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BOUNDARY-DOMAIN INTEGRAL EQUATION SYSTEMS TO THE DIRICHLET BVP FOR AN INCOMPRESSIBLE STOKES SYSTEM WITH VARIABLE VISCOSITY IN 2D

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ABSTRACT. The Dirichlet problem for the steady-state Stokes system of partial differential equations for an incompressible viscous fluid with variable viscosity coefficient is considered in a two-dimensional bounded domain. Using an appropriate parametrix, this problem is reduced to two systems of direct segregated boundary-domain integral equations (BDIEs). The BDIEs in 2D have special properties in comparison with the three dimensional case, because of the logarithmic term in the parametrix for the associated partial differential equation. Consequently, we need to set conditions on the function spaces or on the domain to ensure the invertibility of the corresponding parametrix-based hydrodaynamic single layer potential and hence, guarantying the unique solvability of BDIEs. Equivalence of the obtained BDIE systems to the original Dirichlet BVP and unique solvability of BDIE systems is shown. Invertibility of the corresponding boundary-domain integral operators is proved in appropriate Sobolev-Slobodetski (Bessel potential) spaces.

Key words/phrases: boundary-domain integral equations, equivalence, invertibility, parametrix, Stokes system

INTRODUCTION

The Stokes system of partial differential equations is derived from the linearised steady-state Navier-Stokes system. The study of the Stokes equations is useful in itself; it also gives us an opportunity to introduce several tools necessary for a treatment of the full Navier-Stokes equations, see e.g. Temam (1979), Chapter I. The Stokes system has applications on modelling the motion of a laminar viscous fluid which are useful in science and engineering.

Boundary integral equations and the hydrodynamic potential theory for the Stokes system with constant viscosity have been extensively studied nowadays, see e.g. Ladyzhenskaya (1969), Hsiao and Wendland (2008), Rjasanow and Steinbach (2007), Steinbach (2007), Kohr and Wendland (2006), Domínguez and Sayas (2006) and Hsiao and Kress (1985).

Boundary-domain integral equation systems for the incompressible Stokes system with variable viscosity in 3D have been investigated in Mikhailov and Portillo (2015) and BDIE systems for the compressible Stokes system with variable viscosity in 3D have been analysed in Mikhailov and Portillo (2019). But this is not the case for BDIE systems for the Stokes system with variable viscosity in 2D. The BDIEs in the two-dimensional case have special properties in comparison with the three dimension because of the logarithmic term in the parametrix for the associated partial differential equation. Consequently, we need to set conditions on the function spaces or on the domain for the invertibility of corresponding parametrix-based hydrodaynamic single layer potential and hence the unique solvability of BDIEs.

In this paper, using an appropriate parametrix the Dirichlet boundary value problem for the steady-state Stokes system of partial differential equations for an incompressible viscous fluid with variable viscosity coefficient in two-dimensional bounded domain is reduced to systems of direct segregated boundary-domain integral equations (BDIEs). Following similar approach used in Tamirat and Mikhailov (2015) to analyze BDIE systems for BVPS associated with variable-coefficient scalar elliptic PDEs, equivalence of the obtained BDIE systems to the original Dirichlet BVP and unique solvability of BDIE systems is shown. Invertibility of the corresponding boundary-domain integral operators is proved in appropriate Sobolev-Slobodetski (Bessel potential) spaces.

Formulation of the Boundary Value Problem

Let $\Omega = \Omega^+$ be an open bounded two-dimensional region of $\mathbb{R}^2$ and let $\Omega^- = \mathbb{R}^2 \backslash \overline{\Omega^+}$ The boundary $\partial\Omega$ be a simply connected, closed and infinitely smooth curve.

Let $\boldsymbol{v}$ be the velocity vector field, p the pressure scalar field and $\mu \in C^\infty(\Omega)$ be the variable kinematic viscosity of the fluid such that $\mu(\boldsymbol{x}) > c > 0$. For an arbitrary couple $(p, \boldsymbol{v})$ the stress tensor operator, σ_{ij}, and the Stokes operator, $\boldsymbol{A}_j$, for incompressible fluid are defined as:

$$\sigma_{ij}(p, \boldsymbol{v})(\boldsymbol{x}) := -\delta_i^j p + \mu(\boldsymbol{x}) \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right),$$

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$$A_j(p,v)(x):=\frac{\partial}{\partial x_i}\sigma_{ij}(p,v)(x)=\frac{\partial}{\partial x_i}\left(\mu(x)\left(\frac{\partial v_i}{\partial x_j}+\frac{\partial v_j}{\partial x_i}\right)\right)-\frac{\partial p}{\partial x_j}, j,i\in\{1,2\},$$

where δ_i^j is Kronecker symbol. Here and henceforth we assume the Einstein summation in repeated indices from 1 to 2. We also denote the Stokes operator as $\boldsymbol{A}=\{A_j\}_{j=1}^2$. We will also use the following notation for derivative operators: $\partial_j=\partial_{x_j}:=\frac{\partial}{\partial x_j}$ with $j=1,2$; $\nabla:=(\partial_1,\partial_2)$.

In what follows $H^s(\Omega)=H_2^s(\Omega), H^s(\partial\Omega)$ are the Bessel potential spaces, where s is a real number (see, e.g., (Lions and Magenes, 1973; McLean, 2000)). We recall that H^s coincide with the Sobolev-Slobodetski spaces W_2^s for any non-negative s. We denote by $\widetilde{H}^s(\Omega)$ the subspace of $H^s(R^2)$,$\widetilde{H}^s(\Omega)=\{g: g\in H^s(R^2), supp(g)\subset\overline{\Omega}\}$; similarly, $\widetilde{H}^s(S_1)=\{g: g\in H^s(\partial\Omega), supp(g)\subset\overline{S_1}\}$ is the Sobolev space of functions having support in $S_1\subset\partial\Omega$. We will also use the notations $\boldsymbol{H}^s(\Omega)=[H^s(\Omega)]^2$, $\boldsymbol{L}^2(\Omega)=[L^2(\Omega)]^2$, $L_*^2(\Omega)=L^2(\Omega)/R=\{q\in L^2(\Omega): \int_\Omega q\,dx=0\}$, $\boldsymbol{D}(\Omega)=[D(\Omega)]^2$ for 2-dimensional vector space.

Let $\boldsymbol{H}_{div}^1(\Omega)=\{v\in H^1(\Omega): div v=0\}$ be the divergence-free Sobolev space.

We will also make use of the following space (see, e.g., (Costabel, 1988; Chkadua et al, 2009; Mikhailov and Portillo, 2019)).

$$\boldsymbol{H}^{1,0}(\Omega;\boldsymbol{A}):=\{(p,v)\in L^2(\Omega)\times\boldsymbol{H}^1(\Omega):\boldsymbol{A}(p,v)\in\boldsymbol{L}^2(\Omega)\}$$
$$\boldsymbol{H}_{div}^{1,0}(\Omega;\boldsymbol{A}):=\{(p,v)\in L^2(\Omega)\times\boldsymbol{H}_{div}^1(\Omega):\boldsymbol{A}(p,v)\in\boldsymbol{L}^2(\Omega)\}$$
$$\boldsymbol{H}_*^{1,0}(\Omega;\boldsymbol{A}):=\{(p,v)\in L_*^2(\Omega)\times\boldsymbol{H}^1(\Omega):\boldsymbol{A}(p,v)\in\boldsymbol{L}^2(\Omega)\}$$

endowed with the norm

$$\|(p,v)\|^2_{\boldsymbol{H}^{1,0}(\Omega;A)}:=\|p\|^2_{L^2(\Omega)}+\|v\|^2_{\boldsymbol{H}^s(\Omega)}+\|\boldsymbol{A}(p,v)\|^2_{\boldsymbol{L}^2(\Omega)}.$$

Let us define a space

$$\boldsymbol{H}_{div,*}^{1,0}(\Omega;A):=\{(p,v)\in L^2_*(\Omega)\times\boldsymbol{H}_{div}^1(\Omega):A(p,v)\in\boldsymbol{L}^2(\Omega)\}$$

endowed with the same norm

$$\|(p,v)\|^2_{\boldsymbol{H}_{div,*}^{1,0}(\Omega;A)}:=\|p\|^2_{L^2(\Omega)}+\|v\|^2_{\boldsymbol{H}^1(\Omega)}+\|A(p,v)\|^2_{\boldsymbol{L}^2(\Omega)}.$$

The operator $\boldsymbol{A}$ acting on (p,v) is well defined in the weak sense provided $\mu(x)\in L^\infty(\Omega)$ as

$$\langle\boldsymbol{A}(p,v),\boldsymbol{u}\rangle_\Omega:=-E((p,v),u),\forall\boldsymbol{u}\in\widetilde{\boldsymbol{H}}^1{}_{div}(\Omega),$$

where the form $E:\boldsymbol{H}^1{}_{div}\times\widetilde{\boldsymbol{H}}^1{}_{div}(\Omega)\longrightarrow\boldsymbol{R}$, and the function E are defined as

$$E(v,u):=\int_\Omega E(v,u)(x)dx, \tag{2.1}$$
$$E(v,u)(x):=\frac{\mu(x)}{2}\left(\frac{\partial u_i(x)}{\partial x_j}+\frac{\partial u_j(x)}{\partial x_i}\right)\left(\frac{\partial v_i(x)}{\partial x_j}+\frac{\partial v_j(x)}{\partial x_i}\right).$$

For sufficiently smooth functions $(p,v)\in H^{s-1}(\Omega^\pm)\times\boldsymbol{H}^s(\Omega^\pm)$ with $s>3/2$, we can define the classical traction operators, $\boldsymbol{T}^{c\pm}=\{T_j^{c\pm}\}_{j=1}^2$ on the boundary $\partial\Omega$ as

$$T_j^{c\pm}(p,v)(x):=[\gamma^\pm\sigma_{ij}(p,v)(x)]n_i(x), \tag{2.2}$$

where $n_i(x)$ denote components of the unit outward normal vector $n(x)$ to the boundary $\partial\Omega$ of the domain and $\gamma^\pm$ is the trace operator from inside and outside Ω.

Traction operator (2.2) can be continuously extended to the canonical traction operator $\boldsymbol{T}^\pm: H_{div}^{1,0}(\Omega^\pm;\boldsymbol{A})\to\boldsymbol{H}^{-\frac{1}{2}}(\partial\Omega)$ defined in the weak form similar to [10] as

$$\langle T^\pm(p,v),w\rangle_{\partial\Omega}:=\pm\int_{\Omega^\pm}\left[A(p,v)(\gamma^{-1}w)+E((p,v),\gamma^{-1}w)\right]dx,$$
$$(p,v)\in\boldsymbol{H}_{div}^{1,0}(\Omega^\pm;\boldsymbol{A}),\forall\in\boldsymbol{H}^{\frac{1}{2}}(\partial\Omega).$$

Here the operator $\gamma^{-1}:\boldsymbol{H}^{\frac{1}{2}}(\partial\Omega)\to\boldsymbol{H}_{div}^1(\mathbb{R}^2)$ denotes a continuous right inverse of the trace operator $\gamma^+: H_{div}^1(\mathbb{R}^2)\to H^{\frac{1}{2}}(\partial\Omega)$.In addition, for $(p,v)\in\boldsymbol{H}^{1,0}{}_{*,div}(\Omega;\boldsymbol{A})$ the traction operator $\boldsymbol{T}^\pm$ are also defined.

Furthermore, if $(p,v)\in\boldsymbol{H}_{div}^{1,0}(\Omega;\boldsymbol{A})$ and $\boldsymbol{u}\in\boldsymbol{H}^1(\Omega)$, the following first Green identity holds (see, e.g. Costabel (1988), Chkadua et al. (2009), Mikhailov and Portillo (2015), Mikhailov and Portillo (2019)

$$\langle T^+(p,v),\gamma^+u\rangle_{\partial\Omega}:=\int_\Omega\left[A(p,v)u+E((p,v),u)(x)\right]dx. \tag{2.3}$$

Equation (2.3) is also defined for $(p,v)\in\boldsymbol{H}_{div,*}^{1,0}(\Omega;\boldsymbol{A})$ and $\boldsymbol{u}\in\boldsymbol{H}^1(\Omega)$. Applying the identity (2.3) to the pairs $(p,v)\in\boldsymbol{H}_{div}^{1,0}(\Omega;\boldsymbol{A})$ and $(q,u)\in\boldsymbol{H}_{div}^{1,0}(\Omega;\boldsymbol{A})$ with exchanged roles and subtracting the one from the other, we arrive at the second Green identity (see, e.g. McLean (2000), Mikhailov (2011), Mikhailov and Portillo (2015), Mikhailov and Portillo (2019)) as follow:

$$\int_\Omega\left[A_j(p,v)u_j-A_j(q,u)v_j\right]dx=\int_{\partial\Omega}\left[T_j(p,v)u_j-T_j(q,u)v_j\right]dS_x. \tag{2.4}$$

Equation (2.4) is also defined for $(p,v)\in\boldsymbol{H}_{div,*}^{1,0}(\Omega;\boldsymbol{A})$ and $(q,u)\in\boldsymbol{H}_{div,*}^{1,0}(\Omega;\boldsymbol{A})$.

We shall derive and investigate the BDIE systems for the following Dirichlet boundary value problem. *Given the functions,* $g\in L^2(\Omega)$ *and* $\boldsymbol{f}\in\boldsymbol{L}^2(\Omega)$, *find a couple of functions* $(p,v)\in\boldsymbol{H}^{1,0}{}_{*,\ div}(\Omega;\boldsymbol{A})$ *satisfying,*

$$\boldsymbol{A}(p,v)(x)=\boldsymbol{f}(x), x\in\Omega, \tag{2.5}$$
$$\gamma^+v(x)=\boldsymbol{\varphi}_0(x), x\in\partial\Omega. \tag{2.6}$$

From the divergence theorem it follows that $\boldsymbol{\varphi}_0$ must satisfy the compatibility condition,

$$\int_{\partial\Omega}\boldsymbol{\varphi}_0(x)\cdot\boldsymbol{n}(x)dS_x=0 \tag{2.7}$$

where $\boldsymbol{n}(\boldsymbol{x})$ is the outer normal vector defined for almost all $\boldsymbol{x} \in \partial\Omega$ (Rjasanow and Steinbach (2007), equ. 1.87).

