([a,b], \mathbb{R})$ and q, $r \in C([a,b], \mathbb{R})$. (Billingham* et al.*, 2003)*

Definition 2 (Small and Large Sets). *A set $A = \{a_n\}$ is said to be, (Wadhwa, 1975)*

(i). small if and only if $\displaystyle\sum_{n=1}^{\infty} \frac{1}{a_n}$ *converges,*

(ii). large whenever $\displaystyle\sum_{n=1}^{\infty} \frac{1}{a_n} = \infty$.

Definition 3 (Asymptotic Equality of Sets). *Set A is said to be asymptotically equal to set B, written $A \sim B$, whenever the symmetric difference $A \bigtriangleup B$ is finite. (Sonnenschein, 1971)*

Definition 4 (Asymptotic Equality of Functions). *Two functions f(x) and g(x) are said to be asymptotically equal (as $x \to x_0$) if and only if* $\displaystyle\lim_{x\to x_0} \frac{f(x)}{g(x)} = 1$. *(Ribenboim, 2004)*

Definition 5 (Counting Function). *For $A \in 2^{\mathbb{N}}$, the Counting Function (denoted $A(n)$) is the number of elements in set A that are less than or equal to $n \in \mathbb{N}$ and is given by $A(n) = \sum_{i=1}^{n} \chi_A(i)$; where $\chi_A(i)$ is the characteristic function of the set A. (Sonnenschein, 1971)*

Definition 6 (Asymptotic Density). *A function $\delta : 2^{\mathbb{N}} \to \mathbb{R}$ given by:*

$$\delta(A) = \lim_{n\to\infty} \frac{A(n)}{n}$$

is called Asymptotic (Natural) Density. (Pekara, 1972)

Definition 7 (Relative Asymptotic Density). *The relative asymptotic density of a set A with respect to a base set B, denoted $\delta_B(A)$ is given by:*

$$\delta_B(A) = \lim_{n\to\infty} \frac{A(n)}{B(n)}$$

where $A(n)$ and $B(n)$ represent $\#(A)$ and $\#(B)$ that are less or equal to n respectively, with $B(n) \neq 0$. (Sonnenschein, 1971)

The lemmas we state and prove in this section play a foundational role in the proofs of the results stated in the subsequent section.

Lemma 1. *The set $S = \{\lambda_n^+\}$ of the positive eigenvalues of (1) is a small set.*

Proof. Let S = $\{\lambda_n^+\}$ be the set of positive eigenvalues of (1).

From Atkinson and Mingarelli (1987),

$$\lambda_n^+ \sim \frac{n^2\pi^2}{(\int_a^b \sqrt{(\frac{r(x)}{p(x)})_\pm} dx)^2}.$$

By the definition of asymptotic equality (of functions), this implies that;

$\sum_{n=1}^{\infty} \frac{1}{\lambda_n^+}$ and $\sum_{n=1}^{\infty} \frac{1}{\frac{n^2\pi^2}{(\int_a^b \sqrt{(\frac{r(x)}{p(x)})_{\pm}}dx)^2}}$ diverge (or converge) at the same rate (Thomas Jr., 1973).

Applying the solution of the Basel Problem (Euler, 1737; Ribenboim, 2004) we have,

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{1}{\frac{n^2\pi^2}{K^2}} &= \sum_{n=1}^{\infty} \frac{K^2}{n^2\pi^2} \\ &= \frac{K^2}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \\ &= (\frac{K^2}{\pi^2})\,(\frac{\pi^2}{6}) \\ &= \frac{K^2}{6} \end{aligned}$$

where K $= \int_a^b \sqrt{(\frac{r(x)}{p(x)})_+}dx.$

Observe that, $\frac{K^2}{6} \in \mathbb{R}$ as K $\in \mathbb{R}$. (K $\in \mathbb{R}$ since $\frac{1}{p}$, r, $\in$ L(a, b) by assumption.)

Hence,

$\sum_{n=1}^{\infty} \frac{1}{\frac{n^2\pi^2}{(\int_a^b \sqrt{(\frac{r(x)}{p(x)})_{\pm}}dx)^2}}$ is a convergent series.

And hence, $\sum_{n=1}^{\infty} \frac{1}{\lambda_n^+}$ also converges. (Thomas Jr., 1973)

Therefore, S = $\{\lambda_n^+\}$ of the positive eigenvalues of (1) is a small set.

