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A computation on the decomposition factors of D-modules over a hyperplane arrangement in space

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ABSTRACT: Let m be a positive integer, $\alpha_i : \mathbb{C}^n \longrightarrow \mathbb{C}$, for $i = 1, 2, ..., m$, be linear forms and $H_i = \{P \in \mathbb{C}^n : \alpha_i(P) = 0\}$ be the corresponding hyperplane for each $i = 1, \ldots, m$. The linear forms $\alpha_1, ..., \alpha_m$ define a hyperplane arrangement and $\mathrm{X} = \mathbb{C}^{\mathrm{n}} \setminus \mathrm{V}(\alpha)$, where $\alpha = \prod_{i=1}^{m} \alpha_i$ and $\mathrm{V}(\alpha) = \{\mathrm{p} \in \mathbb{C}^{\mathrm{n}} : \alpha(\mathrm{p}) = 0\}$. The coordinate ring $\mathcal{O}_{\mathrm{X}}$ of X is the localization $\mathbb{C}[x_1, \ldots, x_n]_\alpha$ and the ring $\mathcal{O}_{\mathrm{X}} = \mathbb{C}[\mathrm{x}_1, \ldots, \mathrm{x}_{\mathrm{n}}]_\alpha$ is a holonomic A_n-module, where A_n is the n-th Weyl algebra, hence it has finite length. In this work, we will compute the number of decomposition factors of the A_3-module $\mathbb{C}[x]_\alpha$, where α defines a central hyperplane arrangement in space, in terms of the no-broken circuits and describe the decomposition factors in terms of their supports.

Keywords/phrases: D-modules, Decomposition Factors, Support of a D-module, Length of a D-module, No-broken Circuits

Introduction

Let m be a positive integer, $\alpha_i : \mathbb{C}^n \longrightarrow \mathbb{C}$, for $i = 1, 2, ..., m$, be linear forms and $H_i = \{P \in \mathbb{C}^n : \alpha_i(P) = 0\}$ be the corresponding hyperplane for each $i = 1, \ldots, m$. The linear forms $\alpha_1, ..., \alpha_m$ define a hyperplane arrangement and $\mathrm{X} = \mathbb{C}^{\mathrm{n}} \setminus \mathrm{V}(\alpha)$, where $\alpha = \prod_{i=1}^{m} \alpha_i$ and $\mathrm{V}(\alpha) = \{\mathrm{p} \in \mathbb{C}^{\mathrm{n}} : \alpha(\mathrm{p}) = 0\}$. The coordinate ring $\mathcal{O}_{\mathrm{X}}$ of X is the localization $\mathbb{C}[x_1, \ldots, x_n]_\alpha$ and the ring $\mathcal{O}_{\mathrm{X}} = \mathbb{C}[\mathrm{x}_1, \ldots, \mathrm{x}_{\mathrm{n}}]_\alpha$ is a holonomic A_n-module, where A_n is the $n-$th Weyl algebra. Since holonomic A_n-modules have finite length (see Björk, 1993 and Coutinho, 1995), and hence it has finite length. We will use the notation $\mathbb{C}[x] = \mathbb{C}[x_1, \ldots, x_n]$ in our discussions.

In general, D-module is a module over a ring D of differential operators. The major interest of such D-modules is as an approach to the theory of linear partial differential equations (see Björk, 1979) and algebraic D-modules are modules over the Weyl algebra A_n over a field $\mathbb{K}$ of characteristic zero (see Coutinho, 1995).

The number of decomposition factors of a twisted form of $\mathbb{C}[x]_\alpha$ and their descriptions in the plane case defined by a central hyperplane arrangement are computed by Abebaw and Bøgvad (2010) and the number of decomposition factors of a twisted form of $\mathbb{C}[x]_\alpha$ and their descriptions in the hyperplane arrangement in the general position are computed by Abebaw and Bøgvad (2012).

The length and multiplicity of the local cohomology with support in a hyperplane are computed by Oaku (2015).

In this work, we compute the number of decomposition factors of the A_3-module $\mathbb{C}[x]_\alpha$, where α defines a central hyperplane arrangement in space, in terms of the no-broken circuits and describe the decomposition factors in terms of their supports.