Theorem 2.1 *The Dirichlet BVP (2.5)-(2.6) has a unique solution in the space* $\boldsymbol{H}^{1,0}_{div,*}(\Omega;\boldsymbol{A})$ *satisfying (2.7).*

Proof. Let (p_1, v_1) and (p_2, v_2) are in $\boldsymbol{H}^{1,0}_{div,*}(\Omega;\boldsymbol{A})$ that satisfy the BVP (2.5)-(2.6). Then $(p,\boldsymbol{v}) := (p_2, v_2) - (p_1, v_1)$ also belongs to $\boldsymbol{H}^{1,0}_{div,*}(\Omega;\boldsymbol{A})$ and satisfy the following homogeneous Dirichlet BVP

$$\boldsymbol{A}(p,\boldsymbol{v})(\boldsymbol{x}) = 0, \boldsymbol{x} \in \Omega, \tag{2.8}$$

$$\gamma^+ \boldsymbol{v}(\boldsymbol{x}) = 0, \boldsymbol{x} \in \partial\Omega \tag{2.9}$$

The first Green identity (2.3) holds for any $\boldsymbol{u} \in \boldsymbol{H}^1_{div}(\Omega)$ and for any pair $(p,\boldsymbol{v}) \in \boldsymbol{H}^{1,0}_{div,*}(\Omega;\boldsymbol{A})$. Then due to (2.8)-(2.9) we have, $0 = \int_\Omega E(\boldsymbol{v},\boldsymbol{u})(\boldsymbol{x})d\boldsymbol{x} := E(\boldsymbol{v},\boldsymbol{u})$, that is

$$E(v,u) = \int_\Omega E(v,u)(x)dx = \int_\Omega \frac{\mu(x)}{2}\left(\frac{\partial u_i(x)}{\partial x_j} + \frac{\partial u_j(x)}{\partial x_i}\right)\left(\frac{\partial v_i(x)}{\partial x_j} + \frac{\partial v_j(x)}{\partial x_i}\right)dx = 0.$$

Now if we choose $u = \boldsymbol{v}$, then we get, $E(\boldsymbol{v},\boldsymbol{v}) = 0$. As $\mu(\boldsymbol{x}) > 0$, the only possibility is that $\boldsymbol{v}(\boldsymbol{x}) = \boldsymbol{a} + \boldsymbol{b}\boldsymbol{x}$, i.e, $\boldsymbol{v}$ is a rigid movement (see [4, Eq.(2.2.11)]). Nevertheless, taking into account the Dirichlet condition (2.9), we deduce that $\boldsymbol{v} \equiv \boldsymbol{0}$ and hence $\boldsymbol{v}_1 = \boldsymbol{v}_2$.

Considering now $\boldsymbol{v} = \boldsymbol{0}$ and keeping in mind equation (2.8), we have $\boldsymbol{A}(p,\boldsymbol{v})(\boldsymbol{x}) = \boldsymbol{0}$ and then we get $\nabla p = 0$. Since $p \in L^2_*(\Omega)$, we have that $p = 0$.

Parametrix and parametrix-based hydrodynamic potentials

Parametrix and remainder

The Stokes operator $\boldsymbol{A}$ becomes its counterpart with constant-coefficient, $\mathring{\boldsymbol{A}}$ when $\mu = 1$. Its fundamental solution is defined by the pair of distributions $(\mathring{q}^k, \mathring{u}^k)$, where u^k_j represent components of the incompressible velocity fundamental solution and q^k represent the components of the pressure fundamental solution (see, e.g. Ladyzhenskaya (1969), Hsiao and Wendland (2008), Rjasanow and Steinbach (2007), Steinbach (2007)).

Let us consider the following system

$$\Delta \mathring{u}^k_j - \frac{\partial \mathring{q}^k}{\partial x_j} = \delta(\boldsymbol{x}-\boldsymbol{y})\delta^k_j \ \ in \ \Omega, j, k = 1,2,$$

$$div(\mathring{u}^k) = 0 \ \ in \ \ \Omega.$$

Applying Fourier transforms to the above system we get,

$$\mathring{u}^k_j(\boldsymbol{x},\boldsymbol{y}) = \frac{1}{4\pi}[\delta^k_j \log\frac{|x-y|}{r_0} - \frac{(x_j-y_j)(x_k-y_k)}{|x-y|^2}$$

$$\mathring{q}^k(\boldsymbol{x},\boldsymbol{y}) = \frac{-(x_k-y_k)}{2\pi|x-y|^2} \text{ for } r_0 > 0.$$

Therefore the pair $(\mathring{q}^k, \mathring{u}^k)$ satisfies

$$\frac{\partial}{\partial x_k}\mathring{q}^k(x,y) = \sum_{i=1}^2 \frac{\partial^2}{\partial x_k^2}\left(-\frac{1}{2}\log|x-y|\right) = -\delta(x-y) \tag{3.1}$$

$$A_j(x;\mathring{q}^k,\mathring{u}^k)(x,y) = \sum_{i=1}^2 \frac{\partial^2 \mathring{u}^k_j}{\partial x_i^2} - \frac{\partial \mathring{q}^k}{\partial x_j} = \delta^k_j\delta(x-y), div_x \mathring{u}^k(x,y) = 0 \tag{3.2}$$

Let us denote $\mathring{\sigma}_{ij}(p,\boldsymbol{v}) := \sigma_{ij}(p,\boldsymbol{v})|_{\mu=1}$. For the case $\mu = 1$ and with fundamental solution of the operator $\mathring{\boldsymbol{A}}$, the stress tensor $\mathring{\sigma}_{ij}(\mathring{q}^k, \mathring{u}^k)(\boldsymbol{x}-\boldsymbol{y})$ reads

$$\mathring{\sigma}_{ij}(\boldsymbol{x};\mathring{q}^k,\mathring{u}^k)(\boldsymbol{x}-\boldsymbol{y}) = \frac{1}{\pi}\frac{(x_i-y_i)(x_j-y_j)(x_k-y_k)}{|x-y|^4}.$$

Indeed,

$$\mathring{\sigma}_{ij}(\boldsymbol{x};\mathring{q}^k,\mathring{u}^k)(\boldsymbol{x}-\boldsymbol{y}) = -q^k\delta_{ij} + \left(\frac{\partial u^k_i}{\partial x_j} + \frac{\partial u^k_j}{\partial x_i}\right)$$

$$= \frac{x_k-y_k}{2\pi|x-y|^2}\delta_{ij} + \frac{\partial}{\partial x_i}(\frac{1}{4\pi}[\delta^k_j \log\frac{|x-y|}{r_0} - \frac{(x_j-y_j)(x_k-y_k)}{|x-y|^2})$$

$$+ \frac{\partial}{\partial x_j}(\frac{1}{4\pi}[\delta^k_j \log\frac{|x-y|}{r_0} - \frac{(x_i-y_i)(x_k-y_k)}{|x-y|^2}),$$

which shows that

$$\mathring{\sigma}_{ij}(\boldsymbol{x};\mathring{q}^k,\mathring{u}^k)(\boldsymbol{x}-\boldsymbol{y}) = \frac{1}{\pi}\frac{(x_i-y_i)(x_j-y_j)(x_k-y_k)}{|x-y|^4}.$$

The boundary traction $\mathring{T}^c_j(\boldsymbol{x};\mathring{q}^k,\mathring{u}^k)$ is then

$$\mathring{T}^c_j(x;\mathring{q}^k,\mathring{u}^k)(x,y) := \mathring{\sigma}_{ij}(\mathring{q}^k,\mathring{u}^k)(x-y)n_i(x) = \frac{1}{\pi}\frac{(x_i-y_i)(x_j-y_j)(x_k-y_k)}{|x-y|^4}n_i(x).$$

Let us define a pair of functions as

$$q^k(\boldsymbol{x},\boldsymbol{y}) = \frac{\mu(x)}{\mu(y)}\mathring{q}^k(\boldsymbol{x},\boldsymbol{y}) = \frac{\mu(x)}{\mu(y)}\frac{y_k-x_k}{2\pi|x-y|^2}, j,k \in \{1,2\} \tag{3.3}$$

$$u^k_j(x,y) = \frac{1}{\mu(y)}\mathring{u}^k_j(x,y) = \frac{1}{4\pi\mu(y)}(\delta^k_j \log\frac{|x-y|}{r_0} - \frac{(x_j-y_j)(x_k-y_k)}{|x-y|^2}) \tag{3.4}$$

Then

$$\sigma_{ij}(x;q^k,u^k)(x-y) = -\delta^j_i q^k + \mu(x)\left(\frac{\partial \mathring{u}^k_i}{\partial x_j} + \frac{\partial \mathring{u}^k_j}{\partial x_i}\right)$$

$$= -\delta^j_i\frac{\mu(x)}{\mu(y)}\mathring{q}^k + \mu(x)\left(\frac{\partial\left(\frac{1}{\mu(y)}\mathring{u}^k_i\right)}{\partial x_j} + \frac{\partial\left(\frac{1}{\mu(y)}\mathring{u}^k_j\right)}{\partial x_i}\right)$$

$$= \frac{\mu(x)}{\mu(y)}\left(-\delta^j_i\mathring{q}^k + \left(\frac{\partial \mathring{u}^k_i}{\partial x_j} + \frac{\partial \mathring{u}^k_j}{\partial x_i}\right)\right) = \frac{\mu(x)}{\mu(y)}\sigma_{ij}(\mathring{q}^k,\mathring{u}^k)(\boldsymbol{x}-\boldsymbol{y}).$$

Thus,

$$\sigma_{ij}(\boldsymbol{x};q^k,u^k)(\boldsymbol{x}-\boldsymbol{y}) = \frac{\mu(x)}{\mu(y)}\mathring{\sigma}_{ij}(\mathring{q}^k,\mathring{u}^k)(\boldsymbol{x}-\boldsymbol{y})$$

and

$$T^c_j(x;q^k,u^k)(x,y) := \sigma_{ij}(x;q^k,u^k)(x-y)n_i(x) = \frac{\mu(x)}{\mu(y)}\mathring{T}^c_j(x;\mathring{q}^k,\mathring{u}^k)(x,y). \tag{3.5}$$

Substituting (3.3)-(3.4) into Stokes system we have,

$$\boldsymbol{A}_j(\boldsymbol{x};q^k;\boldsymbol{u}^k)(\boldsymbol{x},\boldsymbol{y}) = \frac{\partial}{\partial \boldsymbol{x}_i}\Big(\sigma_{ij}(\boldsymbol{x};q^k,u^k)(\boldsymbol{x}-\boldsymbol{y})\Big)$$

$$= \frac{\partial}{\partial \boldsymbol{x}_i}\left(\frac{\mu(\boldsymbol{x})}{\mu(\boldsymbol{y})}\mathring{\sigma}_{ij}(\mathring{q}^k,\mathring{u}^k)(\boldsymbol{x}-\boldsymbol{y})\right)$$

$$= \frac{\mu(x)}{\mu(y)}\frac{\partial}{\partial x_i}\left(\mathring{\sigma}_{ij}(\mathring{q}^k,\mathring{u}^k)(x-y)\right) + \frac{\partial}{\partial x_i}\left(\frac{\mu(x)}{\mu(y)}\right)\mathring{\sigma}_{ij}(\mathring{q}^k,\mathring{u}^k)(x-y)$$

$$= \frac{\mu(x)}{\mu(y)}\mathring{A}_j(q^k,u^k)(x) + \frac{1}{\mu(y)}\frac{\partial(\mu(x))}{\partial x_i}\mathring{\sigma}_{ij}(\mathring{q}^k,\mathring{u}^k)(x-y)$$

$$= \frac{\mu(\boldsymbol{x})\delta(\boldsymbol{x}-\boldsymbol{y})\delta_j^k}{\mu(y)} + \frac{1}{\mu(y)}\frac{\partial(\mu(x))}{\partial \boldsymbol{x}_i}\mathring{\sigma}_{ij}(\mathring{q}^k,\mathring{u}^k)(\boldsymbol{x}-\boldsymbol{y})$$

$$= \frac{\mu(\boldsymbol{y})\delta(\boldsymbol{x}-\boldsymbol{y})\delta_j^k}{\mu(\boldsymbol{y})} + \frac{1}{\mu(\boldsymbol{y})}\frac{\partial(\mu(\boldsymbol{x}))}{\partial \boldsymbol{x}_i}\mathring{\sigma}_{ij}(\mathring{q}^k,\mathring{u}^k)(\boldsymbol{x}-\boldsymbol{y})$$

$$= \delta_j^k\delta(\boldsymbol{x}-\boldsymbol{y}) + \frac{1}{\mu(\boldsymbol{y})}\frac{\partial(\mu(\boldsymbol{x}))}{\partial \boldsymbol{x}_i}\mathring{\sigma}_{ij}(\mathring{q}^k,\mathring{u}^k)(\boldsymbol{x}-\boldsymbol{y})$$

Thus,

$$\boldsymbol{A}_j(\boldsymbol{x};q^k;\boldsymbol{u}^k)(\boldsymbol{x},\boldsymbol{y}) = \delta_j^k\delta(\boldsymbol{x}-\boldsymbol{y}) + R_{kj}(\boldsymbol{x},\boldsymbol{y}), \quad (3.6)$$

where

$$R_{kj}(\boldsymbol{x},\boldsymbol{y}) = \frac{1}{\mu(y)}\frac{\partial(\mu(x))}{\partial x_i}\mathring{\sigma}_{ij}(\mathring{q}^k,\mathring{u}^k)(\boldsymbol{x}-\boldsymbol{y}) = O(|\boldsymbol{x}-\boldsymbol{y}|^{-1})$$

is a weakly singular remainder. This implies that the pair $(q^k,\boldsymbol{u}^k)$ is a parametrix for the Stokes operator $\boldsymbol{A}$.

