SINET: *Ethiop. J. Sci.*, 41(1):40–45, 2018 ISSN: 0379–2897 (PRINT)

ISSN: 0379–2897

Review article

A REVIEW OF ENUMERATION METHODS ON INTEGER PARTITION

Samuel Asefa Fufa and **Fikre Bogale Petros**

Department of Mathematics, Addis Ababa University, P.O. Box 1176, Addis Ababa, Ethiopia
E-mail: samuel.asefa@aau.edu.et

ABSTRACT: A partition of a non-negative integer n is a way of writing n as a sum of a non-decreasing sequence of parts. The paper provides the study of some properties of integer partitions. In particular, we are interested to show the number of partitions of n in which the summand k appears at most k times is equal to the number of partitions in which the part1 appears any times and the other part$k$ appears at most $k-2$ times by using a generating function and algebraic construction.

Key words/phrases: Integer partitions, Partition identity, Partitions into parts of different sizes, Ferrer's diagram, Generating function

BACKGROUND ON PARTITIONS

Euler was the first in studying integer's partition counting problem. After Euler, many mathematicians have been interested in integer partitions and their interesting properties. One may require that the parts to be distinct, odd or even, or that n to be split into exactly k parts, and soon. One of the most difficult questions about integer partitions was determining the asymptotic properties of the number of partitions of the integer n when n gets larger and larger.Even though the values of the number of partitions of the integer n have been computed for large values of n, no pattern has been discovered to date this question was finally answered by(G. H. Hardy and S. Ramanujan,1918;Hans Rademacher, 1936; G. H. Hardy and E. M, 1956; Donald E. Knuth, 1973; Albert Nijenhuis and Herbert S. Wilf, 1978; and Kathleen M. O'Hara, 1988).

What is an integer partition? An integer partition is finding the number of ways of writing the positive integer n as a sum of positive integers, where the order of the summands does not matter. That is, $4 = 3 + 1$ or $4 = 1 + 3$ will count as only one way of writing 4 as a sum of the positive integers 1 and 3.As the order of the summands does not matter, we do not loss generality if we assume that they are in weakly decreasing order.

Definition 1.1:

If n is a positive integer, then a partition of n is a non-increasing sequence of positive integers $\lambda_1,\ \lambda_2, \ldots, \lambda_k$ so that $\lambda_1 + \lambda_2 + \ldots + \lambda_k = n$, G.E. Andrews (1984). In this case the λ_i are called the summands or the parts and the quantity n is called the size. We let the$P(n)$ to denote the number of partitions of the integer n. The number of partitions of n into exactly k parts is denoted by $P_k(n)$. For convenience, we define $P(0) = 1$ and we take $P(n) = 0$ for all negative values of n.The number of ways of writing n as the sum of 1 integer, as the sum of $n-1$ integers, or as the sum of n integers is unique, so $P_1(n) = P_{n-1}(n) = P_n(n) = 1$.

Example 1.2: $P(5) = 7$, and here are all 7 of the partitions of the integer $n = 5$:

$$\begin{aligned} 5 &= 5 \\ &= 4+1 \\ &= 3+2 \\ &= 3+1+1 \\ &= 2+2+1 \\ &= 2+1+1+1 \\ &= 1+1+1+1+1 \end{aligned}$$

Thus,$P_1(5) = 1,\ P_2(5) = 2, P_3(5) = 2,\ P_4(5) = 1,$ and $P_5(5) = 1$.

We use $\emptyset$ to represent the empty, or zero, partition.Partitions of integer for $n = 0, 1, 2, 3$ and 4 are shown in the following table.

