# SOME RESULTS ON THE COMMUTATIVITY OF PRIME NEAR-RINGS USING GENERALIZED DERIVATIONS

## Tadesse Wubshet[1] and Tilahun Abebaw[2*]

[1]Department of Mathematics, Woldia University, Ethiopia. E-mail: tadessewebeshet@gmail.com

[2]Department of Mathematics, Addis Ababa University, Ethiopia. E-mail: tilahun.abebaw@aau.edu.et

**ABSTRACT:** Given a right near ring N, an additive mapping D:N $\longrightarrow$ N is said to be a derivation on N, if $D(xy) = D(x)y + xD(y)$ for all $x, y \in$N and an additive mapping $F : N \longrightarrow N$ satisfying $F(xy) = F(x)y + xD(y)$ for all $x, y \in$N, is called generalized derivation on N associated with the derivation D. The aim of this paper is to study the commutativity of a near-ring using some properties of generalized derivations on the given near-ring and we proved the following results. (a) The commutativity of 3-torsion free prime near-ring N with a generalized derivation F associated with non-zero idempotent derivation D on N satisfying the conditions $F^2[x, y] - [x, y] = 0$ for all $x, y \in N$ and $F^2(xoy) - (xoy) = 0$ for all $x, y \in N$ and (b) commutativity of 5-torsion free prime near ring N with a generalized derivation F associated with non-zero idempotent derivation D on N satisfying the conditions $F^2[x, y] + [x, y] = 0$ for all $x, y \in N$ and $F^2(xoy) + (xoy) = 0$ for all $x, y \in N$ are proved in this article. These results may help us to study more about the commutativity of general near-rings.

**Keywords/phrases:** commutativity, derivation, 5-torsion free, generalized derivation, near-ring, prime near-ring, 3-torsion free.

## INTRODUCTION

Posner (1957) started to study about derivations in rings and proved the existence of a non-zero centralizing derivation on a prime ring that forces the given ring to be commutative. Since then, several authors have studied about derivations on rings (see Bell and Kappe (1989) and Rahman (2002)) and Bell and Daif (1985) studied about the derivations and commutativity of prime rings. Bell (1997) studied about derivations in near-rings and Jamal (2008) in his MPhil Thesis studied about derivations in near-rings. The idea of generalized derivation on rings was introduced by Hvala (1998) and several authors have studied about generalized derivations in near-rings and prime near-rings and commutativity of near-rings using generalized derivations (see Gölbaşi (2006), Boua and Oukhtite (2011),

Ali *et al.* (2013), Khan Hasnain (2013) and also Satyanarayana and Prasad (2013)). Boua *et al.* (2018) also studied about derivations and generalized derivations satisfying certain differential identities on Jordan ideals and Lie ideals of 3-prime near-rings. The commutativity of a 2,3 – torsion free prime near ring N which admits a generalized derivation F with some conditions on F was proved by Reddy *et al.* (2016).

In this paper, N will denote a zero symmetric right abelian near-ring and $Z(N)$ denotes a multiplicative center for N. For all $x, y \in N$ as usual $[x, y] = xy - yx$, called the Lie product of $x$ and $y$ and also $xoy = xy + yx$, called Jordan product of $x$ and $y$. A near-ring N is called prime, if $xNy = \{xny : n \in N\} = \{0\}$ for $x, y \in N$, then $x = 0$ or $y = 0$. We refer to Satyanarayana and Prasad (2013) for the basic definitions and properties of near-ring.

---

*Author to whom correspondence should be addressed.

An additive mapping D:N⟶N is called a derivation of N if $D(xy) = xD(y) + D(x)y$ holds for all $x, y \in$ N and an additive mapping F:N⟶N is said to be

(1) a right generalized derivation associated with a derivation D of N if $F(xy) = F(x)y + xD(y)$ and

(2) a left generalized derivation associated with a derivation D of N if $F(xy) = xF(y) + D(x)y$.

A generalized derivation F is a generalized derivation associated with a derivation D if it is a right as well as a left generalized derivation associated with a derivation D.

The purpose of this work is to prove some results about the commutativity of near-rings using conditions on generalized derivations of near-rings that are proposed by Khan and Hasnain (2013).