Parametrix-based volume and surface potentials

Let ρ and $\boldsymbol{\rho}$ be sufficiently smooth scalar and vector functions on $\overline{\Omega}$, e.g., $\rho \in D(\overline{\Omega}), \boldsymbol{\rho} \in \boldsymbol{D}(\overline{\Omega})$. For $\boldsymbol{y} \in \mathbb{R}^2$, similar to (Mikhailov and Portillo, 2015; Mikhailov and Portillo, 2019), let us define the parametrix-based Newton-type and remainder vector potentials operators for the velocity,

$$[\boldsymbol{U\rho}]_k(\boldsymbol{y}) = U_{kj}\rho_j(\boldsymbol{y}) := \int_\Omega u_j^k(\boldsymbol{x},\boldsymbol{y})\rho_j(\boldsymbol{x})d\boldsymbol{x}$$

$$[\mathbf{R}\rho]_k(\boldsymbol{y}) = \mathrm{R}_{kj}\rho_j(\boldsymbol{y}) := \int_\Omega \mathrm{R}_{kj}(\boldsymbol{x},\boldsymbol{y})\rho_j(\boldsymbol{x})d\boldsymbol{x}, \quad \boldsymbol{y} \in \mathbb{R}^2$$

and the scalar Newton-type and remainder potentials for the pressure,

$$\mathcal{Q}\rho(\boldsymbol{y}) = \boldsymbol{Q}\cdot\rho(\boldsymbol{y}) = Q_j\rho_j(\boldsymbol{y}) := -\int_\Omega q^j(x,y)\rho_j(\boldsymbol{x})d\boldsymbol{x} \quad (3.7)$$

$$R^{\bullet}\rho(\boldsymbol{y}) = -2\langle\partial_i\mathring{q}^j(.,\boldsymbol{y}),\rho_i\partial_j\mu\rangle_\Omega - 2\rho_i(\boldsymbol{y})\partial_i\mu(\boldsymbol{y}) \quad (3.8)$$

$$= -2v.p.\int_\Omega \frac{\partial_i\mathring{q}^j(x,y)}{\partial x_i}\frac{\partial\mu(x)}{\partial x_i}\rho_j(\boldsymbol{x})dx - \rho_j(\boldsymbol{y})\frac{\partial\mu(y)}{\partial y_j}. \quad (3.9)$$

The integral in (3.9) is understood as a 2D strongly singular integral in the Cauchy sense. The equality (3.9) hold by the using the same procedure as in 3D case, Mikhailov and Portillo (2019).

For the velocity, the parametrix-based single layer and double layer potentials are defined for $\boldsymbol{y} \notin \partial\Omega$ as:

$$[\boldsymbol{V\rho}]_k(\boldsymbol{y}) = V_{kj}\rho_j(\boldsymbol{y}) := -\int_{\partial\Omega} u_j^k(\boldsymbol{x},\boldsymbol{y})\rho_j(\boldsymbol{x})dS_{\boldsymbol{x}}$$

$$[\boldsymbol{W\rho}]_k(\boldsymbol{y}) = W_{kj}\rho_j(\boldsymbol{y}) := -\int_{\partial\Omega} T_j^+(\boldsymbol{x};q^k,\boldsymbol{u}^k)(\boldsymbol{x},\boldsymbol{y})\rho_j(\boldsymbol{x})dS_{\boldsymbol{x}},$$

and for pressure in the variable coefficient Stokes system, the single layer and double layer potentials are defined for $\boldsymbol{y} \notin \partial\Omega$ as:

$$\Pi^s\boldsymbol{\rho}(\boldsymbol{y}) = \Pi_j^s\rho_j(\boldsymbol{y}) := \int_{\partial\Omega} \mathring{q}^j(\boldsymbol{x},\boldsymbol{y})\rho_j(\boldsymbol{x})dS_{\boldsymbol{x}}$$

$$\Pi^d\boldsymbol{\rho}(\boldsymbol{y}) = \Pi_j^d\rho_j(\boldsymbol{y}) := 2\int_{\partial\Omega}\frac{\partial\mathring{q}^j(x,y)}{\partial n(x)}\mu(\boldsymbol{x})\rho_j(\boldsymbol{x})dS_{\boldsymbol{x}}.$$

The corresponding boundary integral (pseudo-differential) operators of direct surface values of the single layer potential and the double layer potential, the traction of the single layer potential and the double layer potential are

$$[\mathcal{V}\boldsymbol{\rho}]_k(\boldsymbol{y}) = \mathcal{V}_{kj}\rho_j(\boldsymbol{y}) := -\int_{\partial\Omega} u_j^k(\boldsymbol{x},\boldsymbol{y})\rho_j(\boldsymbol{x})dS_{\boldsymbol{x}}, \quad \boldsymbol{y} \in \partial\Omega$$

$$[\mathcal{W}\boldsymbol{\rho}]_k(y) = \mathcal{W}_{kj}\rho_j(y) := -\int_{\partial\Omega} T_j^+(\boldsymbol{x};q^k,\boldsymbol{u}^k)(\boldsymbol{x},\boldsymbol{y})\rho_j(\boldsymbol{x})dS_{\boldsymbol{x}}, \quad \boldsymbol{y} \in \partial\Omega$$

$$[\mathcal{W}'\rho]_k(y) = \mathcal{W}'_{kj}\rho_j(y) := -\int_{\partial\Omega} T_j^+(y;q^k,\boldsymbol{u}^k)(\boldsymbol{x},\boldsymbol{y})\rho_j(\boldsymbol{x})dS_{\boldsymbol{x}}, \quad \boldsymbol{y} \in \partial\Omega$$

$$\boldsymbol{\mathcal{L}}^{\pm}\boldsymbol{\rho}(\boldsymbol{y}) := \boldsymbol{T}^{\pm}(\Pi^d\boldsymbol{\rho},\boldsymbol{W\rho})(\boldsymbol{y}), \quad \boldsymbol{y} \in \partial\Omega$$

where $\boldsymbol{T}^{\pm}$ are the traction operators (see, e.g. Mikhailov and Portillo (2015), Mikhailov and Portillo (2019)).

The parametrix-based integral operators depending on the variable coefficient, μ, can be expressed in terms of the corresponding integral operators for the constant coefficient case, $\mu = 1$, marked by $\mathring{\ }$, as in, Mikhailov and Portillo (2015) and (Mikhailov and Portillo, 2019, Eq.(4.6)-(4.11)). The proof of the relations (3.10)-(3.17) is given in the Appendix.

$$\boldsymbol{U\rho} = \frac{1}{\mu}\mathring{\boldsymbol{U}}\boldsymbol{\rho}, \quad (3.10)$$

$$[R\rho]_k = \frac{-1}{\mu}\left[\frac{\partial}{\partial y_j}\mathring{U}_{ki}(\rho_j\partial_i\mu)(y) + \frac{\partial}{\partial y_i}\mathring{U}_{kj}(\rho_j\partial_i\mu) - \mathring{Q}_k(\rho_j\partial_j\mu)\right], \quad (3.11)$$

$$\mathcal{Q}\rho = \frac{1}{\mu(y)}\mathring{\mathcal{Q}}(\mu\rho), \quad \boldsymbol{R}^{\bullet}\boldsymbol{\rho} = -2\frac{\partial}{\partial y_i}\mathring{Q}_j(\rho_j\partial_i\mu) - \rho_j\frac{\partial\mu}{\partial y_i}, \quad (3.12)$$

$$\boldsymbol{V\rho} = \frac{1}{\mu}\mathring{\boldsymbol{V}}\boldsymbol{\rho}, \quad \boldsymbol{W\rho} = \frac{1}{\mu}\mathring{\boldsymbol{W}}(\boldsymbol{\mu\rho}), \quad (3.13)$$

$$\mathcal{V}\boldsymbol{\rho} = \frac{1}{\mu}\mathring{\mathcal{V}}\boldsymbol{\rho}, \quad \mathcal{W}\boldsymbol{\rho} = \frac{1}{\mu}\mathring{\mathcal{W}}(\boldsymbol{\mu\rho}), \quad (3.14)$$

$$\Pi^s\boldsymbol{\rho} = \mathring{\Pi}^s\boldsymbol{\rho}, \quad \Pi^d\boldsymbol{\rho} = \mathring{\Pi}^d(\boldsymbol{\mu\rho}), \quad (3.15)$$

$$[\mathcal{W}'\rho]_k = [\mathring{\mathcal{W}}'\rho]_k - \left(\frac{\partial_i\mu}{\mu}[\mathring{\mathcal{V}}\rho]_k + \frac{\partial_k\mu}{\mu}[\mathring{\mathcal{V}}\rho]_i\right)n_i, \quad (3.16)$$

$$\hat{\boldsymbol{\mathcal{L}}}(\tau) := \boldsymbol{\mathcal{L}}(\mu\tau). \quad (3.17)$$

Note that the constant-coefficient velocity potentials $\mathring{\boldsymbol{U}}\boldsymbol{\rho}$, $\mathring{\boldsymbol{V}}\boldsymbol{\rho}$ and $\mathring{\boldsymbol{W}}\boldsymbol{\rho}$ are divergence-free in $\Omega^{\pm}$, the corresponding potentials $\boldsymbol{U\rho}$, $\boldsymbol{V\rho}$ and $\mathcal{W}\boldsymbol{\rho}$ are not divergence-free for the variable coefficient $\mu(\boldsymbol{y})$, see, e.g. Mikhailov and Portillo (2019). For the constant-coefficient potentials we have the following well-known relations:

$$\mathring{A}\left(\mathring{\Pi}^s\rho, \mathring{V}\rho\right) = 0, \quad \mathring{A}\left(\mathring{\Pi}^d\rho, \mathring{W}\rho\right) = 0, \quad \mathring{A}\left(\mathcal{Q}\rho, \mathring{U}\rho\right) = \rho \quad (3.18)$$

Theorem 3.1 *Let $s \in \mathbb{R}$, the following operators are continuous:*

$$\boldsymbol{V}: \boldsymbol{H}^s(\partial\Omega) \to \boldsymbol{H}^{s+\frac{3}{2}}(\Omega), \quad \boldsymbol{W}: \boldsymbol{H}^s(\partial\Omega) \to \boldsymbol{H}^{s+\frac{1}{2}}(\Omega), \tag{3.19}$$
$$\mathcal{V}: \boldsymbol{H}^s(\partial\Omega) \to \boldsymbol{H}^{s+1}(\partial\Omega), \quad \mathcal{W}: \boldsymbol{H}^s(\partial\Omega) \to \boldsymbol{H}^{s+1}(\partial\Omega), \tag{3.20}$$
$$\mathcal{L}^{\pm}: \boldsymbol{H}^s(\partial\Omega) \to \boldsymbol{H}^{s-1}(\partial\Omega), \quad \mathcal{W}': \boldsymbol{H}^s(\partial\Omega) \to \boldsymbol{H}^{s+1}(\partial\Omega), \tag{3.21}$$
$$(\Pi^s, V): H^{-\frac{1}{2}}(\partial\Omega) \to H^{1,0}(\Omega;A), \quad (\Pi^d, W): H^{\frac{1}{2}}(\partial\Omega) \to H^{1,0}(\Omega;A) \tag{3.22}$$
$$(\Pi^s, V): H^{-\frac{1}{2}}(\partial\Omega) \to H^{1,0}_{div,*}(\Omega;A), (\Pi^d, W): H^{\frac{1}{2}}(\partial\Omega) \to H^{1,0}_{div,*}(\Omega;A). \tag{3.23}$$

Moreover, the following operators are compact,

$$\boldsymbol{\mathcal{V}}: \boldsymbol{H}^s(\partial\Omega) \to \boldsymbol{H}^s(\partial\Omega), \tag{3.24}$$
$$\boldsymbol{\mathcal{W}}: \boldsymbol{H}^s(\partial\Omega) \to \boldsymbol{H}^s(\partial\Omega), \tag{3.25}$$
$$\mathcal{W}': \boldsymbol{H}^s(\partial\Omega) \to \boldsymbol{H}^s(\partial\Omega). \tag{3.26}$$

Proof. The continuity of the operators for the constant coefficient case is proved in [4]. Consequently, from the relations (3.10)-(3.16) follows the continuity of variable coefficient operators (3.19) - (3.21) as well and the continuity of the operators (3.22) and (3.23) can be proved similar to (Mikhailov and Portillo (2019), Theorem 4.3). The compactness of operators (3.24) - (3.26) is implied by the Rellich compactness embedding theorem.

Theorem 3.2 *Let Ω be a bounded open region $\mathbb{R}^2$ with closed, infinitely smooth boundary $\partial\Omega$. The following operators are continuous:*

$$\boldsymbol{U}: \widetilde{\boldsymbol{H}}^s(\Omega) \to \boldsymbol{H}^{s+2}(\Omega), s \in \mathbb{R}, \tag{3.27}$$
$$\boldsymbol{U}: \boldsymbol{H}^s(\Omega) \to \boldsymbol{H}^{s+2}(\Omega), s > \frac{-1}{2}, \tag{3.28}$$
$$\boldsymbol{R}: \widetilde{\boldsymbol{H}}^s(\Omega) \to \boldsymbol{H}^{s+1}(\Omega), s \in \mathbb{R}, \tag{3.29}$$
$$\boldsymbol{R}: \boldsymbol{H}^s(\Omega) \to \boldsymbol{H}^{s+1}(\Omega), s > \frac{-1}{2}, \tag{3.30}$$
$$\boldsymbol{Q}: \widetilde{H}^s(\Omega) \to \boldsymbol{H}^{s+1}(\Omega), s \in \mathbb{R}, \tag{3.31}$$
$$\boldsymbol{Q}: H^s(\Omega) \to \boldsymbol{H}^{s+1}(\Omega), s > \frac{-1}{2}, \tag{3.32}$$
$$Q: \widetilde{\boldsymbol{H}}^s(\Omega) \to H^{s+1}(\Omega), s \in \mathbb{R}, \tag{3.33}$$
$$Q: \boldsymbol{H}^s(\Omega) \to H^{s+1}(\Omega), s > \frac{-1}{2}, \tag{3.34}$$
$$R^{\bullet}: \widetilde{\boldsymbol{H}}^s(\Omega) \to H^s(\Omega), s > \frac{-1}{2}, \tag{3.35}$$
$$R^{\bullet}: \boldsymbol{H}^s(\Omega) \to H^s(\Omega), s > \frac{-1}{2}. \tag{3.36}$$
$$(Q, \boldsymbol{U}): \boldsymbol{H}^s(\Omega) \to \boldsymbol{H}^{s+2,0}(\Omega; \boldsymbol{A}), s \geq 0, \tag{3.37}$$
$$(\boldsymbol{R}^{\bullet}, \boldsymbol{R}): \boldsymbol{H}^s(\Omega) \to \boldsymbol{H}^{s+1,0}(\Omega; \boldsymbol{A}), s \geq 1 \tag{3.38}$$

Proof. We use similar procedure as in (Mikhailov and Portillo (2019), Theorem 4.1). Since the surface $\partial\Omega$ is infinitely differentiable, the operators $\boldsymbol{U}$ and $\boldsymbol{Q}$ are respectively pseudodifferential operators of order -2 and -1, (see, e.g.,(Hsiao and Wendland (2008), Lemma 5.6.6. and Section 9.1.3]). Then, the continuity of the operators $\boldsymbol{U}$ and $\boldsymbol{Q}$ from the 'tilde spaces' immediately follows by virtue of the mapping properties of the pseudodifferential operators. Alternatively, these mapping properties are well studied for the constant coefficient case, i.e. operators $\mathring{U}$ and $\mathring{Q}$, see, e.g. Hsiao and Wendland (2008), Lemma 5.6.6). Consequently, the respective mapping properties for the remainder operators (3.29) and (3.35) immediately follow by considering the relation (3.11).