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Preliminaries and Notations

In this section, preliminary results on the topic and appropriate notations for the purpose of our use are given.

For a positive integer $n \geq 1$, let $x_1, ..., x_n, \partial_1, ..., \partial_n$ be the operators on $\mathbb{C}[x_1, x_2, ..., x_n]$ which are defined on a polynomial f in the polynomial ring $\mathbb{C}[x_1, x_2, ..., x_n]$ by the formulas $x_i(f) = x_i f$ for all $i = 1, ..., n$ and $\partial_i(f) = \frac{\partial f}{\partial x_i}$ for all $i = 1, ..., n$ and these are linear operators of $\mathbb{C}[x_1, x_2, ..., x_n]$. The $n-$ th Weyl Algebra A_n is a $\mathbb{C}$ sub algebra of $\mathrm{End}_{\mathbb{C}}\mathbb{C}[x]$ generated by operators $x_1, x_2, ..., x_n$ and $\partial_1, \partial_2, ..., \partial_n$ on the polynomial ring $\mathbb{C}[x]$, that is, A_n is given by

$$A_n = \mathbb{C}\langle x_1, x_2, ..., x_n, \partial_1, \partial_2, ..., \partial_n\rangle.$$

For the sake of consistency, we write $\mathrm{A}_0 = \mathbb{C}$ and a multi-index α is an element of $\mathbb{N}^n$, say $\alpha = (\alpha_1, ..., \alpha_n)$ and by x^α we mean the monomial $x_1^{\alpha_1}...x_n^{\alpha_n}$.

Some basic properties of the $n-$th Weyl Algebra A_n are given in the following proposition.

Proposition 1 (Björk (1979), Coutinho (1995))**.** For a positive integer $n \geq 1$.

(i) the Weyl algebra A_n is not commutative.
(ii) The algebra A_n is a domain.
(iii) The only elements of A_n that have an inverse are constants.
(iv) The set $B = \{x^\alpha \partial^\beta | \alpha, \beta \in \mathbb{N}^n\}$ is a basis of A_n as a vector space over $\mathbb{C}$.

With the actions of x_i and ∂_i on the polynomials in $\mathbb{C}[x]$, the polynomial ring $\mathbb{C}[x]$ becomes and A_n-module.

Lemma 1 (Björk (1979), Coutinho (1995))**.** The A_n-module $\mathbb{C}[x]$ is simple.

Let X be a smooth affine algebraic variety (X will be $\mathbb{C}^n$ or an open subset of $\mathbb{C}^n$ which is the complement of a union of hyperplanes defined by forms). We denote by $\mathrm{D_X}$ the ring of differential operators on X and if $\mathrm{X} = \mathbb{C}^\mathrm{n}$, this is the same as A_n. If X is an affine open subset of $\mathbb{C}^n$ defined by $0 \neq f \in \mathbb{C}[x]$, then $\mathrm{D_X} = \mathbb{C}[\mathrm{x}]_\mathrm{f} \otimes_{\mathbb{C}[\mathrm{x}]} \mathrm{A_n}$ and in this case we use the notation $\mathrm{O_X} = \mathbb{C}[\mathrm{x}]_\mathrm{f}$.

If M is a $\mathrm{D_X}$-module then it can be viewed as an $\mathrm{O_X}$-module and hence has an annihilator, $\mathrm{Ann_{O_X}M}$.

Definition 1. $\mathrm{V(Ann_{O_X}M)}$ is called the support of M, and is denoted by SuppM. (With V(I) for an ideal $\mathrm{I} \subset \mathrm{O_X}$ means the closed subvariety of zeroes defined by I.)

We have the following examples that can illustrate our definition of support.

(1) For $\mathrm{M_1} = \mathbb{C}[\mathrm{x,y}]_{\mathrm{xy}}/(\mathbb{C}[\mathrm{x,y}]_\mathrm{x} + \mathbb{C}[\mathrm{x,y}]_\mathrm{y})$, $\mathrm{SuppM_1} = \mathrm{V(x,y)} = (0,0)$.
(2) For $\mathrm{M_2} = \mathbb{C}[\mathrm{x,y}]_\mathrm{x}/\mathbb{C}[\mathrm{x,y}]$, $\mathrm{SuppM_2} = \mathrm{V(x)} = \{(0,\mathrm{y}) : \mathrm{y} \in \mathbb{C}\}$.
(3) For $\mathrm{M_3} = \mathbb{C}[\mathrm{x,y}]$, $\mathrm{SuppM_3} = \mathrm{V}(0) = \mathbb{C}^2$.