## Preliminaries

For the purpose of our works, let us start our discussions by giving formal definitions of near-rings, derivations of near-rings and generalized derivations on near-rings. Detail works and some results can be found in the works of several authors and some of these are: Gölbaşi (2006), Boua and Oukhtite (2011), Ali *et al.* (2013), Khan and Hasnain (2013) and Satyanarayana and Prasad (2013).

The following definition is Definition 2.1.1 in Satyanarayana and Prasad (2013).

**Definition 1.** A non-empty set N together with two binary operations $''+''$, called addition and $''.''$, called multiplication, is said to be a right (resp. left) near-ring if it satisfies the following.

(1) $(N, +)$ is a group;
(2) $(N, .)$ is a semigroup and
(3) $''.''$ is right (resp. left) distributive over $''+''$, that is, for all $x, y, z \in N$, $(x+y)z = xz + yz$ (resp. $x(y+z) = xy + xz$ for all $x, y, z \in N$..

In our discussions, by a near-ring we mean right near-ring, unless stated otherwise. Now, let us define different types of near-rings and study their properties.

The following definitions are Definition 2.1.10 and Definition 2.1.13 in Satyanarayana and Prasad (2013).

**Definition 2.** Let $(N, +, .)$ be a near-ring. Then N is said to be

(a) prime, if $nNm = \{0\}$ for $n, m \in$N, then $n = 0$ or $m = 0$, where $nNm = \{ntm : t \in N\}$;
(b) abelian, if $n + m = m + n$ for all $n, m \in N$.
(c) an $n-$torsion free for some positive integer $n$, if $nt = 0$ for $t \in N$, then $t = 0$.
(d) The set $N_0 = \{n \in N : n0 = 0\}$ is called the zero-symmetric part of N and a near-ring N is called a zero symmetric near ring, if $N_0 = $ N.

**Theorem 1.** Let $(N, +, .)$ be a near-ring with identity. If $n(-1) = -n$ for all $n \in$N, then $(N, +)$ is abelian.

*Proof.* Let (N,+,.) be a near-ring with identity 1. Suppose $n(-1) = -n$ for all $n \in N$. We want to show that for any $n, m \in N$, $n + m = m + n$. Let $n, m \in N$. Then $n + m - m - n = 0$. This implies $(n + m) + (m(-1) + n(-1)) = 0$.
By the same reasoning, we then have that

$$(n + m) + (m + n)(-1) = 0$$

and hence $(n + m) - (m + n) = 0$. This implies $n + m = m + n$ and then N is abelian.

Our main objective in this article is to study about the commutativity of near-rings by using generalized derivations. For these purposes, let us revise some concepts about derivations and generalized derivations on near-rings.

### Derivations in Near-Rings.

Some basic results concerning the study of derivations in near-rings can be obtained from the results studied in the works by Gölbaşi (2006) and Jamal (2008).

**Definition 3.** Let $(N, +, .)$ be a near-ring. An additive mapping D:N⟶N is said to be a derivation on N if $D(xy) = D(x)y + xD(y)$ for all $x, y \in$N and a derivation D is called an idempotent derivation if $D^2 = D$

**Definition 4.** Let $(N, +, .)$ be near ring. The set $\{x \in N : xn = nx \text{ for all } n \in N\}$ is called the center of N denoted by $Z(N)$.

The following are results that are proved by several authors about commutativity of near-rings and derivations on near-rings.

**Lemma 1** (Ali *et al.* (2013), Lemma 1)**.** Let $(N, +, .)$ be a prime near ring. If N admits a non-zero derivation D for which $D(N) \subseteq Z(N)$, then $N$ is a commutative ring.

**Theorem 2** (Gölbaşi (2006), Lemma 3)**.** Let $(N, +, .)$ be a prime near-ring. If N admits a non zero derivation D such that for all $x, y \in N$, $D([x, y]) = 0$, then N commutative.

**Theorem 3** (Gölbaşi (2006), Theorem 1)**.** Let $(N, +, .)$ be prime near-ring which admits a nonzero derivation $D$. Then the following assertions are equivalent.

(i) $D[x, y] = [D(x), y]$ for all $x, y \in$ N.
(ii) $[D(x), y] = [x, y]$ for all $x, y \in N$.
(iii) N is a commutative ring.