For the remaining part of the proof, we shall that $s \in \left(\frac{-1}{2}, \frac{1}{2}\right)$. In this case, $H^s(\Omega) = \widetilde{H}^s(\Omega)$. Hence, the continuity of the operator (3.28) immediately follows from the continuity of (3.27).

Let us consider now that $s \in \left(\frac{1}{2}, \frac{3}{2}\right)$. Then, let $\boldsymbol{g} = (g_1, g_2)$, $\boldsymbol{g} \in \boldsymbol{H}^s(\Omega)$. It is well known that $\partial_i g_j \in H^{s-1}(\Omega)$ and that $\gamma^+ g \in \boldsymbol{H}^{s-\frac{1}{2}}(\partial\Omega)$ due to the continuity of the ∂_i operator and the trace theorem. Consequently, it is possible to use the representation obtained by integrating by parts, (see, e.g. Chkadua et al. (2009), Theorem 3.8)),

$$\partial_i \mathring{U}_{kj} g_j = \mathring{U}_{kj}(\partial_i g_j) + \mathring{V}_{kj}(\gamma^+ g_j n_i), i, j, k \in \{1, 2\} \tag{3.39}$$

where n_i denotes the components of the normal vector to the surface $\partial\Omega$ directed outwards the domain.

Due to the mapping properties of $\boldsymbol{V}$ and $\boldsymbol{U}$ in Theorems 3.1 and 3.2, we deduce that $\partial_i \mathring{U} \ g \in \boldsymbol{H}^{s+1}(\Omega)$ is continuous for $i \in \{1,2\}$. Consequently, from relations (3.10) and (3.13), for $s \in \left(\frac{1}{2}, \frac{3}{2}\right)$, immediately follows the continuity of the operator (3.28). Furthermore, by induction on $k \in \mathbb{N}$, using the representation in identity (3.39) and the fact that the operator (3.28) is continuous for $s \in \left(k - \frac{1}{2}, k + \frac{1}{2}\right)$, follows that the operator (3.28) is also continuous for $s \in \left(k - \frac{1}{2}, k + \frac{1}{2}\right)$. The continuity of the operator (3.28) for the cases $s = k + \frac{1}{2}$ is proved by applying the theory of interpolation of Bessel potential spaces, see, e.g. Triebel (1978), Chapter 4.

Continuity of the operator (3.32) and hence (3.34) can be proved following a similar argument. Continuity of the remainder operators (3.30) and (3.36)) immediately follows from the continuity of operators (3.28) and (3.32) by relations (3.11) and (3.12).

Also the Continuity of the operator (3.37), (3.38) can be proved similar as in Mikhailov and Portillo (2019), Theorem 4.1.

Theorem 3.3 *Let* $s \in \mathbb{R}$. The following operators

$$\Pi^s: \boldsymbol{H}^{s-\frac{3}{2}}(\partial\Omega) \to H^{s-1}(\Omega), \quad s \in \mathbb{R},$$

$$\Pi^d: \boldsymbol{H}^{s-\frac{1}{2}}(\partial\Omega) \to H^{s-1}(\Omega), \quad s \in \mathbb{R},$$

$$\Pi^s: \boldsymbol{H}^{\frac{-1}{2}}(\partial\Omega) \to L^2_*(\Omega),$$

$$\Pi^d: \boldsymbol{H}^{\frac{1}{2}}(\partial\Omega) \to L^2_*(\Omega)$$

are continuous.

Proof. For constant coefficient case, $\mu = 1$, the corresponding operators are proved in (Hsiao and Wendland (2008), Lemma 5.6.6). Due to relations (3.15), the theorem holds true.

Theorem 3.4 *Let* $\boldsymbol{\tau} \in \boldsymbol{H}^{\frac{1}{2}}(\partial\Omega)$ *and* $\boldsymbol{\rho} \in \boldsymbol{H}^{\frac{-1}{2}}(\partial\Omega)$. *Then, the following jump relations hold*

$$\gamma^{\pm}\boldsymbol{V\rho} = \boldsymbol{\mathcal{V}\rho}, \quad \gamma^{\pm}\boldsymbol{W\tau} = \mp\frac{1}{2}\boldsymbol{\tau} + \boldsymbol{\mathcal{W}\tau}, \tag{3.40}$$

$$\boldsymbol{T}^{\pm}(\Pi^s\boldsymbol{\rho}, \boldsymbol{V\rho}) = \pm\frac{1}{2}\boldsymbol{\rho} + \boldsymbol{\mathcal{W}'\rho}. \tag{3.41}$$

Proof. For constant coefficient case, $\mu = 1$, the jump properties for the corresponding operators are proved in Hsiao and Wendland (2008), Lemma 5.6.5). Due to relations (3.13) and (3.16), the theorem holds for (3.40) and (3.41) as well.

Proposition 3.5 *The following operators are compact,*

$$\boldsymbol{R}: \boldsymbol{H}^s(\Omega) \to \boldsymbol{H}^s(\Omega), \quad R^{\bullet}: \boldsymbol{H}^s(\Omega) \to H^{s-1}(\Omega), \quad s \in \mathbb{R}$$

$$\gamma^{+}R: H^s(\Omega) \to H^{s-\frac{1}{2}}(\partial\Omega), \quad T^{\pm}(R^{\bullet}, R): H^{1,0}(\Omega; A) \to H^{\frac{-1}{2}}(\partial\Omega),$$

$$\boldsymbol{T}^{\pm}(R^{\bullet}, \boldsymbol{R}): \boldsymbol{H}^{1,0}_{div,*}(\Omega; \boldsymbol{A}) \to \boldsymbol{H}^{\frac{-1}{2}}(\partial\Omega).$$

Proof. The proof of the compactness for the operators $\boldsymbol{R}$, $\gamma^{+}\boldsymbol{R}$ and $R^{\bullet}$ immediately follows from Theorem 3.2 and the trace theorem along with the Rellich compact embedding theorem. To prove the compactness of the operator $\boldsymbol{T}^{\pm}(R^{\bullet}, \boldsymbol{R})$ we consider a function $\boldsymbol{g} \in \boldsymbol{H}^1(\Omega)$. Then, $(R^{\bullet}\boldsymbol{g}, \boldsymbol{Rg}) \in \boldsymbol{H}^1(\Omega)$ and hence, $(R^{\bullet}\boldsymbol{g}, \boldsymbol{Rg}) \in \boldsymbol{H}^{1,0}(\Omega; \boldsymbol{A})$.

The traction operator $\boldsymbol{T}^{\pm}$ is the composite of a differential operator, with respect to the first variable and with respect to the second variable, and the trace operator $\gamma^{\pm}$ which reduces the regularity by $1/2$ according to the Trace Theorem. Therefore, $\boldsymbol{T}^{\pm}(R^{\bullet}\boldsymbol{g}, \boldsymbol{Rg}) \in \boldsymbol{H}^{\frac{1}{2}}(\partial\Omega)$. Then, the compactness follows from the Rellich compact embedding $\boldsymbol{H}^{\frac{1}{2}}(\partial\Omega) \subset \boldsymbol{H}^{\frac{-1}{2}}(\partial\Omega)$.

Invertibility of the hydrodynamic single layer potential operator in 2D

Suppose that $\boldsymbol{\rho} = \boldsymbol{T}(p, \boldsymbol{v})$ where $(p, \boldsymbol{v}) \epsilon \boldsymbol{H}^{1,0}{}_*(\Omega)$. The single layer potential operator is a Fredholm of index zero.In 3D case, for $\boldsymbol{\rho} \in \boldsymbol{H}^{\frac{-1}{2}}(\partial\Omega)$, if $\boldsymbol{V\rho}(\boldsymbol{y}) = \boldsymbol{0}$, $\boldsymbol{y} \in \Omega$, then $\boldsymbol{\rho} = \boldsymbol{0}$. But this is not generally true for 2D case. It is well known (Domínguez and Sayas (2006), 696, p.707] for some 2D domains the kernel of the operator $\overset{\circ}{\boldsymbol{\mathcal{V}}} : \boldsymbol{H}^{\frac{-1}{2}}(\partial\Omega) \to \boldsymbol{H}^{\frac{1}{2}}(\partial\Omega)$ is non-zero, which is by the first relation in (3.14) implies that the kernel of the operator $\boldsymbol{\mathcal{V}}$ is nontrivial as well. The following example is from Cialdea et al (2013) and Lemma 1) and illustrates this fact.

Theorem 4.1 *Take the density function* $\rho_j^m = \delta_{jm}$ *and* $\Omega = B(0, R)$ *to be a disc of radius* R *centered at the origin and* $\partial\Omega = \partial B(0, R)$ *be the circular boundary of the disc. We want to show that*

$$\mu(y)\mathcal{V}_{kj}\rho_j^m(y) = \overset{\circ}{\mathcal{V}}\rho_j^m(y) = \frac{-R}{2}\delta_{km}\left(2\log\frac{R}{r_0} - 1\right),\ |y| \le R, k, j, m \in \{1,2\}.$$

Indeed,

$$\mathcal{V}_{kj}\rho_j^m(y) = -\sum_{j=1}^{2}\left(\frac{1}{4\pi}\int_{|x|=R}\{\delta_j^k \log\frac{|x-y|}{r_0} - \frac{(x_j-y_j)(x_k-y_k)}{|x-y|^2}\}\,\delta_j^m\, dS_x\right)$$

$$= -\sum_{j=1}^{2}\left(\frac{\delta_k^m}{4\pi}\int_{|x|=R}\log|x-y|dS_x + \frac{1}{4\pi}\int_{|x|=R}\frac{(x_m-y_m)(x_k-y_k)}{|x-y|^2}dS_x\right)$$

$$+\sum_{j=1}^{2}\frac{\delta_k^m}{4\pi}\int_{|x|=R}\log r_0\, dS_x$$

First consider the integral $\int_{|x|=R}\log|\boldsymbol{x}-\boldsymbol{y}|\,dS_{\boldsymbol{x}}$ and let $u(\boldsymbol{y}) = \int_{|\boldsymbol{x}|=R}\log|\boldsymbol{x}-\boldsymbol{y}|\,dS_{\boldsymbol{x}}$. Let us fix $\boldsymbol{y}_0 \in \partial B(0, R)$. For any $\boldsymbol{y} \in \partial B(0, R)$

$$u(\boldsymbol{y}) = \int_{|x|=R}\log|\boldsymbol{x}-\boldsymbol{y}_0|\,dS_{\boldsymbol{x}} = \int_{xVR}\log|\boldsymbol{x}-\boldsymbol{y}|\,dS_{\boldsymbol{x}}$$

Since $\log|\boldsymbol{x}-\boldsymbol{y}|$ is harmonic in $B(0, R)$, constant on $\partial B(0, R)$ and continuous in $B(0, R)$.Then it is constant in $B(0, R)$.

$$u(\boldsymbol{y}) = u(0) = \int_{|x|=R}\log|\boldsymbol{x}|\,dS_x = 2\pi R\log R, for\ |\boldsymbol{y}| \le R$$

Second consider the integral $\int_{|x|=R}\frac{(x_m-y_m)(x_k-y_k)}{|x-y|^2}dS_x$ for each $m, k \in \{1,2\}$ and $|\boldsymbol{y}| \le R$. Before calculate the above integral let us show first

$$\int_{|x|=R}|y-x|^2\log|y-x|\,dS_x = 2\pi R(R^2\log R \quad + (1+\log R)|y|^2),\ |y| \le R$$

To show this let $I(\boldsymbol{y}) = \int_{|\boldsymbol{x}|=R}|\boldsymbol{y}-\boldsymbol{x}|^2\log|\boldsymbol{y}-\boldsymbol{x}|\,dS_{\boldsymbol{x}}$ and fix $\boldsymbol{y}_0 \in \partial B(0, R)$. For any $\boldsymbol{y} \in \partial B(0, R)$ we have $\int_{|x|=R}\boldsymbol{y}-\boldsymbol{x}^2\log|\boldsymbol{y}-\boldsymbol{x}|\,dS(x) = \int_{|x|=R}|\boldsymbol{y}_0-\boldsymbol{x}|^2\log|\boldsymbol{y}_0-\boldsymbol{x}|\,dS_{\boldsymbol{x}}$ and then $\boldsymbol{u}$ is constant on $\partial B(0, R)$. Moreover

$$\Delta I(\boldsymbol{y}) = 4\int_{|x|=R}(\log|\boldsymbol{y}-\boldsymbol{x}| + 1)dS_{\boldsymbol{x}}$$

and then also ΔI is constant on $\partial B(0, R)$. Since ΔI is harmonic in $B(0, R)$ and continuous on $B(0, R)$, it is constant in $B(0, R)$ and then

$$\Delta I(y) = \Delta I(0) = 4\int_{|x|=R}(\log|x| + 1)dS_x = 8\pi R(1 + \log R)$$

The function $I(\boldsymbol{y}) - 2\pi R(1 + \log R)|\boldsymbol{y}|^2$ is continuous on $B(0,R)$, harmonic in $B(0,R)$ and constant on $\partial B(0,R)$. Then it is constant in $B(0,R)$ and

$$I(y) - 2\pi R(1 + \log R)|y|^2 = I(0) = \int_{|x|=R} |x|^2 \log|x|\, dS_x = 2\pi R(R^2 \log R)$$

so that,

$$I(\boldsymbol{y}) = 2\pi R(R^2 \log R + (1 + \log R)|\boldsymbol{y}|^2)$$

for $|\boldsymbol{y}| \le R$

Since,

$$\partial_{y_1} \int_{|x|=R} |y-x|^2 \log|y-x| dS_x = \int_{|x|=R} \left(2(y_1 - x_1)\log|y-x| + (y_1 - x_1)\right) dS_x,$$

$$\partial_{y_1 y_1} \int_{|x|=R} |y-x|^2 \log|y-x| dS_x = \int_{|x|=R} \left(2\log|y-x| + \frac{2(y_1-x_1)^2}{|y-x|^2} + 1\right) dS_x,$$