Definition 2. Let R be a ring and M be an R module. If $0 = M_0 \subset M_1 \subset \cdots \subset M_n = M$ is composition series of M, then the set $DF(M) := \{M_i/M_{i-1}\}_{i=1}^n$ of R modules is called the set of decomposition factors of M.

The next definition is about the length of modules and only modules with finite number of decomposition factors are considered in this article.

Definition 3. Let R be a ring and M be an R module. Then we define the length of M over R by: $c(M) = 0$ if $M = \{0\}$ and $c(M) = n$ for $M \neq \{0\}$ and if there exists composition series $0 = M_0 \subset M_1 \subset M_2 \subset \cdots \subset M_n = M$ of M such that M_i/M_{i-1} is simple and non-zero R-module, for each $i = 1, \ldots, n$. If M is simple, $c(M) = 1$.

The following theorem is about the length of a module in terms of its submodule.

Theorem 1 (Abebaw and Bøgvad (2010))**.** Let R be a ring, M be an R module and N be a submodule of M. Then

(1) $DF(M) = DF(N) \cup DF(M/N)$
(2) $c(M) = c(N) + c(M/N)$

The length of a module which is a direct sum of two modules is the sum of the lengths of the direct summands.

Theorem 2 ([1]). Let R be a ring and M be an R-module. If M_1 and M_2 are submodules of an R module M, then $c(M_1 \oplus M_2) = c(M_1) + c(M_2)$.

Corollary 1 (Abebaw and Bøgvad (2010)). Let R be a ring and M be an R-module. If $M_1, M_2, \dots, M_n$ are submodules of an R module M, then $c(M_1 \oplus M_2 \oplus \cdots \oplus M_n) = c(M_1) + c(M_2) + \cdots + c(M_n)$.

Decomposition factors of the A_n−module $\mathbb{C}[x]_\alpha$.

Let $\alpha_1, ..., \alpha_m$ be linear forms in $\mathbb{C}[x]$. In this subsection we find the number of decomposition factors of $\mathbb{C}[x]_\alpha$, which is equivalent to analyzing expressions in partial fractions for functions in $\mathbb{C}[x]_\alpha$. Let us proceed in the following way.

To every subset

$$S = \{\alpha_{i_1}, \dots, \alpha_{i_d}\} \subset \Delta = \{\alpha_1, \dots, \alpha_m\}$$

that consists of linearly independent forms, choose coordinates $z_{d+1}, \dots, z_n$ such that $\alpha_{i_1}, \dots, \alpha_{i_d}, z_{d+1}, \dots, z_n$ are linear coordinates in space. In order to simplify the notations let us denote $\alpha_{i_k} = z_k, k = 1, 2, ..., d$.
Let $A_S = \mathbb{C}[z_{d+1}, \dots, z_n]$ be the corresponding ring of polynomials and define $R_S = \{h \in \mathbb{C}[x]_\alpha : h = \frac{g}{\prod_{j=1}^{d} z_j^{m_j}}; g \in A_S, m_j > 0, \forall j\}$. We use these modules for certain subsets S called no-broken circuits defined below.
Consider the following sequence of A_n-modules

$$0 \subset R_0 (= \mathbb{C}[x]) \subset R_1 \subset \cdots \subset R_r = \mathbb{C}[x]_\alpha,$$

where $r \leq n$ and R_k is the subspace of $\mathbb{C}[x]_\alpha$ which is generated by monomials in $x_1, ..., x_n, \alpha_1^{-1}, ..., \alpha_m^{-1}$ such that at most k of $\alpha_1, ..., \alpha_m$ have strictly negative exponents for all $k = 0, 1, ..., r$. Clearly each R_k is an A_n-submodule of $\mathbb{C}[x]_\alpha$.

The following result is the main theorem in this section.