**Theorem 4** (Rahman (2002), Theorem 2)**.** Let $(N, +, .)$ be a 2-torsion free prime near-ring which admits a non-zero derivation D. Then the following assertion are equivalent.

(i) $D([x, y]) \in Z(N)$ for all $x, y \in N$.
(ii) $N$ is a commutative ring .

**Lemma 2** (Rahman (2002), Lemma 2.2)**.** Let $(N, +, .)$ be a prime near-ring.

(i) If $z \in Z(N) \backslash \{0\}$, then $z$ is not a zero divisor.
(ii) If $Z(N)$ contains a non-zero element $z$ for which $z + z \in Z(N)$, then $(N, +)$ is abelian.
(iii) Let D be a non-zero derivation on N. Then $x\mathrm{D}(\mathrm{N}) = \{0\}$ implies $x = 0$ and $\mathrm{D}(\mathrm{N})x = \{0\}$ implies $x = 0$.
(iv) If N is 2-torsion free and $D$ is a derivation on $N$ such that $D^2 = 0$, then $D = 0$.

**Generalized Derivations and Commutativity of Prime Near-Rings.**

The concept of generalized derivation was introduced in Hvala (1998). In this section N will denote a zero symmetric abelian near-ring.

**Definition 5.** Let $(N, +, .)$ be a near-ring. An additive mapping F:N⟶N is said to be

(a) a right generalized derivation associated with a derivation D, if there exists a derivation D on N and $F(xy) = F(x)y + xD(y)$ for all $x, y \in$N.
(b) a left generalized derivation associated with derivation D, if there exists a derivation D on N and $F(xy) = xF(y) + D(x)y$ for all $x, y \in N$.
(c) a generalized derivation associated with a derivation D, if there exists a derivation D on N and F is both right and left generalized derivation associated with the derivation D.

**Definition 6.** Let $(N, +, .)$ be a near-ring. An additive mapping g:N⟶N satisfying $g(xy) = g(x)y$ is called left multiplier and every left multiplier is a generalized derivation.

In the following two lemmas, we give some basic properties of derivations and generalized derivations on near rings which we will use them in our proofs of the main results of this article.

**Lemma 3.** Let $(N, +, .)$ be a prime near-ring and D be a derivation of $N$. Then for all $x, y \in$N,

(a) if $D([x, y]) = [x, y]$, then $D([x, yx]) = D([x, y])x$;
(b) If $D(xoy) = xoy$, then $D(xo(yx)) = D(xoy)x$.

*Proof.* Let us prove (a) and then (b) can be proved similarly.

Suppose $D([x, y]) = [x, y]$ for $x, y \in N$. By definition, $D([x, yx]) = x(yx) - (yx)x$. This implies $(xy)x - (yx)x = (xy - yx)x$. Then $D([x, yx]) = [x, y]x = D([x, y])x$.

**Lemma 4.** Let $(N, +, .)$ be a near-ring and F be a generalized derivation associated with a derivation D of $N$. Then for all $x, y \in N$,

(a) if $F([x, y]) = [x, y]$, then $F([x, yx]) = F([x, y])x$;
(b) if $F(xoy) = xoy$, then $F(xo(yx)) = F(xoy)x$.

*Proof.* Let us prove (a) and then (b) can be proved similarly.

Suppose $F([x, y]) = [x, y]$ for $x, y \in N$. By definition, $F([x, yx]) = x(yx) - (yx)x$. This implies $(xy)x - (yx)x = (xy - yx)x$ and hence $F([x, yx]) = [x, y]x = F([x, y])x$.

In order to prove our main results in this article, we will make extensive use of the following results obtained from the studies by Jamal (2008) and also results obtained from the studies by Khan and Hasnian (2013).

**Definition 7.** Let $(N, +, .)$ be a near-ring a nonempty subset U of N is called a semigroup right ideal (respectively semigroup left ideal) of N if $UN \subset U$ (respectively $NU \subset U$).

**Remark 1.** Let $(N, +, .)$ be a near-ring. A nonempty subset U of N is called a semigroup ideal if it is both a semigroup right ideal and a semigroup left ideal.