Lemma 2. *Let P denote the set of prime numbers and $\pi(n)$ denote the number of primes that are no larger than n. Then the asymptotic density of the primes is*

$$\delta(P) \ = \ \lim_{n\to\infty} \frac{\pi(n)}{n} \ = \ 0.$$

Proof. The reader is referred to (Pinsky, 2014).

Lemma 3. *The asymptotic density of the set of positive eigenvalues of (1) in the set of natural numbers is 0. That is, if $S = \{\lambda_n^+\}$ is the set of positive eigenvalues of (1), then $\delta(S) = 0$.*

Proof. From Lemma 1, S is a small set. As the set of Prime numbers is a large set in the set of natural numbers ($\sum_{n=1}^{\infty} \frac{1}{p_n} = \infty$. For a proof refer to Ribenboim (2004).), this implies that the set S is sparser than the set of prime numbers in the set of natural numbers.

Thus, $0 \leq \delta(S) \leq \delta(P) = 0$ (by Lemma 2).

And hence, $\delta(S) = 0$.

Lemma 4. *If G and H are divergent sequences with $G \sim H$, then $\delta_H(G) = 1$.*

Proof. By the definition of asymptotic equality (of sets),
G $\sim$ H $\Rightarrow$ ($\exists$ M $\in \mathbb{N}$) such that

($\forall$ n $\geq$ M and $\exists$ k $\in \mathbb{N}$)

$|G(M) - H(M)| =$ k.

$\Rightarrow$ ($\forall$ n $\geq$ M) (G(n) = H(n) $\pm$ k)

Hence,

$$\lim_{n\to\infty}\frac{G(n)}{H(n)} = \lim_{n\to\infty}\frac{H(n)\pm k}{H(n)} = 1.$$

Thus, by the definition of relative density;

$$\lim_{n\to\infty}\frac{G(n)}{H(n)} = 1 \Leftrightarrow \delta_H(\mathrm{G}) = 1.$$

Lemma 5. *Let $S = \{\lambda_n^+\}$ be the set of positive eigenvalues of (1). If*

$$R = \{\frac{n^2\pi^2}{(\int_a^b\sqrt{(\frac{r(x)}{p(x)})_+}dx)^2}\}$$

then, S(n) $\sim$ R(n) $= \lfloor\frac{n^{\frac{1}{2}}K}{\pi}\rfloor$

where $K = \int_a^b\sqrt{(\frac{r(x)}{p(x)})_+}dx.$

Proof. From Atkinson and Mingarelli (1987) we have S $\sim$ R.

By Lemma 4, S $\sim$ R $\Rightarrow$ δ_R(S) = 1.

Now, from the definition of relative density,

$$\delta_R(\mathrm{S}) = 1 \Rightarrow \lim_{n\to\infty}\frac{S(n)}{R(n)} = 1.$$

By the definition of asymptotic equality (the definition of asymptotic equality is applicable here as S(n) and R(n) are both non-negative integers for any n $\in$ $\mathbb{N}$.),

$$\lim_{n\to\infty}\frac{S(n)}{R(n)} = 1 \Rightarrow \mathrm{S(n)} \sim \mathrm{R(n)}.$$

Thus, if R(m) = $\#\{T : T \leq m,\ m \in \mathbb{N}\}$,

where T $= \frac{n^2\pi^2}{K^2}$,

then R(m) = n.

Where n is the largest positive integer for which T $\leq$ m.

Now, $\frac{n^2\pi^2}{K^2} \leq$ m and

$[\frac{n^2\pi^2}{K^2} =$ m $\Leftrightarrow \frac{n^2\pi^2}{K^2} \in \mathbb{Z}^+]$

$\Rightarrow n^2 \leq \frac{mK^2}{\pi^2} \Rightarrow$ n $\leq \frac{m^{\frac{1}{2}}K}{\pi}$

Therefore, n $= \lfloor\frac{m^{\frac{1}{2}}K}{\pi}\rfloor$

Hence, for a natural number n,

R(n) $= \lfloor\frac{n^{\frac{1}{2}}K}{\pi}\rfloor$.

Consequently,

[S(n) $\sim$ R(n) and R(n) $= \lfloor\frac{n^{\frac{1}{2}}K}{\pi}\rfloor$]

$\Rightarrow$ S(n) $\sim \lfloor\frac{n^{\frac{1}{2}}K}{\pi}\rfloor$

RESULTS

The following results are the main findings of this paper.

Theorem 1. *Any subset of the set of positive integers with asymptotic density strictly greater than zero is not the spectrum of (1).*

Theorem 2. *The set of all the terms of an arithmetic sequence*

$$\{am+b : a \in \mathbb{Z}^+, b \in \mathbb{Z}^+\cup\{0\}, m = 1,2,3,...\}$$

is not the spectrum of (1).