It implies that,

$$\int_{|x|=R} \frac{(y_1 - x_1)^2}{|y-x|^2} dS_x = \frac{1}{2}\partial_{y_1 y_1} \int_{|x|=R} |y-x|^2 \log|y-x|\, dS_x - \int_{|x|=R} (\log|y-x| + \frac{1}{2}) dS_x$$

Therefore,

$$\int_{|x|=R} \frac{(y_1-x_1)^2}{|y-x|^2} dS_x = 2\pi R(1+\log R) - 2\pi R\log R - \pi R = \pi R, \quad |y| \le R$$

In similar way,

$$\int_{|x|=R} \frac{(x_2-y_2)^2}{|y-x|^2} dS_x = 2\pi R(1+\log R) - 2\pi R\log R - \pi R = \pi R, \quad |y| \le R$$

$$\partial_{y_2 y_1} \int_{|x|=R} |y-x|^2 \log|y-x| dS_x = \int_{|x|=R} \frac{2(y_1-x_1)(y_2-x_2)}{|y-x|^2} dS_x,$$

and

$$\int_{|x|=R} \frac{(y_1-x_1)(y_2-x_2)}{|y-x|^2} dS_x = \frac{1}{2}\partial_{y_2 y_1} \int_{|x|=R} |y-x|^2 \log|y-x| dS_x$$

$$= \frac{1}{2}\partial_{y_2}\left(2\pi R(R\log R + (1+\log R)2y_1)\right) = 0$$

$$\int_{|\boldsymbol{x}|=R} \frac{(x_m-y_m)(x_k-y_k)}{|\boldsymbol{x}-\boldsymbol{y}|^2} dS_{\boldsymbol{x}} = \delta_k^m \pi R.$$

Hence,

$$\mu(y)V_{kj}\rho_j^m(y) = \mathring{\mathcal{V}}\,\rho_j^m(y) = \sum_{j=1}^{2}\left(\frac{-\delta_k^m}{4\pi}(2\pi R\log R) + \frac{1}{4\pi}\delta_k^m(\pi R)\right) + \delta_k^m R\log r_0$$

$$= \frac{-\delta_k^m R}{2}\left(2\log\frac{R}{r_0} - 1\right). \blacksquare$$

Corollary 4.2 *If we set* $R = r_0 \exp\left(\frac{1}{2}\right)$ *in Example 1, with* $\mu(\boldsymbol{y}) \neq 0$, *we get,* $[\boldsymbol{V\rho}]_k(\boldsymbol{y}) = 0$ *in* $\overline{\Omega}$.

In order to have invertibility for the single layer potential operator in 2D, we define the subspace $\boldsymbol{H}_*^s(\partial\Omega)$ of the space $\boldsymbol{H}^s(\partial\Omega)$, see e.g., (Domínguez and Sayas (2006), Appendix A, in particular $s = -\frac{1}{2}$ and $\frac{1}{2}$),

$$\boldsymbol{H}_*^s(\partial\Omega) := \{\boldsymbol{\rho} \in \boldsymbol{H}^s(\partial\Omega) : \langle\rho_i, 1\rangle_{L^2(\partial\Omega)} = 0 \text{ for } i = 1,2\}, \quad (4.1)$$

where the norm in $\boldsymbol{H}_*^s(\partial\Omega)$ is induced norm of $\boldsymbol{H}^s(\partial\Omega)$.

The boundary integral operator, $\mathring{\mathcal{V}}$ is a Fredholm operator of index zero on $\boldsymbol{H}^{\frac{-1}{2}}(\partial\Omega)$ as indicated in Domínguez and Sayas (2006), Lemma A.2) and also $\mathcal{V}: \boldsymbol{H}_*^{\frac{-1}{2}}(\partial\Omega) \to \boldsymbol{H}_*^{\frac{1}{2}}(\partial\Omega)$ by the relation (3.14).

Theorem 4.3 *If* $\boldsymbol{\Psi} \in \boldsymbol{H}_*^{\frac{-1}{2}}(\partial\Omega)$ *satisfies* $\mathcal{V}\boldsymbol{\Psi} = \boldsymbol{0}$ *on* $\partial\Omega$, *then* $\boldsymbol{\Psi} = \boldsymbol{0}$.

Proof. Let us proof by using similar procedure as in [8, Corollary 8.11]. The single layer potential $(\mathring{p}, \mathring{v}) = (\mathring{\Pi}^s\Psi, \mathring{V}\Psi)$ satisfies

$$\Delta\mathring{v} - \nabla\mathring{p} = 0 \text{ in } \Omega^{\pm}, \quad (4.2)$$

$$div(\mathring{v}) = 0 \text{ in } \Omega^{\pm}, \quad (4.3)$$

$$\gamma^{\pm}\mathring{v} = 0 \text{ on } \partial\Omega. \quad (4.4)$$

For the exterior problem, we use the following growth conditions at infinity,

$$\mathring{v}(\boldsymbol{x}) = \boldsymbol{A}\log|\boldsymbol{x}| + O(1),$$

$$\mathring{p} = O(|\boldsymbol{x}|^{-1}) \text{as } |\boldsymbol{x}| \to \infty,$$

where $\boldsymbol{A} = \int_{\partial\Omega} \boldsymbol{\Psi} ds_{\boldsymbol{x}}$, (see, e.g.,(Hsiao and Wendland (2008), Eq.(2.3.18), Eq.(2.3.19) and Eq.(2.3.22))). Since $\boldsymbol{\Psi} \in \boldsymbol{H}_*^{\frac{-1}{2}}(\partial\Omega)$, i.e., $\int_{\partial\Omega} \boldsymbol{\Psi} ds_{\boldsymbol{x}} = 0$, it follows that $\mathring{v} = \boldsymbol{0}$ and $\mathring{p} = 0$ in Ω^-.

For the interior problem, using first Green identity, we get, $\mathring{v} = 0$ and using interior part of (4.2), we have that $\nabla\mathring{p} = 0$ in Ω. Since $\mathring{p} \in L_*^2(\Omega)$, then $\mathring{p} = 0$. Consequently, $\boldsymbol{\Psi} = \mathring{T}^+(\mathring{\Pi}^s\Psi, \mathring{V}\Psi) - T^-(\mathring{\Pi}^s\Psi, \mathring{V}\Psi) = 0$. Thus, $\boldsymbol{\Psi} = \boldsymbol{0}$. That is, from $\mathring{\mathcal{V}}\boldsymbol{\Psi} = \boldsymbol{0}$ follows that $\boldsymbol{\Psi} = \boldsymbol{0}$ and relation (3.14) implies for the operator $\mathcal{V}$ as well.

Theorem 4.4 *Let* $\Omega \subset \mathbb{R}^2$ be a bounded domain. Then the single layer potential $\mathcal{V}: \boldsymbol{H}_*^{\frac{-1}{2}}(\partial\Omega) \to \boldsymbol{H}_*^{\frac{1}{2}}(\partial\Omega)$ *is* invertible.

Proof. Due to Domínguez and Sayas (2006), Lemma A.2) the operator $\mathring{\mathcal{V}}$ is Fredholm of index zero and the first relation in (3.14) implies that so is operator $\mathcal{V}$. Theorem 4.3 implies the injectivity of operetor $\mathcal{V}$ and hence the invertibility of operator $\mathcal{V}$.

To prove the $\boldsymbol{H}^{\frac{-1}{2}}(\partial\Omega)$- ellipticity of the single-layer potential operator for the Stokes system we first introduce the fundamental solution for $r_0 > 0$

$$\mathring{u}_j^k(\boldsymbol{x},\boldsymbol{y}) = \frac{1}{4\pi}[\delta_j^k \log\frac{|\boldsymbol{x}-\boldsymbol{y}|}{r_0} - \frac{(x_j-y_j)(x_k-y_k)}{|\boldsymbol{x}-\boldsymbol{y}|^2}]$$

$$\mathcal{V}_j^k w_j(\boldsymbol{x},\boldsymbol{y}) = -\int_{\partial\Omega} \frac{1}{4\pi}[\delta_j^k \log\frac{|x-y|}{r_0} - \frac{(x_j-y_j)(x_k-y_k)}{|x-y|^2}]w_j dS_x$$

In Kohr et al. (2013), Appendix], the single layer potential operator $\overset{\circ}{\mathcal{V}}$ is positive and so is $\overset{\circ}{\mathcal{V}}$, that is,

$$\langle \overset{\circ}{\mathcal{V}} \; \boldsymbol{w}, \boldsymbol{w} \rangle_{\partial\Omega} > 0 \tag{4.5}$$

for a non zero $\boldsymbol{w}$ that satisfy $\int_{\partial\Omega} \boldsymbol{w} dS = \boldsymbol{0}$ and follows the theorem.

Consider the following basis of the space of rigid body translations in plane: $\boldsymbol{\mu}^1 = [1,0]^T$, $\boldsymbol{\mu}^2 = [0,1]^T$

Theorem 4.5 *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary $\partial\Omega$. Let $\partial\Omega$ is contained in the interior of a circular disk with a radius R. If $r_0 = Re^{\frac{-1}{2}}$, then $\boldsymbol{\mathcal{V}}$ is $\boldsymbol{H}^{\frac{-1}{2}}(\partial\Omega)$- elliptic.*

Proof. First we show the positivity of $\overset{\circ}{\mathcal{V}}$ and we use a similar procedure as in Vodička and Mantič(2004), Proposition 2]. Let ∂B denote the boundary of the disk with radius R containing $\partial\Omega$. The operator $\overset{\circ}{\mathcal{V}}$ is positive by (4.5). So that

$$\langle \overset{\circ}{\mathcal{V}} \; \boldsymbol{w}, \boldsymbol{w} \rangle_{(\partial\Omega\cup\partial B)} > 0 \tag{4.6}$$

for nonzero $\boldsymbol{w} \in \boldsymbol{H}^{\frac{-1}{2}}(\partial\Omega \cup \partial B)$ satisfying $\int_{\partial\Omega\cup\partial B} w(\boldsymbol{x}) dS_{\boldsymbol{x}} = 0$, $\int_{\partial\Omega} w_j(\boldsymbol{x}) dS_{\boldsymbol{x}} = c_j$ and $\int_{\partial B} w_j(\boldsymbol{x}) dS_{\boldsymbol{x}} = 2\pi\omega_i$ for a chosen ω_i.

Decomposing the integral in (4.6) yields

$$0 < \langle \overset{\circ}{\mathcal{V}}_{jk} w_j, w_j \rangle_{\partial\Omega} + \int_{\partial\Omega} w_j(x)\left[-\int_{\partial B} \overset{\circ}{u}_{jk}(x,y) w_j dS_y\right] dS_x$$
$$+ \int_{\partial B} w_j\left[-\int_{\partial\Omega} \overset{\circ}{u}_{jk}(x,y) w_j(y) dS_y\right] dS_x$$
$$+ \int_{\partial B} w_j(\boldsymbol{x})\left[-\int_{\partial\Omega} \overset{\circ}{u}_{jk}(\boldsymbol{x},\boldsymbol{y}) w_j(\boldsymbol{y}) dS_{\boldsymbol{y}}\right] dS_{\boldsymbol{x}}. \tag{4.7}$$

The in equation (4.7), the second, third and fourth integrals becomes respectively

$$\frac{-1}{4\pi}\left[-2\log\frac{R}{r_0} + 1\right](c_1^2 + c_2^2),$$
$$\frac{-1}{4\pi}\left[-2\log\frac{R}{r_0} + 1\right](c_1^2 + c_2^2)$$

and

$$\frac{1}{4\pi}\left[-2\log\frac{R}{r_0} + 1\right](c_1^2 + c_2^2).$$

Hence, equation (4.7) becomes

$$0 < \langle \overset{\circ}{\mathcal{V}} \; \boldsymbol{w}, \boldsymbol{w} \rangle_{\partial\Omega} - \frac{1}{4\pi}\left[-2\log\frac{R}{r_0} + 1\right](c_1^2 + c_2^2). \tag{4.8}$$

Also equation (4.8) can be written as

$$0 < \langle \overset{\circ}{\mathcal{V}} \; \boldsymbol{w}, \boldsymbol{w} \rangle_{\partial\Omega} - \frac{1}{4\mu\pi}\left[-2\log\frac{R}{r_0} + 1\right](c_1^2 + c_2^2). \tag{4.9}$$

The second member in (4.9) vanishes for $R = r_0 e^{\frac{1}{2}}$, therefore (4.9) must be positive for any nonzero w. From the positivity and Gårding inequality which is indicated from [19, Eq.(A.15)] for the scaled one, so it is also $\boldsymbol{\mathcal{V}}$. We find that $\boldsymbol{\mathcal{V}}$ is $H^{\frac{-1}{2}}(\partial\Omega)$- elliptic due to Lemma 5.2.5 in [4].

Theorem 4.6 *Let* $\Omega \subset \mathbb{R}^2$ have $diam(\Omega) < R = r_0 e^{\frac{1}{2}}$, then the operator $\boldsymbol{\mathcal{V}}$ has a bounded inverse on $\boldsymbol{H}^{\frac{-1}{2}}(\partial\Omega)$.

Proof. By Theorem 4.5 the operator $\boldsymbol{\mathcal{V}}$ is $H^{\frac{-1}{2}}(\partial\Omega)$-elliptic and due to Theorem 3.1 it is also continuous, that is, bounded. Hence, by Lax-Milgram Lemma $\mathcal{V}$ has a bounded inverse.