**Lemma 5** (Khan and Hasnian(2013), Lemma B). Let $(N, +, .)$ be a prime near-ring and $U \neq \{0\}$ be a semigroup ideal of $N$. If $U \subseteq Z(N)$, then $N$ is commutative.

**Lemma 6** (Khan and Hasnian (2013), Lemma C). Let $(N, +, .)$ be a prime near-ring and let $U \neq \{0\}$ be a semigroup ideal of $N$. If $m \in N$ such that $mU = 0$ or $Um = 0$, then $m = 0$.

**Lemma 7** (Jamal (2008), Lemma 3). Let $(N, +, .)$ be a prime near-ring and let $U \neq \{0\}$ be a semigroup ideal of $N$. If $n, m$ are elements of $N$ such that $mUn = 0$, then $m = 0$ or $n = 0$.

**Lemma 8** (Khan and Hasnian (2013), Lemma D). Let $(N, +, .)$ be a prime near-ring and $U \neq \{0\}$ a semigroup ideal of $N$. If $D$ is a derivation on $N$ such that $D(U) = 0$, then $D = 0$.

**Lemma 9** (Khan and Hasnian (2013)). Let $(N, +, .)$ be a prime near-ring, U$\neq \{0\}$ be a semigroup ideal of N and F be a non-zero generalized derivation such that F is left multiplier. If $F(x) = x$ for all $x \in$U then $F(n) = n$ for all $n \in$N.

**Theorem 5** (Khan and Hasnian(2013)). Let $(N, +, .)$ be a non-commutative prime near-ring, $U$ be a non-zero semigroup ideal of $N$ and $F \neq 0$ be a generalized derivation associated with $D$. If $F[x, y] - [x, y] = 0$ for all $x, y \in U$, then $F$ is the identity mapping on N.

**Corollary 1** (Khan and Hasnian (2013)). Let $(N, +, .)$ is a prime near-ring.

(a) Let $U$ be a nonzero semigroup ideal of $N$. If $N$ admits a generalized derivation $F$ associated with a derivation $D$ such that $F[x, y] - [x, y] = 0$ for all $x, y \in U$, then $N$ is commutative or $F$ is identity map.

(b) If $N$ admits a generalized derivation $F$ associated with a nonzero derivation $D$ such that $F[x, y] - [x, y] = 0$ for all $x, y \in N$, then $N$ is commutative.

**Theorem 6** (Khan and Hasnian (2013)). Let $(N, +, .)$ be a non commutative prime near-ring, $U$ be a nonzero semigroup ideal of $N$ and $F \neq 0$ a generalized derivation associated with a derivation $D$ on $N$ such that $F[x, y] + [x, y] = 0$ for all $x, y \in U$, then $F(n) = -n$ for all $n \in N$.

**Corollary 2** (Khan and Hasnian (2013)).

(i) Suppose $(N, +, .)$ is a prime near-ring and $U$ a nonzero semigroup ideal of $N$. If $N$ admits a generalized derivation $F$ associated with a derivation $D$ on N such that $F[x, y] + [x, y] = 0$ for all $x, y \in$U , then $N$ is commutative or $F(n) = -n$.

(ii) Suppose $(N, +, .)$ is a prime near-ring and If $N$ admits a generalized derivation $F$ associated with $D \neq 0$ such that $F[x, y] + [x, y] = 0$ for all $x, y \in N$,then $N$ is commutative.

**Theorem 7** (Khan and Hasnian (2013), Theorem 3). Let $(N, +, .)$ be a noncommutative prime near-ring, $U$ be a nonzero semigroup ideal of $N$ and $F \neq 0$ a generalized derivation associated with $D$ of $N$. If $n(-1) = -n$ for all $n \in N$ and $F(xoy) - (xoy) = 0$ for all $x, y \in U$, then $F$ is identity map.

The following results are about the commutativity of near-rings involving generalized derivations satisfying certain conditions they can be proved using Theorem 7.

**Corollary 3** (Khan and Hasnian (2013)). Let $(N, +, .)$ is a prime near-ring.

(a) Let U be a nonzero semigroup ideal of N. If N admits a generalized derivation F associated

with D such that $n(-1) = -n$ for all $n \in$N and $F(xoy) - (xoy) = 0$ for all $x, y \in$U, then N is commutative or F is an identity map.