Corollary 2.1. *None of the following three sets is the spectrum of (1):*

2.1.1. The set of natural numbers.

2.1.2. The set of even natural numbers.

2.1.3. The set of odd natural numbers.

Theorem 3. *Any subset $\{a_n\}$ of the set of positive integers which is a large set is not the spectrum of (1).*

Corollary 3.1. *The set of prime numbers is not the spectrum of (1).*

Theorem 4. *The set $D = \{m^2\colon\ m = 1,2,3, ...\}$ of square natural numbers is the spectrum of (1) implies that $K = \pi$,*

where $K = \displaystyle\int_a^b \sqrt{(\frac{r(x)}{p(x)})_+}dx.$

Theorem 5. *The set $F = \{m^t\colon\ t \geq 3,\ t \in \mathbb{N},\ m = 1,2,3, ...\}$ is not the spectrum of (1).*

Theorem 6. *If a set A which is a subset of the set of positive integers is a spectrum of (1), then the following three conditions are satisfied:*

6.1. $\delta(A) = 0$.

6.2. $\delta_S(A) = 1$.

6.3. A is a small set.

Remark 6.1. *It should be noted that Theorem 6, provides a necessary condition in order for a subset of the set of positive integers to be a spectrum of (1).*

Remark 6.2. *Among the three statements (necessary conditions), only the condition in Theorem 6.2 is potentially a sufficient condition.*

The condition in Theorem 6.1 fails to be a sufficient condition because; although the asymptotic density of the set of Prime numbers is 0, by Corollary 3.1 it is not a spectrum of (1).

Furthermore, the condition in Theorem 6.3 fails to be a sufficient condition because; although the set of square natural numbers is a small set (Ribenboim, 2004), it is nevertheless not a spectrum of (1) whenever $K \neq \pi$. (Refer to the proof of Theorem 4.)

As an aside, we now state (and prove) the following theorem involving the Erdos's conjecture (the Erdos - Turan conjecture) on arithmetic progressions which states that; if the sum of the reciprocals of the elements of a set A of natural integers diverges, then A contains arbitrarily long arithmetic progressions.

Theorem 7. *Provided that the converse of Erdos's conjecture on arithmetic progressions holds, the spectrum of (1) doesn't contain arbitrarily long arithmetic progressions.*

Proof. Let S = $\{\lambda_n^+\}$ be the spectrum of (1).

From Lemma 1, S is a small set.

Suppose the converse of Erdos's conjecture holds.

But then its contrapositive implies that the spectrum of (1) doesn't contain arbitrarily long arithmetic progressions.

PROOFS

Proof. (**Theorem 1**)
Let A $\subseteq$ N with $\delta(A) > 0$.

By Lemma 3, δ(S) = 0.

Hence, $\delta(A) > 0 \Rightarrow \delta(A) > \delta(S)$

$\Rightarrow \delta(A) \neq \delta(S)$.

Now, $\delta(A) \neq \delta(S) \Rightarrow A \nsim S$. (For sets A and B; A $\sim$ B $\Rightarrow$ $\delta(A) = \delta(B)$. (Sonnenschein, 1971))

And, $A \nsim S \Rightarrow A \neq S$. (For sets A and B; A = B $\Rightarrow$ A $\sim$ B.)

Therefore, set A is not the spectrum of (1) whenever $\delta(A) > 0$.

Proof. (**Theorem 2**)
Let A = {am + b: a $\in \mathbb{Z}^+$, b $\in \mathbb{Z}^+ \cup \{0\}$, m = 1, 2, 3, ...}.

Since, $\lim_{n\to\infty} \dfrac{n}{an+b} = \dfrac{1}{a}$ and a > 0 by assumption, $\dfrac{1}{a}$ is defined and $\dfrac{1}{a} > 0$.

Then, by a property of asymptotic density, $\delta(A)$ exists and $\delta(A) = \frac{1}{a}$. (Pekara, 1972) (Let A = $\{a_n\}$ be an infinite sequence of positive integers. If $\lim_{n\to\infty} \dfrac{n}{a_n}$ converges then so does δ(A) = $\lim_{n\to\infty} \dfrac{A(n)}{n}$ and both converge to the same value. (Sonnenschein, 1971))

Consequently, $\delta(A) > 0$.

By Theorem 1, $\delta(A) > 0$ implies that A is not the spectrum of (1).