Table 1.Values of $P(n)$ and partitions of n.

n	$P(n)$	Partitions, λ, of n
0	1	$\emptyset$
1	1	1
2	2	2, 1+1
3	3	3, 2+1, 1+1+1
4	5	4, 3+1, 2+2, 2+1+1, 1+1+1+1

We can also use a recurrence relation to find the partition numbers. We will find a recurrence relation to compute the $P_k(n)$, and then

$$P(n) = \sum_{k=1}^{n} P_k(n).$$

For example,

$$P(5) = \sum_{k=1}^{5} P_k(5)$$
$$= P_1(5) + P_2(5) + P_3(5) + P_4(5) + P_5(5) = 1 + 2 + 2 + 1 + 1 = 7.$$

Now consider the partitions of n into k parts. Some of these partitions contain no $1s$, like $3 + 3 + 4 + 6$, a partition of 16 into 4 parts. Subtracting 1 from each part, we get a partition of $n - k$ into k parts; for example, this is $2 + 2 + 3 + 5$. The remaining partitions of n into k parts contain a 1. If we remove the 1, we are left with a partition of $n - 1$ into $k - 1$ parts.

This gives us a $1 - 1$ correspondence between the partitions of n into k parts, and the partitions of $n - k$ into k parts together with the partitions of $n - 1$ into $k - 1$ parts. This leads us to the following theorem:

Theorem1.3: $P_k(n) = P_k(n-k) + P_{k-1}(n-1)$

Thus we can use Theorem 1.3 to find $P_k(n)$recursively.

Example 1.4: Find$P_3(8)$.

Solution: $P_3(8) = P_3(8-3) + P_2(7)$

$$= P_3(5) + P_2(7)$$
$$= 2 + P_2(7)$$
$$= 2 + P_2(7-2) + P_1(6)$$
$$= 2 + P_2(5) + P_1(6)$$
$$= 2 + 2 + 1 = 5.$$

Thus, the 5 partitions of 8 with 3 summands are

$$8 = 6 + 1 + 1$$
$$= 5 + 2 + 1$$
$$= 4 + 3 + 1$$
$$= 4 + 2 + 2$$
$$= 3 + 3 + 2$$

Ferrers diagram

Another useful way to think of a partition is with a Ferrers diagram. The Ferrers diagram of an integer partition gives us a very useful tool for visualizing partitions, and sometimes for proving identities. Each integer in the partitions is represented by a row of dots, and the rows are ordered from longest on the top to shortest at the bottom. The first row corresponds to the largest part; the second row corresponds to the second largest part, and so on. It is constructed by stacking left-justified rows of cells, where the number of cells in each row corresponds to the size of a part, (Albert Nijenhuis and Herbert S. Wilf, 1978 and Kathleen M. O'Hara, 1988).

Definition 2.1: The Ferrers diagram of a partition$\lambda = \lambda_1 \geq \lambda_2 \ldots \geq \lambda_k$ of n is the left – justified array of dots(stars) obtained by having λ_1 dots in the first (top) row, $\lambda_2$dots in the second row, and so on through λ_k dots in the final (bottom) row.

For example, the partition $3 + 3 + 4 + 5$ would be represented by

```
* * * * *
 * * * *
  * * *
  * * *
```

The conjugate of a partition is the one corresponding to the Ferrers diagram produced by flipping the diagram for the original partition a cross the main diagonal, thus turning rows into columns and vice versa. For the diagram above, the conjugate is

```
* * * *
* * * *
* * * *
  * *
   *
```

with the corresponding partition $1 + 2 + 4 + 4 + 4$.

A partition is self-conjugate if it is equal to its conjugate, or in other words, if its Ferrers diagram is symmetric about the diagonal. For example, the Ferrers diagram for the partition

20 = 6 + 4 + 4 + 4 + 1 + 1 is self-conjugate.

As an example of the use of Ferrers diagram in partition theory, we prove the following:

Theorem 2.2: The number of partitions of the integer n whose largest part is $k$is equal to the number of partitions of n with k parts.

Proof: The number of partitions of n into at most k parts is equal to the Ferrers diagram of size n with at most k rows and the partitions of n into parts not larger than k is equal to the Ferrers diagram with at most k columns. Now if we interchange the rows and columns (taking conjugates) we have a one – to - one correspondence between the two kinds of partitions.