(b) Let F be a generalized derivation on N associated with a non-zero derivation D on N such that $n(-1) = -n$ for all $n \in$N and $F(xoy) - (xoy) = 0$ for all $x, y \in$N. Then N is commutative.

**Theorem 8** (Khan and Hasnian (2013)). Let $(N, +, .)$ be a non-commutative prime near-ring, $U$ be a nonzero semigroup ideal of N and $F \neq 0$ a generalized derivation associated with $D$ of $N$. If $n(-1) = -n$ for all $n \in N$ and $F(xoy) + (xoy) = 0$ for all $x, y \in U$, then $F(n) = -n$ for all $n \in N$.

The following results are about the commutativity of near-rings involving generalized derivations satisfying certain conditions they can be proved using Theorem 8.

**Corollary 4** (Khan and Hasnian (2013)). Let $(N, +, .)$ be a prime near-ring.

(a) If U is a non-zero semigroup ideal of N and if N admits a generalized derivation F associated with D such that $n(-1) = -n$ for all $n \in$N and $F(xoy) + (xoy) = 0$ for all $x, y \in$U, then N is commutative or $F(n) = -n$ for all $n \in$N.

(b) If $N$ admits a generalized derivation $F$ associated with $D \neq 0$ such that $n(-1) = -n$ for all $n \in N$ and $F(xoy) + (xoy) = 0$ for all $x, y \in N$, then $N$ is commutative.

### Main Results

In this section, the main results of this article are given and these results are based on the results in Khan and Hasnain (2013) about the commutativity of near-rings involving powers of generalized derivations associated with idempotent derivations.

**Theorem 9.** Let $(N, +, .)$ be 3-torsion free prime near-ring with a generalized derivation $F$ associated with a non-zero idempotent derivation $D$ of N. If $F^2[x, y] - [x, y] = 0$ for all $x, y \in N$, then $N$ is commutative.

*Proof.* Let $(N, +, .)$ be a 3-torsion free prime near-ring which admits a generalized derivation F associated with a nonzero idempotent derivation D of N.

Suppose $F^2[x, y] = [x, y]$ for $x, y \in N$. Now, replacing $y$ by $yx$ gives us

$$F^2[x, yx] = F^2[x, y]x, \qquad (0.1)$$

since $[x, yx] = [x, y]x$ and from the right side of Equation (0.1) we have

$$F^2[x, yx] = F^2[x, y]x + F[x, y]D(x) \qquad (0.2)$$
$$+ F[x, y]D(x) + [x, y]D^2(x).$$

Using the relation $F^2[x, yx] = F^2[x, y]x$ and Equation (0.2), we have the identity.

$$F[x, y]D(x) + F[x, y]D(x) + [x, y]D^2(x) = 0. \quad (0.3)$$

On the other hand, the derivation D is idempotent, which implies that Equation (0.3) becomes $F[x, y]D(x) + F[x, y]D(x) + [x, y]D(x) = 0$ and using the right distributive property of multiplication on addition we have the following equation.

$$(F[x, y] + F[x, y] + [x, y])D(x) = 0 \qquad (0.4)$$

Applying F on both sides of Equation (0.4), since D is idempotent and using Equation (0.4) once again, we have the identity,

$$[x, y]D(x) + [x, y]D(x) + F[x, y]D(x) = 0. \quad (0.5)$$

Now, equating Equation (0.4) with Equation(0.5) and simplifying the resulting expression, we have the following expression,

$$F[x, y]D(x) - [x, y]D(x) = 0 \text{ for all } x, y \in N. \quad (0.6)$$

Now, equating Equations (0.5) and (0.6) since both of them are equal to zero for all $x, y \in N$ and simplifying the expression gives us $3[x, y]D(x) = 0$. But from our assumption, N is a 3-torsion free prime near ring. This implies $[x, y]D(x) = 0$ and thus

$$xyD(x) = yxD(x). \qquad (0.7)$$

Now replacing $y$ by $yz$ in Equation (0.7) gives us $xyzD(x) = yzxD(x) = yxzD(x)$ and hence $[x, y]zD(x) = 0$ for all $x, y, z \in N$. That is, $[x, y]ND(x) = 0$ for all $x, y \in$N.