Theorem 3. Considering the notations given above, we have

$$R_k / R_{k-1} = \oplus_W \oplus_S R_S$$

where W runs over the subspaces of dimension k generated by elements of Δ and S runs over certain subsets of k elements of Δ (the so called no-broken circuits, see definition below) which generate W.

The proof of Theorem 3 can be found in De Concini and Procesi (2006) and also in De Concini and Procesi (2010), whose exposition is what we follow to make our computations and some parts of it are indicated below.

Basic Lemma.

The following lemma is one of the basic and most powerful tools that we use to prove our main results. The proof is included as it can give us ideas how to make our computations.

Lemma 2 (De Concini and Procesi (2010)). Let $\alpha_1, \dots, \alpha_k, \alpha_{k+1}$ be non-zero linear forms with $\alpha_1 = \sum_{j=2}^{k+1} c_j \alpha_j$. Then we have

$$\frac{1}{\prod_{j=1}^{k+1} \alpha_j} = \sum_{j=2}^{k+1} c_j \frac{1}{\alpha_1^2 \prod_{i=1}^{j-1} \alpha_i \prod_{i=j+1}^{k+1} \alpha_i}$$

Proof.

$$\frac{1}{\prod_{j=1}^{k+1} \alpha_j} = \frac{\alpha_1}{\alpha_1^2 \prod_{j=2}^{k+1} \alpha_j} = \sum_{j=2}^{k+1} c_j \frac{\alpha_j}{\alpha_1^2 \prod_{j=2}^{k+1} \alpha_j}.$$

Let d be the dimension of the vector space that the non-zero linear forms $\alpha_1, \dots, \alpha_m$ generate. Then the following is established.

The following proposition gives a partial fraction expression for the expression $\frac{1}{\prod_{j=1}^{m} \alpha_i^{h_j}}$.

Proposition 2 (De Concini and Procesi(2010)). Every expression $\frac{1}{\prod_{j=1}^{m} \alpha_i^{h_j}}$ can be expressed as linear combinations of expressions $\frac{1}{\prod_{j=1}^{d} \alpha_{i_j}^{m_j}}$ with $\alpha_{i_1}, \alpha_{i_2}, \dots, \alpha_{i_d}$ linearly independent and $\sum_{j=1}^{d} m_j = \sum_{i=1}^{m} h_i$.

Proof. Using the given ordering we can take the first linearly dependent elements that appear in the product with non-zero exponents.

Using Lemma 2, we can substitute the product of these terms with a sum which results the vector of exponents is increased in the lexicographical order maintaining the same sum. In each term the space generated by the factors remains the same.

Clearly this recursive procedure terminates after a finite number of steps, when all the summands are of the required type.

No-broken circuits.

We will systemize the procedure in the proof of Proposition 2.

Definition 4. Let $\alpha_1, \ldots, \alpha_m$ be non-zero linear forms. Let $\alpha_{i_1}, \alpha_{i_2}, \ldots, \alpha_{i_h}$, $i_1 < i_2 < ... < i_h$ be an ordered sublist of linearly independent elements. We say that the sublist is a broken circuit if there exists an integer $k \leq h$ and an integer $i < i_k$ such that the elements $\alpha_i, \alpha_{i_k}, \ldots, \alpha_{i_h}$ are linearly dependent, otherwise it is called no-broken circuit.

Lemma 3 (De Concini and Procesi (2006))**.** If $\alpha_{i_1}, \alpha_{i_2}, \ldots, \alpha_{i_h}$ is a broken circuit, then $\frac{1}{\prod_{j=1}^{h} \alpha_{i_j}}$ is a linear combination of expressions $\frac{1}{\prod_{j=1}^{m} \alpha_j^{h_j}}$ with the vector of exponents lexicographically bigger than the vector of exponents of $\frac{1}{\prod_{j=1}^{h} \alpha_{i_j}}$.