Generating function

A useful and important tool to study partitions is generating functions. A generating function for a sequence is a formal power series whose nth coefficient corresponds to the nth term of the sequence.

Definition 3.1:$f(x)$ is a generating function for the sequence $a_0,\ a_1,\ a_2, \dots$ if

$$f(x) = \sum_{n=0}^{\infty} a_n x^n.$$

There is no simple formula for $P(n)$, but it is possible to find a generating function for them. We seek a product of factors so that when the factors are multiplied out, the coefficient of x^n is $P(n)$. We will now derive Euler's generating function$\varepsilon(x)$ for the sequence $\{P(n)\}_{n=0}^{\infty}$. In other words, we are looking a function which gives us

$$\varepsilon(x) = \sum_{n=0}^{\infty} p(n)x^n.$$

Consider the expression

$$(1 + x + x^2 + x^3 + \cdots)(1 + x^2 + x^4 + x^6 + \cdots) \dots (1 + x^k + x^{2k} + x^{3k} + \cdots) \dots$$
$$= \prod_{k=1}^{\infty} \sum_{i=0}^{\infty} x^{ik}.$$

When this product is expanded, we pick one term from each factor in all possible ways, with the further condition that we only pick a finite number of "non-1" terms. For example, if we pick x^3 from the first factor, x^3 from the third factor, x^{15} from the fifth factor, and $1s$ from all other factors, we get x^{21}. In this context of the product, this represents to the partition $1 + 1 + 1 + 3 + 5 + 5 + 5$. That is, three $1s$, one 3, and three $5s$.

Each factor is a geometric series; the k^{th} factor is

$$1 + x^k + (x^k)^2 + (x^k)^3 + \cdots = \frac{1}{1 - x^k},$$

So the generating function can be written $\prod_{k=1}^{\infty} \frac{1}{1-x^k}$.

These observations lead to Euler's Theorem.

Theorem 3.2:

$$\varepsilon(x) = \sum_{n=0}^{\infty} p(n)x^n = \frac{1}{1-x} \cdot \frac{1}{1-x^2} \cdot \frac{1}{1-x^3} \cdots$$
$$= \prod_{k=1}^{\infty} \frac{1}{1-x^k}$$

Note that if we are interested in some particular $P(n)$, we do not need the entire infinite product, or even any complete factor, since no partition of n can be use any integer greater than n, and also cannot use more than $\frac{n}{k}$ copies of k.

Example 3.3: Find $P(8)$. We expand

$$(1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + x^8)(1 + x^2 + x^4 + x^6 + x^8)(1 + x^3 + x^6)(1 + x^4 + x^8)(1 + x^5)(1 + x^6)(1 + x^7)(1 + x^8)$$
$$= 1 + x + 2x^2 + 3x^3 + 5x^4 + 7x^5 + 11x^6 + 15x^7 + 28x^8 + \cdots + x^{56},$$

So $P(8) = 22$.

We can use a generating function to find $P_k(n)$.

Let $P_{\le k}$ be the class of integer partitions with at most k summands.

$$P_{\le k}(x) = \prod_{i=1}^{k} \frac{1}{1 - x^i}$$

Hence, the generating function of partitions with exactly k summands is

$$P_k(x) = P_{\le k}(x) - P_{\le k-1}(x)$$
$$= \prod_{i=1}^{k} \frac{1}{1 - x^i} - \prod_{i=1}^{k-1} \frac{1}{1 - x^i}$$
$$P_k(x) = \frac{x^k}{(1-x)(1-x^2)\dots(1-x^k)}.$$

For example, consider the partitions of n with exactly 3 parts. A generating function for the number of partitions with exactly three parts is

$$\frac{1}{(1-x)(1-x^2)(1-x^3)} - \frac{1}{(1-x)(1-x^2)}$$
$$= \frac{1}{(1-x)(1-x^2)}\left(\frac{1}{1-x^3} - 1\right)$$
$$= \frac{x^3}{(1-x)(1-x^2)(1-x^3)}$$
$$= x^3 + x^4 + 2x^5 + 3x^6 + 4x^7 + 5x^8 + 7x^9 + \cdots$$

Thus, there are 5 partitions of 8 with exactly 3 parts as we did it in example 1.4.