From the primeness of N and because D is a non-zero derivation on N, we have $[x, y] = 0$ for

all $x, y \in N$ and hence $xy = yx$ for all $x, y \in N$. That is, N is commutative and this completes the proof of the theorem.

**Theorem 10.** Let $(N, +, .)$ be a 3-torsion free prime near-ring with identity with a generalized derivation $F$ associated with non-zero idempotent derivation $D$. If $n(-1) = -n$ for all $n \in N$ and $F^2(xoy) - (xoy) = 0$ for all $x, y \in N$, then $N$ is commutative.

*Proof.* Let $(N, +, .)$ be a 3-torsion free prime near-ring with identity and F be generalized derivation associated with non-zero idempotent derivation $D$ on $N$.

Suppose $F^2(xoy) - (xoy) = 0$ for all $x, y \in N$, that is, $F^2(xoy) = (xoy)$.

Now, if we replace $y$ by $yx$, then we get $F^2(xo(yx)) = (xoy)x$.

On the other hand,

$$
\begin{aligned}
F^2(xoyx) &= F^2(xoy)x \\
&\quad + F(xoy)D(x) \\
&\quad + F(xoy)D(x) \\
&\quad + (xoy)D^2(x)
\end{aligned}
$$
(0.8)

and since D is idempotent, we have $F(xoy)D(x) + F(xoy)D(x) + (xoy)D(x) = 0$.

By right distributivity of multiplication on addition we have,

$$(F(xoy) + F(xoy) + (xoy))D(x) = 0. \quad (0.9)$$

Applying F on Equation (0.9) and since D is idempotent and using the assumption, that $F^2(xoy) = xoy$, we can easily get

$$((xoy) + (xoy) + F(xoy))D(x) +$$
$$(F(xoy) + F(xoy) + (xoy))D(x) = 0.$$

Substituting identity in Equation (0.9) in this equation gives us the identity

$$((xoy) + (xoy) + F(xoy))D(x) = 0. \quad (0.10)$$

Since both Equation (0.9) and Equation (0.10) are identities for all $x, y \in N$, after equating these equations and simplifying, for all $x, y \in N$ one can get

$$F(xoy)D(x) - (xoy)D(x) = 0. \quad (0.11)$$

Similarly, both Equation (0.10) and Equation (0.11) are identities for all $x, y \in N$, equating these equations and simplifying we can get

$$[(xoy) + (xoy) + (xoy)]D(x) = 3(xoy)D(x) = 0$$

for all $x, y \in N$. By assumption, N is a 3-torsion free prime near-ring. This implies $(xoy)D(x) = 0$, that is,

$$xyD(x) = -yxD(x). \quad (0.12)$$

Replacing $y$ by $yz$ in Equation (0.12) and using the assumption that $n(-1) = -n$ for all $n \in N$, we have $xyzD(x) = -yzxD(x) = yxzD(x)$ and hence $[x, y]zD(x) = 0$ for $x, y, z \in N$. This implies $[x, y]ND(x) = 0$ for $x, y \in N$. By the primeness of N and since D is a non-zero derivation on N, we have $[x, y] = 0$ for all $x, y \in N$. That is $xy = yx$ for all $x, y \in N$ and hence N is commutative.

The next results are on the commutativity of 5-torsion free prime near-rings.

**Theorem 11.** Let $(N, +, .)$ be a 5-torsion free prime near-ring with a generalized derivation $F$ associated with non-zero idempotent derivation $D$. If $F^2[x, y] + [x, y] = 0$ for all $x, y \in N$, then $N$ is commutative.

*Proof.* Let $(N, +, .)$ be a 5-torsion free prime near-ring and F be a generalized derivation on N associated with a non-zero idempotent derivation D.