Proof. From the given hypothesis we have $\alpha_i = \sum_{j=k}^{h} c_j \alpha_{i_j}$, with $i < i_k$. Let us substitute and simplify:

$$\frac{1}{\prod_{j=1}^{h} \alpha_{i_j}} = \frac{\alpha_i}{\alpha_i \prod_{j=1}^{h} \alpha_{i_j}} = \frac{c_k \alpha_{i_k} + \cdots + c_h \alpha_{i_h}}{\alpha_i \prod_{j=1}^{h} \alpha_{i_j}}$$

Simplifying every term in the numerator with the corresponding factor in the denominator we get the desired expressions.

Theorem 4 (De Concini and Procesi (2010))**.** Every expression $\frac{1}{\prod_{j=1}^{m} \alpha_j^{h_j}}$ can be expressed as a linear combination of expressions $\frac{1}{\prod_{j=1}^{d} \alpha_{i_j}^{m_j}}$, with $\alpha_{i_1}, \ldots, \alpha_{i_d}$ a no-broken circuit and $\sum_{j=1}^{d} m_j = \sum_{i=1}^{m} h_i$.

Proof. The fact that an expression of the given type can be written as a linear combination of expressions relative to no-broken circuits can be proved by induction on the lexicographic order of the vector exponents as in Proposition 2 and repeatedly using Lemma 3.

Corollary 2. The space R_S has basis the monomials $\prod_{i=1}^{n} z_i^{h_i}$ such that $h_i \geq 0$ $\forall i > d$ and $h_i < 0$ $\forall i \leq d$ and $\mathbb{C}[x]_\alpha = \sum_S R_S$ as S varies among the no-broken circuits.

Proof. The elements $z_1, z_2, \ldots, z_n$ are linear coordinates in space and R_S is contained in the ring of Laurent polynomials in these variables. These polynomials have as basis all the monomials in the variables with integer exponents. The proposed monomials are thus part of these basis and so linearly independent.
From Theorem 4 it follows immediately that every function f in R can be written as a linear combination of expressions

$$f = \frac{g}{\prod_{j=1}^{d} \alpha_{i_j}^{m_j}}$$

such that $g \in \mathbb{C}[x]$, $m_j > 0$, $\forall j$ and $S = \alpha_{i_1}, \ldots, \alpha_{i_d}$ a no-broken circuit.
We write f as a polynomial in the variables $\alpha_{i_1}, \ldots, \alpha_{i_d}, z_{d+1}, \ldots, z_n$. Simplify the α_i that appear in the numerator and the denominator. Thus with as easy induction we can prove that every element in R is a sum of elements of the spaces R_S.

The following Proposition is an immediate consequence of the above discussions.

Proposition 3 (De Concini and Procesi (2010))**.** Each R_S is irreducible A_n-module and the number of decomposition factors $\mathbb{C}[x]_\alpha$ is equal to the number of no-broken circuits.

Main Results

In this section, we describe the main results obtained in this article when $n = 1$, compute the number of decomposition factors of the A_2-module $\mathbb{C}[x]_\alpha$, where α defines a central hyperplane arrangement in the general position, using a direct sum method and compute the number of decomposition factors of the A_3-module $\mathbb{C}[x]_\alpha$, where α defines a central hyperplane arrangement using the idea of no-broken circuits. The idea of partial fractions is highly involved in our proofs.

Results in the Line and Plane Cases.

Let us start our discussion about the decomposition of the modules using Lemma 3 above for the one variable case.

Theorem 5. Let $\alpha_1 = x$, $\alpha_k = x + c_k$, for $k = 2, \ldots, m$, where $c_2, \ldots, c_m$ are nonzero distinct constants, and $\alpha = \prod_{i=1}^{m} \alpha_i$. Then the A_1-module $\mathbb{C}[x]_\alpha$ has

(i) one decomposition factor with support $\mathbb{C}$;
(ii) one decomposition factor with support the origin;
(iii) one decomposition factor with support $V(x+c_k)$ for each $k = 2, \ldots, m$ and
(iv) has $m + 1$ decomposition factors.