We prove a partition identity through the use of generating functions.

Example 3.4: Let $F(n)$be the number of partitions of n that has no part equal to 1.

Recall that the monomial chosen from the factor $(1 + x + x^2 + x^3 + \cdots)$ indicates the number of 1's in the partition. Since we can only choose 1 from this term, we obtain the following generating function:

$$\sum_{n=0}^{\infty} F(n)x^n = \frac{1}{1-x^2}.\frac{1}{1-x^3}\cdots$$
$$\sum_{n=0}^{\infty} F(n)x^n = \frac{1-x}{1-x}.\frac{1}{1-x^2}.\frac{1}{1-x^3}\cdots$$
$$= \varepsilon(x)(1-x)$$

This generating function yields the following lemma, by matching the coefficients of like powers of x on both sides.

Lemma 3.5: The number of partitions of n with no parts equal to 1is$P(n) - P(n-1)$.

Partitions of integers have some interesting properties, R.P. Stanley (1999).

Theorem 3.6: The number of partitions of n into distinct parts equals the number of partitions of n into odd parts, R.P. Stanley (1999)

Let $P_d(n)$ be the number of partitions of n into distinct parts; let $P_o(n)$ be the number of partitions into odd parts.

Example 3.7: For $n = 6$, the partitions into distinct parts are

$$6 = 6$$
$$= 5 + 1$$
$$= 4 + 2$$
$$= 3 + 2 + 1,$$

So $P_d(6) = 4$, and the partitions into odd parts are

$$6 = 5 + 1$$
$$6 = 3 + 3$$
$$6 = 3 + 1 + 1 + 1$$
$$6 = 1 + 1 + 1 + 1 + 1 + 1$$

So $P_o(6) = 4$.

In fact, for every n , $P_d(n) = P_o(n)$, and we can see this by manipulating generating functions. The generating function for $P_d(n)$ is

$$f_d(x) = (1+x)(1+x^2)(1+x^3)\ldots = \prod_{i=1}^{\infty}(1+x^i).$$

The generating function for $P_o(n)$ is

$$(1 + x + x^2 + x^3 + \cdots)(1 + x^3 + x^6 + x^9 + \cdots)\ldots$$
$$= \prod_{i=0}^{\infty}\frac{1}{1-x^{2i+1}}.$$

We can write

$$f_d(x) = \frac{1-x^2}{1-x}.\frac{1-x^4}{1-x^2}.\frac{1-x^6}{1-x^3}$$

and notice that every numerator is eventually canceled by a denominator, leaving only the denominators containing odd powers of x, so

$$f_d(x) = \frac{1}{1-x}.\frac{1}{1-x^3}.\frac{1}{1-x^5}$$
$$= \prod_{i=0}^{\infty}\frac{1}{1-x^{2i+1}} = f_o(x).$$

Proposition 3.8: The number of partitions of n in which the even summands appear at most once is equal to the partitions of n in which every summand appears at most three times.

Proof: The generating function for the number of partitions of n in which the even summands appear at most once is

$$(1 + x + x^2 + x^3 + \cdots)(1 + x^2)(1 + x^3 + x^6 + x^9 \ldots)(1 + x^4)$$

$$= \frac{1}{1-x}.(1+x^2).\frac{1}{1-x^3}.(1+x^4).\frac{1}{1-x^5}.(1+x^6)$$
$$= \frac{1}{1-x}.\frac{1-x^4}{1-x^2}.\frac{1}{1-x^3}.\frac{1-x^8}{1-x^4}.\frac{1}{1-x^5}.\frac{1-x^{12}}{1-x^6}\cdots$$

$$= \frac{1-x^4}{1-x}.\frac{1-x^8}{1-x^2}.\frac{1-x^{12}}{1-x^3}.\frac{1-x^{16}}{1-x^4}\cdots$$
$$= (1 + x + x^2 + x^3)(1 + x^2 + x^4 + x^6)(1 + x^3 + x^6 + x^9)\ldots$$

which is exactly the generating function that describes the number of partitions of n such that each summand appears at most three times.