Suppose $F^2[x, y] = -[x, y]$ for $x, y \in N$. Now, since $[x, yx] = [x, y]x)$, replacing $y$ by $yx$ gives us

$$F^2[x, yx] = -[x, y]x, \quad (0.13)$$

and we also have that

$$
\begin{aligned}
F^2[x, yx] &= F^2[x, y]x + F[x, y]D(x) \\
&\quad + F[x, y]D(x) + [x, y]D^2(x)
\end{aligned}
$$

and D idempotent. This implies,

$$(F[x, y] + F[x, y] + [x, y])D(x) = 0. \quad (0.14)$$

Applying F on both sides of Equation (0.14), using the assumption $F^2[x, y] = -[x, y]$ we get the following.

$$(-[x, y] - [x, y] + F[x, y])D(x) +$$
$$(F[x, y] + F[x, y] + [x, y])D^2(x) = 0. \quad (0.15)$$

But D is idempotent and using Equation (0.14), we have the identity

$$(-[x,y] - [x,y] + F[x,y])D(x) = 0,$$

which implies

$$F[x,y]D(x) = ([x,y] + [x,y])D(x). \qquad (0.16)$$

Applying F on both sides of Equation (0.16), D is idempotent and $F^2[x,y] = -[x,y]$ and also simplifying these expressions give us

$$\begin{aligned} F[x,y]D(x) &= [x,y]D(x) - [x,y]D(x) \\ &\quad -[x,y]D(x). \qquad (0.17) \end{aligned}$$

Equating Equation (0.16) and Equation (0.17) and simplifying the given equation, we can get the following equation

$$5[x,y]D(x) = 0 \text{ for all } x, y \in N. \qquad (0.18)$$

Since N is a 5-torsion free near-ring, we have that $[x,y]D(x) = 0$, which implies

$$xyD(x) = yxD(x). \qquad (0.19)$$

Now, replacing $y$ by $yz$ in Equation (0.19) gives us $xyzD(x) = yzxD(x)$. Again by using Equation (0.19) we have that $xyzD(x) = yxzD(x)$ which implies that

$$[x,y]zD(x) = 0 \text{ for all } x, y, z \in N.$$

This implies $[x,y]ND(x) = 0$ for all $x, y \in N$. From the primeness of N and since D is a non-zero derivation on N, we have $[x,y] = 0$ for all $x, y \in N$ and hence $xy = yx$ for all $x, y \in N$. That is, N is commutative and this completes the proof of the theorem.

**Theorem 12.** Let $(N, +, .)$ be a 5-torsion free prime near-ring with identity and $F$ be a generalized derivation on N associated with non-zero idempotent derivation $D$. If $n(-1) = -n$ for all $n \in N$ and $F^2(xoy) + (xoy) = 0$ for all $x, y \in N$, then $N$ is commutative.

*Proof.* Let $N$ be a 5-torsion free prime near-ring, $F$ be a generalized derivation on N associated with non-zero idempotent derivation $D$ of N. Now, let us suppose $F^2(xoy) + (xoy) = 0$ for all $x, y \in N$. Then $F^2(xoy) = -(xoy)$. If we replace $y$ by $yx$, we get $F^2(xo(yx)) = -(xoy)x$. On the other hand, D is idempotent and we have

$$\begin{aligned} F^2((xoy)x) &= -(xoy)x + F(xoy)D(x) \\ &\quad +F(xoy)D(x) + (xoy)D(x). \qquad (0.20) \end{aligned}$$

and this leads to have the following identity

$$(F(xoy) + F(xoy) + (xoy))D(x) = 0. \qquad (0.21)$$

Applying F on both sides of Equation (0.21) and using the assumption that D is idempotent gives us that

$$(-(xoy) - (xoy) + F(xoy))D(x) = 0.$$

This implies

$$F(xoy)D(x) = ((xoy) + (xoy))D(x). \qquad (0.22)$$

Again applying F on both sides of Equation (0.22), using the assumptions that

$$F^2(xoy) = -(xoy)$$

and D is idempotent we have

$$F(xoy)D(x) = (-(xoy) - (xoy) - (xoy))D(x). \qquad (0.23)$$

By considering Equation (0.22) and Equation (0.23) and simplifying the given expressions, we easily obtain the identity

$$5(xoy)D(x) = 0, \text{ for all } x, y \in N. \qquad (0.24)$$

Since N is 5-torsion free near-ring we have

$$(xoy)D(x) = 0 \qquad (0.25)$$

Now replacing $y$ by $yz$ in Equation (0.25) and using the same equation once again gives us $xyzD(x) = -yzxD(x) = yxzD(x)$ and hence $[x,y]zD(x) = 0$ for all $x, y, z \in$ N.