The third Green identities

Theorem 5.1 For any $(p, \boldsymbol{v}) \in \boldsymbol{H}^{1,0}_{div}(\Omega; \boldsymbol{A})$ $(or\ \boldsymbol{H}^{1,0}_{div,*}(\Omega; \boldsymbol{A}))$ the following third Green identities hold

$$\boldsymbol{v} + \boldsymbol{R}\boldsymbol{v} - \boldsymbol{V}\boldsymbol{T}^{+}(p, \boldsymbol{v}) + \boldsymbol{W}\gamma^{+}\boldsymbol{v} = \boldsymbol{U}\boldsymbol{A}(p, \boldsymbol{v}) \text{ in } \Omega, \tag{5.1}$$
$$p + R^{\bullet}\boldsymbol{v} - \Pi^{s}\boldsymbol{T}^{+}(p, \boldsymbol{v}) + \Pi^{d}\gamma^{+}\boldsymbol{v} = \overset{\circ}{\mathcal{Q}} \; A(p, \boldsymbol{v}) \text{ in } \Omega. \tag{5.2}$$

Proof. We use similar procedures as in Mikhailov and Portillo (2015), Mikhailov and Portillo (2019) to prove. For an arbitrary fixed $\boldsymbol{y} \in \Omega$, let $B_\epsilon(\boldsymbol{y}) \subset \Omega$ be a ball with a small enough radius ϵ and centre $\boldsymbol{y} \in \Omega$, and let $\Omega_\epsilon(\boldsymbol{y}) = \Omega \backslash B_\epsilon(\Omega)$. Let first $(p, \boldsymbol{v}) \in D(\overline{\Omega}) \times \boldsymbol{D}(\overline{\Omega}) \subset \boldsymbol{H}^{1,0}_{div}(\Omega; \boldsymbol{A})$ (or $\boldsymbol{H}^{1,0}_{div,*}(\Omega; \boldsymbol{A})$).

Let us start from the velocity identity (5.1). For the parametrix, evidently, we have the inclusion $(q^k, \boldsymbol{u}^k) \in \boldsymbol{H}^{1,0}_{div}(\Omega_\epsilon(\boldsymbol{y}); \boldsymbol{A})$ $(or\ \boldsymbol{H}^{1,0}_{div,*}(\Omega_\epsilon(\boldsymbol{y}); \boldsymbol{A}))$. Therefore, we can apply the second Green identity (2.4) in the domain $\Omega_\epsilon(\boldsymbol{y})$ to $(p, \boldsymbol{v})$ and to $(q^k, \boldsymbol{u}^k)$ to obtain

$$\int_{\partial B_\epsilon} \gamma^{+}u^k(x,y).T^{+}(p(x), v(x)) ds_x - \int_{\partial B_\epsilon} \gamma^{+}v(x).T^{+}\left(x; q^k(x,y), u^k(x,y)\right) ds_x$$
$$+ \int_{\partial\Omega} \gamma^{+}u^k(x,y).T^{+}(p(x), v(x)) ds_x - \int_{\partial\Omega} \gamma^{+}v(x).T^{+}\left(x; q^k(x,y), u^k(x,y)\right) ds_x$$
$$+ \int_{\Omega_\epsilon(y)} R_{kj}(\boldsymbol{x}, \boldsymbol{y}) v_j(\boldsymbol{x}) d\boldsymbol{x} = \int_{\Omega_\epsilon(y)} \boldsymbol{A}(p, \boldsymbol{v}).u^k(\boldsymbol{x}, \boldsymbol{y}) d\boldsymbol{x} \tag{5.3}$$

Since all the functions in (5.3) are smooth, the canonical conormal derivatives coincide with the classical ones and it is easy to show that when $\epsilon \to 0$, the first integral in (5.3) tends to 0, the second tends to $-\boldsymbol{v}_k(\boldsymbol{y})$, while integrands in the remaining domain integrals are weakly singular and these integrals tend to the corresponding improper integrals, which leads us to (5.1) for $(p, \boldsymbol{v}) \in D(\overline{\Omega}) \times \boldsymbol{D}(\overline{\Omega}) \subset \boldsymbol{H}^{1,0}_{div}(\Omega; \boldsymbol{A})$ (or $\boldsymbol{H}^{1,0}_{div,*}(\Omega; \boldsymbol{A})$).

Let us now prove the pressure identity (5.2) for $(p, \boldsymbol{v}) \in D(\overline{\Omega}) \times \boldsymbol{D}(\overline{\Omega})$. One can do this using the second Green identity similar to (5.3) but we will employ a slightly different approach. Multiplying equation the Stokes operator by the fundamental pressure vector $\overset{\circ}{q}{}^{j}(\boldsymbol{x}, \boldsymbol{y})$, integrating over the domain Ω and writing it as the bilinear form, which will be then treated in the sense of distributions, we obtain

$$\left\langle \overset{\circ}{}_{q}{}^{j}(.,y),\partial_i\left[\mu(x)\left(\partial_i v_j+\partial_j v_i\right)\right]\right\rangle_{\Omega}-\left\langle \overset{\circ}{}_{q}{}^{j}(.,y),\partial_j p\right\rangle_{\Omega}=\left\langle \overset{\circ}{}_{q}{}^{j}(.,y),A_j(p,v)\right\rangle_{\Omega} \quad (5.4)$$

Applying the first Green identity to the first term, we have,

$$\left\langle \overset{\circ}{}_{q}{}^{j}(.,y),\partial_i\left[\mu(x)\left(\partial_i v_j+\partial_j v_i\right)\right]\right\rangle_{\Omega}=-\left\langle \partial_i\overset{\circ}{}_{q}{}^{j}(.,y),\mu(x)\left(\partial_i v_j+\partial_j v_i\right)\right\rangle_{\Omega}$$
$$+\left\langle \overset{\circ}{}_{q}{}^{j}(.,\boldsymbol{y}),\mu(\boldsymbol{x})\left(\partial_i v_j+\partial_j v_i\right)n_j\right\rangle_{\partial\Omega} \quad (5.5)$$

and also in the second term

$$\left\langle \overset{\circ}{}_{q}{}^{j}(.,y),\partial_j p\right\rangle_{\Omega}=-\left\langle \partial_j\overset{\circ}{}_{q}{}^{j}(.,y),p\right\rangle_{\Omega}+\left\langle \overset{\circ}{}_{q}{}^{j}(.,y),pn_j\right\rangle_{\partial\Omega}=p(y)+\left\langle \overset{\circ}{}_{q}{}^{j}(.,y),pn_j\right\rangle_{\partial\Omega} \quad (5.6)$$

Substituting (5.5) and (5.6) into (5.4) and rearranging terms we get

$$\left\langle \overset{\circ}{}_{q}{}^{j}(.,\boldsymbol{y}),\boldsymbol{A}_j(p,\boldsymbol{v})\right\rangle_{\Omega}=-\left\langle \partial_i\overset{\circ}{}_{q}{}^{j}(.,\boldsymbol{y}),\mu(\boldsymbol{x})\left(\partial_i v_j+\partial_j v_i\right)\right\rangle_{\Omega}$$
$$+\left\langle \overset{\circ}{}_{q}{}^{j}(.,\boldsymbol{y}),\mu(\boldsymbol{x})\left(\partial_i v_j+\partial_j v_i\right)n_j\right\rangle_{\partial\Omega}$$
$$+\left\langle \partial_j\overset{\circ}{}_{q}{}^{j}(.,\boldsymbol{y}),p\right\rangle_{\Omega}-\left\langle \overset{\circ}{}_{q}{}^{j}(.,\boldsymbol{y}),pn_j\right\rangle_{\partial\Omega} \quad (5.7)$$

By (2.2) we obtain

$$\left\langle \overset{\circ}{}_{q}{}^{j}(.,y),\mu\left(\partial_i v_j+\partial_j v_i\right)n_j\right\rangle_{\partial\Omega}-\left\langle \overset{\circ}{}_{q}{}^{j}(.,y),pn_j\right\rangle_{\partial\Omega}=\left\langle \overset{\circ}{}_{q}{}^{j}(.,y),T^{c+}{}_j(p,v)\right\rangle_{\partial\Omega} \quad (5.8)$$

Let us now simplify the first term in the right hand side of (5.7) using the symmetry $\partial_{x_i}\overset{\circ}{}_{q}{}^{j}(\boldsymbol{x},\boldsymbol{y})=\partial_{x_j}\overset{\circ}{}_{q}{}^{i}(\boldsymbol{x},\boldsymbol{y})$ and (3.1) . Then,

$$\left\langle \partial_i\overset{\circ}{}_{q}{}^{j}(.,y),\mu(x)\left(\partial_i v_j+\partial_j v_i\right)\right\rangle_{\Omega}=2\left\langle \partial_i\overset{\circ}{}_{q}{}^{j}(.,y),\mu\partial_j v_i\right\rangle_{\Omega} \quad (5.9)$$

Applying again the first Green identity to the first term in the right hand side of (5.9), we obtain

$$\left\langle \partial_i\overset{\circ}{}_{q}{}^{j}(.,y),\mu\partial_j v_i\right\rangle_{\Omega}=\left\langle \partial_i\overset{\circ}{}_{q}{}^{j}(.,y),\mu n_j\gamma^+v_i\right\rangle_{\partial\Omega}-\left\langle \partial_i\overset{\circ}{}_{q}{}^{j}(.,y),v_i\partial_j\mu\right\rangle_{\Omega}-\left\langle \partial_j\partial_i\overset{\circ}{}_{q}{}^{j}(.,y),v_i\ \mu\right\rangle_{\Omega}$$
$$=\left\langle \partial_i\overset{\circ}{}_{q}{}^{j}(.,y),\mu n_j\gamma^+v_i\right\rangle_{\partial\Omega}-\left\langle \partial_i\overset{\circ}{}_{q}{}^{j}(.,y),v_i\partial_j\mu\right\rangle_{\Omega}-v_i(y)\partial_i\mu(y) \quad (5.10)$$

Now, substitute (5.10) in to (5.9),

$$\left\langle \partial_i q^j(.,y),\mu(\boldsymbol{x})\left(\partial_i v_j+\partial_j v_i\right)\right\rangle_{\Omega}=2\left\langle \partial_i\overset{\circ}{}_{q}{}^{j}(.,\boldsymbol{y}),\mu n_j\gamma^+v_i\right\rangle_{\partial\Omega}$$
$$-2\left\langle \partial_i q^j(.,y),v_i\partial_j\mu\right\rangle_{\Omega}-2v_i(y)\partial_i\mu(y) \quad (5.11)$$

Now, substitute (5.11) and (5.8) into (5.7). As a result, we obtain

$$\left\langle q^j(.,y),A_j(p,v)\right\rangle_{\Omega}=-2\left\langle \partial_i\overset{\circ}{}_{q}{}^{j}(.,y),\mu n_j\gamma^+v_i\right\rangle_{\partial\Omega}+2\left\langle \partial_i q^j(.,y),v_i\partial_j\mu\right\rangle_{\Omega}\partial_i\mu(y)$$
$$+2v_i(y)\partial_i\mu(y)-p(y)+\left\langle \overset{\circ}{}_{q}{}^{j}(.,\boldsymbol{y}),T^{c+}{}_j(p,\boldsymbol{v})\right\rangle_{\partial\Omega} \quad (5.12)$$

Rearranging the terms, taking into account that $T_j^{c+}(p,\boldsymbol{v})=T_j^{+}(p,\boldsymbol{v})$, and using the potential operator notations, we obtain (5.2) for $(p,\boldsymbol{v})\in D(\overline{\Omega})\times\boldsymbol{D}(\overline{\Omega})$.

If the couple $(p,v)\in\boldsymbol{H}^{1,0}_{div}(\Omega;\boldsymbol{A})(or\ \boldsymbol{H}^{1,0}_{div,*}(\Omega;\boldsymbol{A}))$ is a solution of the Stokes PDE (2.5) with variable coefficient, then (5.1) and (5.2) give

$$\boldsymbol{v}+R\boldsymbol{v}-\boldsymbol{V}\boldsymbol{T}^{+}(p,\boldsymbol{v})+W\gamma^{+}\boldsymbol{v}=\boldsymbol{U}\boldsymbol{f},\ in\quad\Omega \quad (5.13)$$
$$p+R^{\bullet}\boldsymbol{v}-\Pi^{s}\boldsymbol{T}^{+}(p,\boldsymbol{v})+\Pi^{d}\gamma^{+}\boldsymbol{v}=\overset{\circ}{}_{Q}\ \boldsymbol{f},in\quad\Omega \quad (5.14)$$

We will also need the trace and traction of the third Green identities (5.13) and (5.14) on $\partial\Omega$.

$$\frac{1}{2}\gamma^{+}\boldsymbol{v}+R^{+}\boldsymbol{v}-\boldsymbol{\mathcal{V}}\boldsymbol{T}^{+}(p,\boldsymbol{v})+\boldsymbol{\mathcal{W}}\gamma^{+}\boldsymbol{v}=\gamma^{+}\boldsymbol{U}\boldsymbol{f}$$
$$\frac{1}{2}T^{+}(p,v)+T^{+}(R^{\bullet},R)v-\mathcal{W}'T^{+}(p,v)+\mathcal{L}^{+}\gamma^{+}v=T^{+}\left(\overset{\circ}{}_{Q}\ f,Uf\right) \quad (5.16)$$

One can prove the following two assertions that are instrumental for proof of equivalence of the BDIEs and the Dirichlet PDE.