Proof. We have the following sequence of A_1-modules $\{0\} \subset \mathbb{C}[x] \subset \mathbb{C}[x]_\alpha$. The A_1-module $\mathbb{C}[x]_\alpha$ is a space generated by monomials in $x, \alpha_1^{-1}, ..., \alpha_m^{-1}$ such that at most one of $\alpha_1, ..., \alpha_m$ has strictly negative exponent. The main important thing in this case is that, using the idea of partial fraction we can write $\frac{1}{\alpha}$ as a linear combination of $\frac{1}{\alpha_k}$ for $k = 1, \ldots, m$ and we have the following isomorphism of A_1-modules

$$\mathbb{C}[x]_\alpha/\mathbb{C}[x] \cong \bigoplus_{k=1}^{m} R_{S_k},$$

where R_{S_k} is the A_1-module generated by $e_{S_k} = \frac{1}{\alpha_k}(\mathrm{mod}\mathbb{C}[x])$ for $k = 1, \ldots, m$.

Then $\mathbb{C}[x]$ and R_{S_k} for $k = 1, \ldots, m$ are all simple A_1-modules and hence

(i) $\mathbb{C}[x] \cong \mathbb{C}[x]/\{0\}$ is a decomposition factor of $\mathbb{C}[x]_\alpha$ with support $V(0) = \mathbb{C}$;
(ii) R_{S_1} is a decomposition factor for $\mathbb{C}[x]_\alpha$ with support $V(x)$;
(iii) R_{S_k} is a decomposition factor for $\mathbb{C}[x]_\alpha$ with support $V(x + c_k)$ for $k = 2, \ldots, m$ and
(iv) The number of decomposition factors of $\mathbb{C}[x]_\alpha$ is given by

$$\begin{aligned} c(\mathbb{C}[x]_\alpha) &= c(\mathbb{C}[x]) + c(\mathbb{C}[x]_\alpha/\mathbb{C}[x]) \\ &= 1 + \sum_{k=1}^{m} c(R_{S_k}) = 1 + m. \end{aligned}$$

That is, $c(\mathbb{C}[x]_\alpha) = m+1$ and this completes the proof.

The set of no-broken circuits of given linear forms is $\{\{\}, x, x + c_2, \ldots, x + c_m\}$. The cardinality of this set is $m + 1$ and this is the same as number of decomposition factors of $\mathbb{C}[x]_\alpha$.

Theorem 6. Let $\alpha_1 = x, \alpha_2 = y, \alpha_k = x + c_k y$ for $k = 3, \ldots, m$,where the c_k's are distinct nonzero constants in $\mathbb{C}$ and $\alpha = \prod_{i=1}^{m} \alpha_i$. Then the A_2-module $\mathbb{C}[x, y]_\alpha$ has

(i) one decomposition factor with support $\mathbb{C}^2$;
(ii) one decomposition factor with support the hyperplane $V(\alpha_k)$ for each $k = 1, \ldots, m$;
(iii) $m - 1$ decomposition factors with each has support the origin and
(iv) $2m$ decomposition factors.

Proof. Consider the sequence of A_2-modules, $R_0 \subset R_1 \subset R_2$, where $R_0 = \mathbb{C}[x, y]$, R_1 is the subspace of $\mathbb{C}[x, y]_\alpha$ generated by monomials in $x, y, \alpha_1^{-1}, ..., \alpha_m^{-1}$ such that at most one of $\alpha_1, ..., \alpha_m$ has strictly negative exponent and $R_2 = \mathbb{C}[x, y]_\alpha$. Then

(i) $\mathbb{C}[x, y] \cong \mathbb{C}[x, y]/\{0\}$ is a decomposition factor for $\mathbb{C}[x, y]_\alpha$ with support $\mathbb{C}^2$;
(ii) we have the following isomorphism of A_2-modules $R_1/R_0 \cong \bigoplus_{k=1}^{m} R_{S_k}$, where R_{S_k} is the simple A_2-module generated by $e_{S_k} = \frac{1}{\alpha_k}$ for $k = 1, \ldots, m$. Thus, R_{S_k} is a decomposition factor of $\mathbb{C}[x, y]_\alpha$ with support $V(\alpha_k)$ for $k = 1, \ldots, m$ and
(iii) the quotient R_2/R_1 is given by $R_2/R_1 \cong \oplus_{k=2}^{m} R_{T_k}$, where R_{T_k} is the simple

A_2-module generate by $e_{T_k} = \frac{1}{\alpha_1\alpha_k}(\text{mod}R_1)$ and R_{T_k} is a decomposition factor with support $V(\alpha_1, \alpha_k) = \{(0,0)\}$ for $k = 2, \ldots, m$. The most important thing in this case is that, using the idea of partial fractions, we can write $\frac{1}{\alpha}$ as a linear combination of $\frac{1}{\alpha_1\alpha_k}$ for $k = 2, \ldots, m$.