This implies $[x,y]ND(x) = 0$ for all $x, y \in$ N. From the primeness of N and since D is a non-zero derivation on N, we have $[x,y] = 0$ for all $x, y \in N$. This implies $xy = yx$ for all $x, y \in N$ and hence N is commutative.

### Conclusions

In this work, we proved the commutativity of a 3-torsion free prime near-ring $(N, +, .)$ with a generalized derivation F associated with non-zero idempotent derivation D on N satisfying the conditions $F^2[x,y] - [x,y] = 0$ for all $x, y \in N$ and

$F^2(xoy) - (xoy) = 0$ for all $x, y \in N$.

The commutativity of a 5-torsion free prime near ring $(N, +, .)$ with a generalized derivation F associated with non-zero idempotent derivation D on N satisfying the conditions $F^2[x, y] + [x, y] = 0$ for all $x, y \in N$ and for all $x, y \in N$, $F^2(xoy) + (xoy) = 0$ are also proved in this article.

To conclude this work, it may give of interest to consider the following problems.

(1) Can we prove the commutativity of the near-ring for any positive integer $n > 2$, with all the conditions given in the results?

(2) Can we leave the conditions 3-torsion free or 5-torsion free and still prove the commutativity of the given prime near-rings?

## Acknowledgments

## References

1 Ali, A., Bell, H.E. and Miyan, P. (2013). Generalized Derivations on Prime Near Rings. *Int. Journal of Math. and Mathematical Sciences*, Vol. **2013**, Art.ID 170749. Hindawi Publishing Corporation.

2 Bell, H.E. (1997). On Derivations in Near-Rings II. *Proceedings of the Conference on Nearrings & Nearfields*, Hamburg, Germany. pp 191-197.

3 Bell, H. E. and DAIF, M. N.(1995). On Derivations and Commutativity in Prime Rings. *Acta Math. Hungar.* **66 (4)**, pp 337-343.

4 Bell,H.E. and Kappe,L.C.(1989) Rings in which derivations satisfying certain algebraic conditions. *Acta.Math.hung.***53(3-4)**,339-346

5 Boua, A., Ali, A. and Huque, I. UL.(2018). Several Algebraic Identities in 3-Prime Near Rings. *Kragujevac Journal of Mathematics*, Vol. **42(2)**, pp 249-258.

6 Boua, A. and Oukhtite, L. (2011). Derivations On Prime Near-rings. *Int. J. Open Problems Compt. Math.*, Vol. **4(2)**. pp. 162-167.

7 Gölbaşi, Ö. (2006). On Generalized Derivations of Prime Near-Rings. *Hacettepe Journal of Mathematics and Statistics* Vol. **35 (2)**. pp 173- 180

8 Hvala, B. (1998). Generalized derivations in rings. *Communications in Algebra*, Vol. **26(4)**, pp. 1147-1166.

9 JamaL, M.R. (2008). A Study Of Derivations in Near-Rings. M.Phil Thesis. *Aligarh Muslim Universty Aligarh, India.*

10 Khan, M. R. and Hasnain, M. M.(2013). On Generalized Derivation On Prime Near-Ring. *Chinese Journal of Mathematics* Vol.**2013**, Article ID 752928, Hindawi Publishing Corporation

11 Posner, E. C.(1957). Derivations in Prime rings. *Proceedings of the American Mathematical Society,* Vol. **8**, pp.1093–1100.

12 Reddy, C. Jaya Subba, Kumar, S. Vasantha and Rao. Mallikarjuna. (2016). Prime Near Rings with Generalized Jordan Derivatives. *International Journal of Mathematics and Computer Applications Research*, Vol. **6(2)**, pp. 61-66.

13 Rehman, N. (2002). On Commutativity of Rings with Generalized Derivations. *Math. J. Okayama Univ.* **44**, pp 43-49

14 Satyanarayana, B. and Prasad, K.S. (2013). Near-Rings, Fuzzy Ideals and Graph Theory. *Taylor & Francis Group, LLC.*