Proof. (**Corollary 2.1.1.**)
Consider the set of natural numbers, $\mathbb{N}$. Then $\mathbb{N}$ can be expressed as:
$\mathbb{N}$ = {n: [n = m] (m = 1, 2, 3, ...)}

Thus, $\mathbb{N}$ = {m} is an arithmetic sequence, a special case of
A = {am + b} with a = 1 and b = 0.

Then, by Theorem 2, $\mathbb{N}$ is not the spectrum of (1).

Proof. (**Corollary 2.1.2.**)
Let E be the set of even positive integers. Then, the set E can be defined as:
E = {k: [k = 2m] (m = 1, 2, 3, ...)}

Thus, E = {2m} is an arithmetic sequence, a special case of A = {am + b} with a = 2 and b = 0.

And hence, by Theorem 2, E is not the spectrum of (1).

Proof. (**Corollary 2.1.3.**)
Let O be the set of odd positive integers. Then, the set O can be defined as:
O = {d: [d = 2m + 1] (m = 0, 1, 2, 3, ...)}

Thus, O = {2m + 1} is an arithmetic sequence, a special case of A = {am + b} with a = 2 and b = 1.

Therefore, by Theorem 2, O is not the spectrum of (1).

Proof. (**Theorem 3**)
Let a_n denote the n^{th} term of the sequence $\{a_n\}$ of positive integers which is a large set. Then, by assumption, $\sum_{n=1}^{\infty} \dfrac{1}{a_n} = \infty$.

Thus, by Lemma 1, the set $\{\lambda_n^+\}$ of the positive eigenvalues of (1) is a small set.

Therefore, any subset $\{a_n\}$ of the set of positive integers; with $\sum_{n=1}^{\infty}\frac{1}{a_n}=\infty$, is not the spectrum of (1).

Proof. (**Corollary 3.1**)
Based on the proof in (Euler, 1732), the sum of the reciprocals of the primes is a divergent series (i.e., the set of prime numbers is a large set).

Therefore, by Theorem 3, the set of prime numbers is not the spectrum of (1).

Proof. (**Theorem 4**)
Let the set of square natural numbers, denoted by D be the spectrum of (1). Suppose $S=\{\lambda_n^+\}$ is the set of positive eigenvalues of (1)

and, let R = $\{\dfrac{n^2\pi^2}{(\int_a^b\sqrt{(\frac{r(x)}{p(x)})_+}dx)^2}\}$.

Now, D = S $\Rightarrow$ D $\sim$ S.

But then, [D $\sim$ S $\wedge$ S $\sim$ R] $\Rightarrow$ D $\sim$ R; since asymptotic density is an equivalence relation.

Applying Lemma 4 yields $\delta_R(\mathrm{D})=1$.

And from Lemma 5, R(n) $=\lfloor\frac{n^{\frac{1}{2}}K}{\pi}\rfloor$.

Consequently,

$$\begin{aligned}\delta_R(D) &= \lim_{n\to\infty}\frac{D(n)}{R(n)}\\ &= \lim_{n\to\infty}\frac{\lfloor n^{\frac{1}{2}}\rfloor}{\lfloor\frac{n^{\frac{1}{2}}K}{\pi}\rfloor}\\ &= \frac{\pi}{K}\end{aligned}$$

where K $=\int_a^b\sqrt{(\frac{r(x)}{p(x)})_+}dx.$

In evaluating the resulting limit involving the floor function, consider the inequality;

$$\begin{aligned}\lim_{n\to\infty}\frac{n^{\frac{1}{2}}-1}{\frac{n^{\frac{1}{2}}K+1}{\pi}} &\le \lim_{n\to\infty}\frac{\lfloor n^{\frac{1}{2}}\rfloor}{\lfloor\frac{n^{\frac{1}{2}}K}{\pi}\rfloor}\\ &\le \lim_{n\to\infty}\frac{n^{\frac{1}{2}}+1}{\frac{n^{\frac{1}{2}}K-1}{\pi}}\end{aligned}$$

And then apply squeezing theorem to evaluate $\delta_R(\mathrm{D})$, as both the given lower and upper bounds for $\delta_R(\mathrm{D})$ converge to $\dfrac{\pi}{K}$. ($\frac{\pi}{K}\in\mathbb{R}$ since K $\neq 0$ as the BVP in (1) is assumed to be regular.)

Hence, $\delta_R(\mathrm{D})=\dfrac{\pi}{K}$.

Now, $\delta_R(\mathrm{D})=\dfrac{\pi}{K}=1\Leftrightarrow$ K $=\pi$.

Based on Lemma 4, S(n) $\sim$ R(n).