Lemma 5.2 *Let* $\boldsymbol{v}\in\boldsymbol{H}^1_{div}(\Omega),p\in L^2(\Omega)(or\ L^2_*(\Omega))$ *satisfy equations.*

$$\boldsymbol{v}+R\boldsymbol{v}-\boldsymbol{V\Psi}+\boldsymbol{W\Phi}=\boldsymbol{U}\boldsymbol{f},in\Omega,$$
$$p+R^{\bullet}\boldsymbol{v}-\Pi^{s}\boldsymbol{\Psi}+\Pi^{d}\boldsymbol{\Phi}=\overset{\circ}{}_{Q}\ \boldsymbol{f},in\Omega.$$

Then $(p,\boldsymbol{v})\in\boldsymbol{H}^{1,0}_{div}(\Omega;\boldsymbol{A})(or\ \boldsymbol{H}^{1,0}_{div,*}(\Omega;\boldsymbol{A}))$ and solve the equations

$$\boldsymbol{A}(y;p,\boldsymbol{v})=\boldsymbol{f}. \quad (5.19)$$

Moreover, the following relations hold true:

$$\boldsymbol{V}\left(\boldsymbol{\Psi}-\boldsymbol{T}^{+}(p,\boldsymbol{v})\right)(y)-\boldsymbol{W}\left(\boldsymbol{\Phi}-\gamma^{+}\boldsymbol{v}\right)(y)=\boldsymbol{0},\ \boldsymbol{y}\in\Omega, \quad (5.20)$$
$$\Pi^{s}\left(\boldsymbol{\Psi}-\boldsymbol{T}^{+}(p,\boldsymbol{v})\right)(y)-\Pi^{d}\left(\boldsymbol{\Phi}-\gamma^{+}\boldsymbol{v}\right)(y)=\boldsymbol{0},\quad y\in\Omega. \quad (5.21)$$

Proof. First of all, let us prove $(p,\boldsymbol{v})\in\boldsymbol{H}^{1,0}_{div}(\Omega;\boldsymbol{A})(or\ \boldsymbol{H}^{1,0}_{div,*}(\Omega;\boldsymbol{A}))$. Since

$$A_j(y;p,v)=\overset{\circ}{}_{A}\ {}_{j}(y;p,\mu v)-\frac{\partial}{\partial y_i}\left(v_j\frac{\partial\mu}{\partial y_i}+v_i\frac{\partial\mu}{\partial y_j}\right)$$
$$\frac{\partial}{\partial y_i}\left(v_j\frac{\partial\mu}{\partial y_i}+v_i\frac{\partial\mu}{\partial y_j}\right)\in L^2(\Omega)$$

We need only to show that $\overset{\circ}{}_{A}\ {}_{j}(y;p,\mu v)\in L^2(\Omega)$. Further, from (5.17) due to (3.10) and (3.13) we have,

$$\mu v=\mu Uf-\mu Rv+\mu V\Psi-\mu W\Phi=\overset{\circ}{}_{U}\ f-\mu Rv+\overset{\circ}{}_{V}\ \Psi-\overset{\circ}{}_{W}\ (\mu\Phi)$$

And from (5.18) due to (3.12) and (3.15) we have,

$$p=Qf-R^{\bullet}v+\Pi^{s}\Psi-\Pi^{d}\Phi=\overset{\circ}{}_{Q}\ f-R^{\bullet}v+\overset{\circ}{}_{\Pi}\ {}^{s}\Psi-\overset{\circ}{}_{\Pi}\ {}^{d}(\mu\Phi).$$

Then by (3.18)

$$\overset{\circ}{}_{A}\ {}_{j}(y;p,\mu v)=\overset{\circ}{}_{A}\ {}_{j}\left(y;\overset{\circ}{}_{Q}\ f-R^{\bullet}v+\overset{\circ}{}_{\Pi}\ {}^{s}\Psi-\overset{\circ}{}_{\Pi}\ {}^{d}(\mu\Phi),\overset{\circ}{}_{U}\ f-\mu Rv+\overset{\circ}{}_{V}\ \Psi-\overset{\circ}{}_{W}\ (\mu\Phi)\right)$$
$$=\overset{\circ}{}_{A}\ {}_{j}\left(\overset{\circ}{}_{Q}\ f,\overset{\circ}{}_{U}\ f\right)-\overset{\circ}{}_{A}\ {}_{j}(R^{\bullet}v,\mu Rv)+\overset{\circ}{}_{A}\ {}_{j}\left(\overset{\circ}{}_{\Pi}\ {}^{s}\Psi,\overset{\circ}{}_{V}\ \Psi\right)-\overset{\circ}{}_{A}\ {}_{j}\left(\overset{\circ}{}_{\Pi}\ {}^{d}(\mu\Phi),\overset{\circ}{}_{W}\ (\mu\Phi)\right)$$
$$=\boldsymbol{f}-\overset{\circ}{}_{A}\ {}_{j}(R^{\bullet}\boldsymbol{v},\mu R\boldsymbol{v})$$

Also $(R^{\bullet}\boldsymbol{v},\mu R\boldsymbol{v})\in\boldsymbol{H}^1_{div}(\Omega)\times\boldsymbol{H}^2_{div}(\Omega)$ for $\boldsymbol{v}\in\boldsymbol{H}^1_{div}(\Omega)$ by using the mapping properties of $\boldsymbol{R}^{\bullet}$ and $\boldsymbol{R}$. Therefore $\overset{\circ}{}_{A}\ {}_{j}(\boldsymbol{y};p,\mu\boldsymbol{v})\in L^2(\Omega)$ and hence $\boldsymbol{A}(p,\boldsymbol{v})\in L^2(\Omega)$. Therefore, $(p,v)\in\boldsymbol{H}^{1,0}_{div}(\Omega;A)(or\ \boldsymbol{H}^{1,0}_{div,*}(\Omega;A))$.

Secondly, let us prove that $(p,\boldsymbol{v})$ solve $\boldsymbol{A}(\boldsymbol{y};p,\boldsymbol{v})=\boldsymbol{f}$. Subtracting (5.17) from identity (5.1), we obtain

$$-\boldsymbol{V\Psi}^{*}+\boldsymbol{W\Phi}^{*}=\boldsymbol{U}\left(\boldsymbol{A}(\boldsymbol{y};p,\boldsymbol{v})\right)\ \text{in}\ \Omega \quad (5.22)$$

Subtracting (5.18) from identity (5.2), we obtain

$$-\Pi^{s}\boldsymbol{\Psi}^{*}+\Pi^{d}\boldsymbol{\Phi}^{*}=\boldsymbol{Q}\left[\boldsymbol{A}(y;p,v)-\boldsymbol{f}\right]\text{in}\quad\Omega \quad (5.23)$$

where $\boldsymbol{\Psi}^{*}:=\boldsymbol{T}^{+}(p,\boldsymbol{v})-\boldsymbol{\Psi}$ and $\boldsymbol{\Phi}^{*}:=\gamma^{+}\boldsymbol{v}-\boldsymbol{\Phi}$.

After Multiplying equality (5.22) by $\mu(\boldsymbol{y})$ and applying relations (3.10) and (3.14), we get

$$-\overset{\circ}{}_{V}\ \boldsymbol{\Psi}^{*}+\overset{\circ}{}_{W}\ (\mu\boldsymbol{\Phi}^{*})=\overset{\circ}{}_{U}\ \left[\boldsymbol{A}(y;p,v)-\boldsymbol{f}\right] \quad (5.24)$$

Applying the stokes operator with $\mu = 1$ to these two previous equations, by (3.18) we obtain $\boldsymbol{A}(\boldsymbol{y}; p, \boldsymbol{v}) = \boldsymbol{f}$ and hence the first equation in (5.19).

Finally, the relations (5.20) and (5.21) follow from the substitution of (5.19) in to (5.22) and (5.23).

Lemma 5.3

(i) Let either $\boldsymbol{\Psi}^* \in \boldsymbol{H}^{\frac{-1}{2}}(\partial\Omega)$ and $diam(\Omega) < r_0 e^{\frac{1}{2}}$ or $\boldsymbol{\Psi}^* \in \boldsymbol{H}_*^{\frac{-1}{2}}(\partial\Omega)$. If $\boldsymbol{V\Psi}^*(\boldsymbol{y}) = 0, \ \ \boldsymbol{y} \in \Omega,$ (5.25) then $\boldsymbol{\Psi}^* = \boldsymbol{0}$

(ii) Let $\boldsymbol{\Phi}^* \in \boldsymbol{H}^{\frac{1}{2}}(\partial\Omega)$. If $\boldsymbol{W\Phi}^*(\boldsymbol{y}) = 0, \ \boldsymbol{y} \in \Omega,$ (5.26) then $\boldsymbol{\Phi}^* = \boldsymbol{0}$.

Proof. We will use similar procedures as in [3].

(i) Taking the trace of (5.25) on $\partial\Omega$ and using jump relation (3.40). Then we have

$$\boldsymbol{V\Psi}^* = \boldsymbol{0} \text{on} \ \ \partial\Omega$$

If $\boldsymbol{\Psi}^* \in \boldsymbol{H}^{\frac{-1}{2}}(\partial\Omega)$ and $diam(\Omega) < r_0 e^{\frac{1}{2}}$, then the result follows from the invertability of the single layer potential given by Theorem 4.5. On the other hand, if $\boldsymbol{\Psi}^* \in \boldsymbol{H}_*^{\frac{-1}{2}}(\partial\Omega)$, then the result is implied by Theorem 4.4.

(i) Taking the trace of (5.26) and then by (3.40) gives $\frac{-1}{2}\boldsymbol{\Phi}^* + \boldsymbol{W\Phi}^* = \boldsymbol{0}$on $\partial\Omega,$ due to (3.14), $\frac{-1}{2}\widetilde{\boldsymbol{\Phi}}^* + \boldsymbol{W}\widetilde{\boldsymbol{\Phi}}^* = \boldsymbol{0}$on $\partial\Omega,$ where $\widetilde{\boldsymbol{\Phi}}^* = \mu\boldsymbol{\Phi}^*$. Due to the contraction property of the operator $\frac{-1}{2}\boldsymbol{I} + \boldsymbol{W}$, then $\widetilde{\boldsymbol{\Phi}}^*$ is uniquely solvable and $\mu(\boldsymbol{y}) \neq 0$, $\widetilde{\boldsymbol{\Phi}}^* = \boldsymbol{0}$ implies$\boldsymbol{\Phi}^* = \boldsymbol{0}$.

BDIE systems for Dirichlet BVP

We aim to obtain a segregated boundary-domain integral equation systems for Dirichlet BVP (2.5)-(2.6). We will use similar procedures as in, Tamirat and Mikhailov (2015).

Let us denote the unknown traction as $\boldsymbol{\psi} = \boldsymbol{T}^+(p, \boldsymbol{v}) \in \boldsymbol{H}^{\frac{-1}{2}}(\partial\Omega)$ and will further consider $\boldsymbol{\psi}$ as formally independent on p and $\boldsymbol{v}$. Assuming that the function $(p, \boldsymbol{v})$ satisfies system of PDE (2.5), by substituting the Dirichlet condition in to the third Green identities (5.1),(5.2) and either into its trace (5.15) or into its traction (5.16) on $\partial\Omega$, we can reduce the BVP (2.5)- (2.6) to two different systems of Boundary-Domain Integral Equations for the unknowns $(p, \boldsymbol{v}, \boldsymbol{\psi}) \in \boldsymbol{H}^{1,0}_{div,*}(\Omega; \boldsymbol{A})$.

BDIE System (D1)

From the equations (5.13), (5.14) and(5.15) we obtain

$$p + R^{\bullet}\boldsymbol{v} - \Pi^s\psi = F_0 \text{in} \ \ \Omega,$$
$$\boldsymbol{v} + \boldsymbol{Rv} - \boldsymbol{V}\psi = \boldsymbol{F}\text{in} \ \ \Omega,$$
$$\gamma^+\boldsymbol{Rv} - \boldsymbol{\mathcal{V}}\psi = \gamma^+\boldsymbol{F} - \boldsymbol{\varphi}_0 \text{on} \ \ \partial\Omega,$$

where

$$F_0 := \mathring{Q} \ \ \boldsymbol{f} - \Pi^d\boldsymbol{\varphi}_0, \ \boldsymbol{F} := \boldsymbol{Uf} - \boldsymbol{W\varphi}_0 \qquad (6.4)$$

Using theorems 3.1, 3.2 and 3.3 we have, $(F_0, \boldsymbol{F}) \in \boldsymbol{H}^{1,0}_{div,*}(\Omega; \boldsymbol{A})$.

We denote the right hand side of BDIE system (6.1) -(6.3) as

$$\boldsymbol{\mathcal{F}}^1 := [F_0, \boldsymbol{F}, \gamma^+\boldsymbol{F} - \boldsymbol{\varphi}_0]^T \qquad (6.5)$$

which implies $\boldsymbol{\mathcal{F}}^1 \in \boldsymbol{H}^{1,0}_{div,*}(\Omega; \boldsymbol{A}) \times \boldsymbol{H}^{\frac{1}{2}}(\partial\Omega)$

Note that BDIE system (6.1)-(6.3) can be split into the BDIE system (**D1**), of 2 vector equations (6.2), (6.3)) for 2 vector unknowns, $\boldsymbol{v}$ and $\boldsymbol{\psi}$,and the scalar equation (6.1) that can be used after solving the system to obtain the pressure, p. The system (**D1**) given by equations (6.1) – (6.3) can be written using matrix notation as

$$\boldsymbol{\mathcal{D}}^1 X = \boldsymbol{\mathcal{F}}^1, \qquad (6.6)$$

where $\boldsymbol{\mathcal{X}}$ represents the vector containing the unknowns of the system

$$\boldsymbol{\mathcal{X}} = (p, \boldsymbol{v}, \boldsymbol{\psi}) \in L^2_*(\Omega) \times \boldsymbol{H}^1_{div}(\Omega) \times \boldsymbol{H}^{\frac{-1}{2}}(\partial\Omega)$$

The matrix operator $\boldsymbol{\mathcal{D}}^1$ is defined by

$$\boldsymbol{\mathcal{D}}^1 = \begin{bmatrix} I & R^{\bullet} & -\Pi^s \\ \boldsymbol{0} & \boldsymbol{I} + \boldsymbol{R} & -\boldsymbol{V} \\ \boldsymbol{0} & \gamma^+\boldsymbol{R} & -\boldsymbol{\mathcal{V}} \end{bmatrix}$$

Remark 6.1 *The term* $\boldsymbol{\mathcal{F}}^1 = \boldsymbol{0}$ *if and only if* $(\boldsymbol{f}, \boldsymbol{\varphi}_0) = \boldsymbol{0}$.

Suppose$\boldsymbol{\mathcal{F}}^1 = \boldsymbol{0}$, then . Now multiplying the second equation of (6.4) by μ and applying Stokes operator with $\mu = 1$ to these two equations (6.4), by (3.18) we obtain $\boldsymbol{f} = \boldsymbol{0}$.

In addition, as $\boldsymbol{F} = \boldsymbol{0}$, we get that

$$\gamma^+\boldsymbol{F} - \boldsymbol{\varphi}_0 = \boldsymbol{0} \text{ implies} \boldsymbol{\varphi}_0 = \boldsymbol{0}.$$

Therefore, we obtain that $\boldsymbol{\varphi}_0 = \boldsymbol{0}$ on $\partial\Omega$. And by first equation of (6.4) we obtain $g = 0$.

On the other hand assume that $(\boldsymbol{f}, \boldsymbol{\varphi}_0) = \boldsymbol{0}$. Then immediately we have $\boldsymbol{\mathcal{F}}^1 = \boldsymbol{0}$.