(iv) The number of decomposition factors of $\mathbb{C}[x,y]_\alpha$ is given by,

$$\begin{aligned} c(\mathbb{C}[x,y]_\alpha) &= c(R_0) + c(R_1/R_0) + c(R_2/R_1) \\ &= 1 + \sum_{k=1}^{m} c(R_{S_k}) + \sum_{k=2}^{m} c(R_{T_k}) \\ &= 1 + m + m - 1 = 2m. \end{aligned}$$

That is, $c(\mathbb{C}[x,y]_\alpha) = 2m$.

The set of no-broken circuits defined by α is $\{\{\}, \alpha_1, \ldots, \alpha_m, \alpha_1\alpha_2, \ldots, \alpha_1\alpha_m\}$ and the cardinality of this set is $2m$ which is equal to the number of decomposition factors of $\mathbb{C}[x,y]_\alpha$.

Results in the Space Case.

The main results of the work are on the lengths and decomposition factors of D-modules. Our main results are based on the results in Abebaw and Bógvad (2010) and the idea of partial fractions is heavily involved.

Theorem 7. Let $\alpha_1 = x, \alpha_2 = y, \alpha_3 = z, \alpha_i = x + y + a_i z$ for $i = 4, \ldots, m$ be nonzero district linear forms with nonzero distinct constants a_i in $\mathbb{C}$ for $i = 4, \ldots, m$ and $\alpha = \alpha_1 \cdots \alpha_m$. Then the number of decomposition factors of the A_3-module $\mathbb{C}[x,y,z]_\alpha$ is $m^2 - m + 2$.

Proof. Consider the following sequences of A_3-modules $R_0 \subset R_1 \subset R_2 \subset R_3$, where $R_0 = \mathbb{C}[x,y,z]$, R_1 is the subspace of $\mathbb{C}[x,y,z]_\alpha$ generated by monomials in $x, y, z, \alpha_1^{-1}, ..., \alpha_m^{-1}$ such that at most one of $\alpha_1, ..., \alpha_m$ has strictly negative exponent, R_2 is the subspace of $\mathbb{C}[x,y,z]_\alpha$ generated by monomials in $x, y, z, \alpha_1^{-1}, ..., \alpha_m^{-1}$ such that at most two of $\alpha_1, ..., \alpha_m$ has strictly negative exponents and $R_3 = \mathbb{C}[x,y,z]_\alpha$. Then we have the following isomorphism of A_3-modules.

(a) The module R_1/R_0 is given as a direct sum by: $R_1/R_0 \cong \bigoplus_{k=1}^m R_{S_k}$, where R_{S_k} is the simple A_3-module generate by $e_{S_k} = \frac{1}{\alpha_k}$ for $k = 1, \ldots, m$.

(b) We also have the following quotient module.

$$\begin{aligned} R_2/R_1 \quad &\cong \quad (\oplus_{k=2}^m R_{S_k}) \\ & \quad \bigoplus (\oplus_{k=3}^m R_{T_k}) \\ & \quad \bigoplus (\oplus_{k=4}^m R_{U_k}) \bigoplus N \end{aligned}$$

$$\begin{aligned} \text{where } N \quad = \quad & (\oplus_{j=5}^m R_{U_j^4}) \bigoplus \\ & (\oplus_{j=6}^m R_{U_j^5}) \bigoplus \cdots \bigoplus R_{U(m-1)}. \end{aligned}$$

(1) the module R_{S_k} is the simple A_3-module generate by $e_{S_k} = \frac{1}{\alpha_1\alpha_k}(\text{mod}R_1)$ for $k = 2, \ldots, m$;

(2) the module R_{T_k} is the simple A_3-module generate by $e_{U_k} = \frac{1}{\alpha_2\alpha_k}(\text{mod}R_1)$ for $k = 3, \ldots, m$;

(3) the module R_{U_k} is the simple A_3-module generate by $e_{U_k} = \frac{1}{\alpha_3\alpha_k}(\text{mod}R_1)$ for $k = 4, \ldots, m$;

(4) the module $R_{U_j^4}$ is a simple A_3-module generated by $e_{U_j^5} = \frac{1}{\alpha_4\alpha_j}(\text{mod}R_1)$ for $j = 5, \ldots, m$.