Example 3.9: For $n = 6$, the partitions into even parts appear at most once are

$$\begin{aligned} 6 &= 6 \\ &= 5 + 1 \\ &= 4 + 1 + 1 \\ &= 3 + 3 \\ &= 3 + 2 + 1 \\ &= 2 + 1 + 1 + 1 + 1 \\ &= 1 + 1 + 1 + 1 + 1 + 1 \end{aligned}$$

and the partitions into each part appears at most three times are

$$\begin{aligned} 6 &= 6 \\ &= 5 + 1 \\ &= 4 + 1 + 1 \\ &= 4 + 2 \\ &= 3 + 3 \\ &= 3 + 2 + 1 \\ &= 2 + 2 + 2 \end{aligned}$$

In fact, they are equal in number.

Proposition 3.10: The number of partitions of n in which the summand k appears at most k times is equal to the partitions of n in which the part 1 appears any timesand the other part k appears at most $k - 2$ times.

Proof: The generating function for the number of partitions of n in which the summand k appears at most k times is

$$\begin{aligned} &(1 + x)(1 + x^2 + x^4)(1 + x^3 + x^6 + x^9)(1 + x^4 + x^8 + x^{12} + x^{16}) \dots \\ &= \frac{1 - x^2}{1 - x} \cdot \frac{1 - x^6}{1 - x^2} \cdot \frac{1 - x^{12}}{1 - x^3} \cdot \frac{1 - x^{20}}{1 - x^4} \dots \\ &= \frac{1}{1 - x} \cdot \frac{1 - x^2}{1 - x^2} \cdot \frac{1 - x^6}{1 - x^3} \cdot \frac{1 - x^{12}}{1 - x^4} \cdot \frac{1 - x^{20}}{1 - x^5} \dots \\ &= \frac{1}{1 - x} . 1. (1 + x^3)(1 + x^4 + x^8)(1 + x^5 + x^{10} + x^{15}) \dots \\ &= (1 + x + x^2 + x^3 + \cdots)(1 + x^3)(1 + x^4 + x^8)(1 + x^5 + x^{10} + x^{15}) \dots \end{aligned}$$

which is the generating function describing the number of partitions of n in which the summand 1 appears any times and the other summand k appears at most $k - 2$ times.

Example 3.11: For $n = 5$, the partitions into summand k appears at most k times are

$$\begin{aligned} 5 &= 5 \\ &= 4 + 1 \\ &= 3 + 2 \\ &= 2 + 2 + 1 \end{aligned}$$

and the partitions into summand 1 appears any times, 2 appears no times and other summand k appears at most$k - 2$ times are

$$\begin{aligned} 5 &= 5 \\ &= 4 + 1 \\ &= 3 + 1 + 1 \\ &= 1 + 1 + 1 + 1 + 1 \end{aligned}$$

Thus the two partitions are equal in number.
We can show this fact for $n = 6,\ 7$ and 8.
For $n = 6$, the partitions into summand k appears at most k times are

$$\begin{aligned} 6 &= 6 \\ &= 5 + 1 \\ &= 4 + 2 \\ &= 3 + 3 \\ &= 3 + 2 + 1 \end{aligned}$$

and the partitions into summand 1 appears any times, 2 appears no times and other summand k appears at most$k - 2$ times are

$$\begin{aligned} 6 &= 6 \\ &= 5 + 1 \\ &= 4 + 1 + 1 \\ &= 3 + 1 + 1 + 1 \\ &= 1 + 1 + 1 + 1 + 1 + 1 \end{aligned}$$