And $\delta_R(\mathrm{D})=1\Leftrightarrow$ R(n) $\sim$ D(n). (The equivalence follows from the definition of relative asymptotic density.)

Combining these we have,
[(S(n) $\sim$ R(n))
and
(R(n) $\sim$ D(n) $\Leftrightarrow$ K $=\pi$)]

$\Rightarrow$ (S(n) $\sim$ D(n) $\Leftrightarrow$ K $=\pi$)

Hence, based on the definition of relative asymptotic density and Lemma 4,

(S $\sim$ D $\Leftrightarrow$ K $=\pi$)

$\Rightarrow$ (S $=$ D $\Rightarrow$ K $=\pi$).

Therefore, D is the spectrum of (1) implies that K $=\pi$.

Proof. (**Theorem 5**)
Given $S = \{\lambda_n^+\}$ is the set of positive eigenvalues of (1);

Let $\mathrm{R} = \{\frac{n^2\pi^2}{(\int_a^b \sqrt{(\frac{r(x)}{p(x)})_+ dx})^2}\}$.

and

$\mathrm{F} = \{m^t\text{: t} \geq 3, \text{ t} \in \mathbb{N}, \text{ m} = 1,2,3, \ldots\}$.

From the asymptotic estimate in Atkinson and Mingarelli (1987),

$$\lambda_n^+ \sim \frac{n^2\pi^2}{(\int_a^b \sqrt{(\frac{r(x)}{p(x)})_+ dx})^2}$$

From Lemma 5, $\mathrm{R(n)} = \lfloor \frac{n^{\frac{1}{2}}K}{\pi} \rfloor$.

Consequently,

$$\begin{aligned}\delta_R(F) &= \lim_{n\to\infty} \frac{F(n)}{R(n)} \\ &= \lim_{n\to\infty} \frac{\lfloor n^{\frac{1}{t}} \rfloor}{\lfloor \frac{n^{\frac{1}{2}}K}{\pi} \rfloor}\end{aligned}$$

where $\mathrm{K} = \int_a^b \sqrt{(\frac{r(x)}{p(x)})_+ dx}$.
In evaluating the above limit involving the floor function, consider the inequality;

$$\begin{aligned}\lim_{n\to\infty} \frac{n^{\frac{1}{t}} - 1}{\frac{n^{\frac{1}{2}}K+1}{\pi}} &\leq \lim_{n\to\infty} \frac{\lfloor n^{\frac{1}{t}} \rfloor}{\lfloor \frac{n^{\frac{1}{2}}K}{\pi} \rfloor} \\ &\leq \lim_{n\to\infty} \frac{n^{\frac{1}{t}} + 1}{\frac{n^{\frac{1}{2}}K-1}{\pi}}\end{aligned}$$

And then apply squeezing theorem to evaluate δ_R(F), as both the given lower and upper bounds for δ_R(F) converge to 0.

Hence, $\delta_R(\mathrm{F}) = 0$. Now, by the contrapositive of Lemma 4,

$\delta_R(\mathrm{F}) \neq 1 \Rightarrow \mathrm{F} \nsim \mathrm{R}$.

Thus, $\mathrm{F} \nsim \mathrm{R} \Rightarrow \mathrm{F} \nsim \mathrm{S} \Rightarrow \mathrm{F} \neq \mathrm{S}$.

Therefore, F is not the spectrum of (1).

Proof. (**Theorem 6.1**)
Let $\delta(A) \neq 0$.

Then, based on the definition of asymptotic density, we have,
$\delta(A) > 0$. By Theorem 1, this implies that A is not the spectrum of (1).

Therefore, set A is the spectrum of (1) implies that $\delta(A) = 0$.

Proof. (**Theorem 6.2**)
Let $\delta_S(\mathrm{A}) \neq 1$.

By the contrapositive of Lemma 4, this implies that $\mathrm{A} \nsim \mathrm{S}$.

And, $A \nsim S \Rightarrow A \neq S$.

Therefore, set A is the spectrum of (1) implies that $\delta_S(\mathrm{A}) = 1$.

Proof. (**Theorem 6.3**)
Let A be a large set. Then, by Theorem 3, this implies that A is not the spectrum of (1).

Therefore, set A is the spectrum of (1) implies that A is a small set.

OPEN PROBLEMS

1. Whether the converse of Theorem 4 holds (for the case $K = \pi$) is still (to the authors' knowledge) an open problem.

2. Whether the condition in Theorem 6.2 is a sufficient condition in order for a set of positive integers to be a spectrum of (1) is (to the authors' knowledge) still an open question.

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