$\vdots$

(m-1) the module $R_{U(m-1)}$ is a simple A_3-module generated by

$$e_{U(m-1)} = \frac{1}{\alpha_{(m-1)}\alpha_m}(\text{mod}R_1).$$

(c) Consider the quotient A_3-module R_3/R_2. Then we have the following isomorphism of A_3-modules,

$$R_3/R_2 \cong \bigoplus (\oplus_{j=3}^m R_{S_j}) \bigoplus (\oplus_{j=4}^m R_{T_j}) \bigoplus M,$$

where

$$M = \bigoplus_{j=5}^{m} R_{U_j^4} \bigoplus_{j=6}^{m} R_{U_j^5} \bigoplus \cdots \bigoplus R_{U(m-1)}$$

and have the following simple A_3-modules:

(1) the module R_{S_j} is a simple A_3-module generated by $e_{S_j} = \frac{1}{\alpha_1\alpha_2\alpha_j}(\text{mod}R_2)$ for each $j = 3, \ldots, m$;

(2) the module R_{T_j} is a simple A_3-module generated by $e_{T_j} = \frac{1}{\alpha_1\alpha_3\alpha_j}(\text{mod}R_2)$ for each $j = 4, \ldots, m$ and also

(3) the module$R_{U_j^4}$ is a simple A_3-module generated by $e_{U_j^4} = \frac{1}{\alpha_1\alpha_4\alpha_j}(\text{mod}R_2)$ for $j = 5, \ldots, m$.

(4) the module $R_{U_j^5}$ is a simple A_3-module generated by $e_{U_j^5} = \frac{1}{\alpha_1\alpha_5\alpha_j}(\text{mod}R_2)$ for $j = 6, \ldots, m$.

$\vdots$

(m-2) the module $R_{U(m-1)}$ is a simple A_3-module generated by

$$e_{U(m-1)} = \frac{1}{\alpha_1\alpha_{(m-1)}\alpha_m}(\text{mod}R_2).$$

The most important thing in this case is that, using the idea of partial fractions, we can write $\frac{1}{\alpha}$ as a linear combination of the fractions $\frac{1}{xyz}$, $\frac{1}{xy\alpha_i}$, $\frac{1}{xz\alpha_i}$ and $\frac{1}{x\alpha_i\alpha_j}$, for $i \neq j$ and all $i, j = 4, \ldots, m$.

Then the number of decomposition factors of the A_3-module $\mathbb{C}[x, y, z]_\alpha$ is given by

$$\begin{aligned} c(\mathbb{C}[x, y, z]_\alpha) &= c(R_0) + c(R_1/R_0) \\ &\quad + c(R_2/R_1) + c(R_3/R_2). \end{aligned}$$

So we have $c(R_0) = 1$, $c(R_1/R_0) = m$, $c(R_2/R_1) = (m-1) + (m-2) + \cdots + 1 = \frac{m(m-1)}{2}$ and

$$\begin{aligned} c(R_3/R_2) &= (m-2) + (m-3) + \cdots + 1 \\ &= \frac{(m-1)(m-2)}{2}. \end{aligned}$$

Thus, we have

$$\begin{aligned} c(\mathbb{C}[x, y, z]_\alpha) &= 1 + m + \frac{m(m-1)}{2} + \\ &\quad \frac{(m-1)(m-2)}{2} \\ &= m^2 - m + 2. \end{aligned}$$

That is, $c(\mathbb{C}[x, y, z]_\alpha) = m^2 - m + 2$ and one can easily see that this is the same as the number of no-broken circuits defined by α.

Conclusions

We have computed the number of decomposition factors of an A_3-module defined by a cental hyperplane in space. In the computation, some combinatorics is observed and it is still open to describe the combinatorics involved. One may also study the geometry involved in the support of such decomposition factors.

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