In fact, the two partitions are equal in number.
For $n = 7$, the partitions into summand k appears at most k times are

$$\begin{aligned} 7 &= 7 \\ &= 6 + 1 \\ &= 5 + 2 \\ &= 4 + 3 \\ &= 4 + 2 + 1 \\ &= 3 + 3 + 1 \\ &= 3 + 2 + 2 \end{aligned}$$

and the partitions into summand 1 appears any times, 2 appears no times and other summand k appears at most$k - 2$ times are

$$\begin{aligned} 7 &= 7 \\ &= 6 + 1 \\ &= 5 + 1 + 1 \\ &= 4 + 3 \\ &= 4 + 1 + 1 + 1 \\ &= 3 + 1 + 1 + 1 + 1 \\ &= 1 + 1 + 1 + 1 + 1 + 1 + 1 \end{aligned}$$

In fact, the two partitions are equal in number.

For $n = 8$, the partitions into summand k appears at most k times are

$$\begin{aligned} 8 &= 8 \\ &= 7+1 \\ &= 6+2 \\ &= 5+3 \\ &= 5+2+1 \\ &= 4+4 \\ &= 4+3+1 \\ &= 4+2+2 \\ &= 3+3+2 \\ &= 3+2+2+1 \end{aligned}$$

and the partitions into summand 1 appears any times, 2 appears no times and other summand k appears at most$k - 2$ times are

$$\begin{aligned} 8 &= 8 \\ &= 7+1 \\ &= 6+1+1 \\ &= 5+3 \\ &= 5+1+1+1 \\ &= 4+4 \\ &= 4+3+1 \\ &= 4+1+1+1+1 \\ &= 3+1+1+1+1+1 \end{aligned}$$

$$= 1+1+1+1+1+1+1+1$$

In fact, the two partitions are equal in number.

CONCLUSION

The main purpose of this study is not giving a general formula, a recurrence relation or a bijective proof for the integer partitions, rather studying some of their properties. We explained how generating functions are very useful describing the number of partitions of n in which the summand k appears at most k times. In the future we will work on studying more about finding a bijective proof or a recurrence relation for the integer partitions that we have discussed. We believe this will help for researchers attempting to enumerate and/or understand the structure of integer partitions.

ACKNOWLEDGMENTS

The first author would like to thank Dr. Tomas Vetrik for hosting him at his institution and providing guidance while he was working on this project. He also acknowledges the support he received from DAAD (Germany Academic Exchange Service), Addis Ababa University, and ISP (International Science Program) to conduct this research.

The second author acknowledges Mathematics Department of Addis Ababa University for its financial support and allowing him to use the graduate Mathematics computer laboratory when he was conducting this work.

REFERENCES

1. Albert Nijenhuis and Herbert S. Wilf (1978). *Combinatorial Algorithms*, 2nd Ed., Academic Press, New York.
2. Donald E. Knuth (1973). *Analytic Combinatorics*, The Art of Computer Programming, Vol. 3, Addison-Wesley, Reading.
3. Andrews G. E. (1984). *The Theory of Partitions*, Addison–Wesley, Reading, MA; reissued (1976) by Cambridge University Press, Cambridge.
4. Hardy G. H. and Ramanujan S. (1918). *Asymptotic formulæ* in combinatory analysis, Proc. London Math. Soc.
5. Hardy G. H. and Wright E. M. (1956). *The Theory of Numbers*, Oxford University Press.
6. Hans Rademacher(1937). *On the partition function p(n)*, Proc. London Math. Soc., pp. 241–254.
7. Kathleen M. O'Hara (1988). *Bijections for partition identities*. J. Combin. Theory Ser. A, pp. 13–25.
8. Stanley R. P. (1999). *Enumerative Combinatorics*, vol. 2, Cambridge Stud. Adv. Math., vol. 62, Cambridge University Press, Cambridge.