BDIE System (D2)

From the equations (5.13),(5.14) and its traction (5.16) we obtain

$$p + R^{\bullet}\boldsymbol{v} - \Pi^s\boldsymbol{\psi} = F_0 \text{ in } \Omega,$$
$$\boldsymbol{v} + \boldsymbol{Rv} - V\boldsymbol{\psi} = \boldsymbol{F} \text{ in } \Omega,$$

$$\frac{1}{2}\boldsymbol{\psi} + \boldsymbol{T}^+(R^{\bullet}, \boldsymbol{R})\boldsymbol{v} - \boldsymbol{\mathcal{W}}'\boldsymbol{\psi} = \boldsymbol{T}^+(F_0, \boldsymbol{F}) \text{on } \partial\Omega,$$

where F_0 and $\boldsymbol{F}$ are given by (6.4). In matrix form it can be written as $\mathcal{D}^2\mathcal{X} = \mathcal{F}^2$, where

$$\mathcal{D}^2 = \begin{bmatrix} I & R^{\bullet} & -\Pi^s \\ \boldsymbol{0} & \boldsymbol{I}+\boldsymbol{R} & -\boldsymbol{V} \\ \boldsymbol{0} & \boldsymbol{T}^+(R^{\bullet},\boldsymbol{R}) & \frac{1}{2}\boldsymbol{I}-\boldsymbol{W}' \end{bmatrix}, \quad \mathcal{F}^2 = \begin{bmatrix} F_0 \\ \boldsymbol{F} \\ \boldsymbol{T}^+(F_0,\boldsymbol{F}) \end{bmatrix}$$

Note that BDIE system (6.7)-(6.9) can be split in to the BDIE system (**D2**), of 2 vector equations (6.8), (6.9)) for 2 vector unknowns, $\boldsymbol{v}$ and $\boldsymbol{\psi}$,and the scalar equation (6.7) that can be used, after solving the system, to obtain the pressure, p.

Remark 6.2 *The term* $\mathcal{F}^2 = \boldsymbol{0}$ *if and only if* $(\boldsymbol{f},\boldsymbol{\varphi}_0) = \boldsymbol{0}$.

Indeed, it is evident that $(\boldsymbol{f},\boldsymbol{g},\boldsymbol{\varphi}_0) = 0$ implies $\mathcal{F}^2 = \boldsymbol{0}$. Let now $\mathcal{F}^2 = \boldsymbol{0}$. Lemma 5.2 with $F_0 = 0$ for p and $\boldsymbol{F} = \boldsymbol{0}$ for $\boldsymbol{v}$ applying to equation (6.4) implies that $\boldsymbol{f} = \boldsymbol{0}$ and $\Pi^d\boldsymbol{\varphi}_0 = 0, \boldsymbol{W}\boldsymbol{\varphi}_0 = \boldsymbol{0}$ in Ω. Therefore, by Lemma 5.3(ii) $\boldsymbol{\varphi}_0 = \boldsymbol{0}$ on $\partial\Omega$.

In the following theorem we shall see the equivalence of the original Dirichlet boundary value problem to the boundary domain integral equation systems

Equivalence and invertibility theorems

Theorem 7.1 (EquivalenceTheorem) *Let* $\boldsymbol{f} \in \boldsymbol{L}^2(\Omega), g \in L^2(\Omega), \boldsymbol{\varphi}_0 \in \boldsymbol{H}^{\frac{1}{2}}(\partial\Omega)$ *and satisfy the compatibility condition (2.7) ;*

(i) If some $(p,\boldsymbol{v}) \in \boldsymbol{H}^{1,0}_{div,*}(\Omega;\boldsymbol{A})$ solve the Dirichlet BVP (2.5)-(2.6), then $(p,\boldsymbol{v},\boldsymbol{\psi}) \in \boldsymbol{H}^{1,0}_{div,*}(\Omega;\boldsymbol{A}) \times \boldsymbol{H}^{\frac{-1}{2}}(\partial\Omega)$, where

$$\boldsymbol{\psi} = \boldsymbol{T}^+(p,\boldsymbol{v}) \in \boldsymbol{H}^{\frac{-1}{2}}(\partial\Omega) \tag{7.1}$$

solves the BDIE systems (**D1**) and (**D2**).

(ii) If $(p,\boldsymbol{v},\boldsymbol{\psi}) \in \boldsymbol{H}^{1,0}_{div,*}(\Omega;\boldsymbol{A}) \times \boldsymbol{H}^{\frac{-1}{2}}(\partial\Omega)$ solves the BDIE system (**D1**) and $diam(\Omega) < r_0 e^{\frac{1}{2}}$, then $(p,\boldsymbol{v})$ solves the BDIE system (**D2**)and BVP (2.5) -(2.6), this solution is unique, and $\boldsymbol{\psi}$ satisfies (7.1).

(iii) If $(p,\boldsymbol{v},\psi) \in \boldsymbol{H}^{1,0}_{div,*}(\Omega;\boldsymbol{A}) \times \boldsymbol{H}^{\frac{-1}{2}}(\partial\Omega)$ solves the BDIE system (**D2**) , then $(p,\boldsymbol{v})$ solves the BDIE system (**D1**) and BVP (2.5) -(2.6), this solution is unique, and $\boldsymbol{\psi}$ satisfies (7.1).

Proof (i) Let $(p,\boldsymbol{v}) \in \boldsymbol{H}^{1,0}_{div,*}(\Omega;\boldsymbol{A})$ be a solution of the BVP. Let us define the function $\boldsymbol{\psi}$ by (7.1). Taking into account the Green identities (5.13)- (5.15), we immediately obtain that $(p,\boldsymbol{v},\boldsymbol{\psi})$ solves BDIE systems (**D1**)and (**D2**).

We note that if $(p,\boldsymbol{v},\boldsymbol{\psi}) \in L^2_*(\Omega) \times \boldsymbol{H}^1_{div}(\Omega) \times \boldsymbol{H}^{\frac{-1}{2}}(\partial\Omega)$ solves BDIE systems (**D1**) and (**D2**). Due to the first two equations in the BDIE systems, the hypotheses of Lemma 5.2 are satisfied implying that $(p,\boldsymbol{v}) \in \boldsymbol{H}^{1,0}_{div,*}(\Omega;\boldsymbol{A})$ and solves PDE (2.5) in Ω and also satisfying

$$\boldsymbol{V}(\boldsymbol{\psi} - \boldsymbol{T}^+(p,\boldsymbol{v})) - \boldsymbol{W}(\boldsymbol{\varphi}_0 - \gamma^+\boldsymbol{v}) = \boldsymbol{0} \tag{7.2}$$

(ii) let $(p,\boldsymbol{v},\boldsymbol{\psi}) \in \boldsymbol{H}^{1,0}_{div,*}(\Omega;\boldsymbol{A}) \times \boldsymbol{H}^{\frac{-1}{2}}(\partial\Omega)$ solve BDIE system (**D1**). If we take the trace of the second equation in (**D1**)and subtracting the third equation from it, we arrive at $\gamma^+\boldsymbol{v} = \boldsymbol{\varphi}_0$ on $\partial\Omega$. Therefore, the Dirichlet boundary is satisfied.

Now using Dirichlet condition in (7.2), we have $\boldsymbol{V}$, Lemma 5.3(i) then implies $\boldsymbol{\psi} = \boldsymbol{T}^+(p,\boldsymbol{v})$

(iii) let $(p,\boldsymbol{v},\boldsymbol{\psi}) \in \boldsymbol{H}^{1,0}_{div,*}(\Omega;\boldsymbol{A}) \times \boldsymbol{H}^{\frac{-1}{2}}(\partial\Omega)$ solve BDIE system (**D2**). If we take the traction of the first and second equations in (**D2**) and subtracting the third equation from it, we arrive at $\boldsymbol{\psi} = \boldsymbol{T}^+(p,\boldsymbol{v})$ on $\partial\Omega$. Therefore $\boldsymbol{\psi}$ satisfies (7.1) .

Now inserting $\boldsymbol{\psi} = \boldsymbol{T}^+(p,\boldsymbol{v})$ in (7.2),we have $\boldsymbol{W}$, Lemma 5.3 (ii) then implies $\boldsymbol{\varphi}_0 = \gamma^+\boldsymbol{v}$. therefore, satisfy the Dirichlet Condition.

The uniqueness of the BDIE system solutions follows form Theorem 2.1.

Theorem 7.2 *If* $diam(\Omega) < r_0 e^{\frac{1}{2}}$, then the following operators are invertible

$$\mathcal{D}^1: L^2_*(\Omega) \times \boldsymbol{H}^1_{div}(\Omega) \times \boldsymbol{H}^{\frac{-1}{2}}(\partial\Omega) \to L^2_*(\Omega) \times \boldsymbol{H}^1_{div}(\Omega) \times \boldsymbol{H}^{\frac{1}{2}}(\partial\Omega) \tag{7.3}$$

$$\mathcal{D}^1: \boldsymbol{H}^{1,0}_{div,*}(\Omega;A) \times \boldsymbol{H}^{\frac{-1}{2}}(\partial\Omega) \to \boldsymbol{H}^{1,0}_{div,*}(\Omega;A) \times \boldsymbol{H}^{\frac{1}{2}}(\partial\Omega) \tag{7.4}$$

Proof. Theorem 7.1(ii) implies that operators 7.3 and 7.4 are injective. To see this, let $\mathcal{D}^1\mathcal{X} = \boldsymbol{0}$, then $\mathcal{F}^1 = \boldsymbol{0}$, or by Remark 6.1, which implies $(\boldsymbol{f},g,\boldsymbol{\varphi}_0) = \boldsymbol{0}$. This means $\boldsymbol{A}(p,\boldsymbol{v}) = \boldsymbol{0}$, $\boldsymbol{\varphi}_0 = \gamma^+\boldsymbol{v} = \boldsymbol{0}$, hence by Theorem 7.1(ii), $\boldsymbol{v} = \boldsymbol{0}, p = 0, \boldsymbol{\psi} = \boldsymbol{0}$. Therefore, $\mathcal{X} = \boldsymbol{0}$.

Let us denote

$$\tilde{\mathcal{D}}^1 = \begin{bmatrix} I & 0 & -\Pi^s \\ \boldsymbol{0} & \boldsymbol{I} & -\boldsymbol{V} \\ \boldsymbol{0} & \boldsymbol{0} & -\boldsymbol{V} \end{bmatrix}.$$

Then

$$\tilde{\mathcal{D}}^1: L^2_*(\Omega) \times \boldsymbol{H}^1_{div}(\Omega) \times \boldsymbol{H}^{\frac{-1}{2}}(\partial\Omega) \to L^2_*(\Omega) \times \boldsymbol{H}^1_{div}(\Omega) \times \boldsymbol{H}^{\frac{1}{2}}(\partial\Omega)$$

is continuous which is bounded. It is invertible due to its triangular structure and invertibility of its diagonal operators $I: L^2_*(\Omega) \longrightarrow L^2_*(\Omega)$, $\boldsymbol{I}: \boldsymbol{H}^1_*(\Omega) \longrightarrow \boldsymbol{H}^1_*(\Omega)$ and $-\boldsymbol{V}: \boldsymbol{H}^{\frac{-1}{2}}(\partial\Omega) \longrightarrow \boldsymbol{H}^{\frac{1}{2}}(\partial\Omega)$ (see theorem (4.5)). Due to proposition 3.5 the operator

$\mathcal{D}^1 - \tilde{\mathcal{D}}^1 : L_*^2(\Omega) \times \boldsymbol{H}_{div}^1(\Omega) \times \boldsymbol{H}^{\frac{-1}{2}}(\partial\Omega) \to L_*^2(\Omega) \times \boldsymbol{H}_{div}^1(\Omega) \times \boldsymbol{H}^{\frac{1}{2}}(\partial\Omega)$ which is

$$\mathcal{D}^1 - \tilde{\mathcal{D}}^1 = \begin{bmatrix} 0 & R^{\bullet} & 0 \\ \mathbf{0} & \boldsymbol{R} & \mathbf{0} \\ \mathbf{0} & \gamma^+ \boldsymbol{R} & \mathbf{0} \end{bmatrix}$$

is compact, implying that operator (7.3) is Fredholm operator with zero index (cf. McLean (2000), Theorem 2.27) and then the injectivity of operator (7.3) implies its invertibility.

To prove the invertibility of the operator (7.4), consider the solution $\mathcal{X} = (\mathcal{D}^1)^{-1}\mathcal{F}^1$ of the system (6.6). Here $\mathcal{F}^1 \in \boldsymbol{H}_{div,*}^{1,0}(\Omega; \boldsymbol{A}) \times \boldsymbol{H}^{\frac{1}{2}}(\partial\Omega)$ is an arbitrary right hand side and is the inverse of the operator (7.3) which exists. Applying Lemma 5.2 to the first two equations of the system (6.1)- (6.3), we get that $\mathcal{X} \in \boldsymbol{H}_{div,*}^{1,0}(\Omega; \boldsymbol{A}) \times \boldsymbol{H}^{\frac{-1}{2}}(\partial\Omega)$. Consequently, the operator is also the continuous inverse of the operator (7.4).

The following similar assertion for the operator D^2 holds without the needs of setting condition on the space of functions.

Theorem 7.3 *The operators*

$$\mathcal{D}^2 : L_*^2(\Omega) \times \boldsymbol{H}_{div}^1(\Omega) \times \boldsymbol{H}^{\frac{-1}{2}}(\partial\Omega) \to L_*^2(\Omega) \times \boldsymbol{H}_{div}^1(\Omega) \times \boldsymbol{H}^{\frac{1}{2}}(\partial\Omega), \quad (7.5)$$

$$\mathcal{D}^2 : \boldsymbol{H}_{div,*}^{1,0}(\Omega; \boldsymbol{A}) \times \boldsymbol{H}^{\frac{-1}{2}}(\partial\Omega) \to \boldsymbol{H}_{div,*}^{1,0}(\Omega; \boldsymbol{A}) \times \boldsymbol{H}^{\frac{1}{2}}(\partial\Omega)$$

are invertible.

Proof. Theorem 7.1(iii) implies that operators 7.5 and 7.6 are injective. To see this, let $\mathcal{D}^2\mathcal{X} = \mathbf{0}$, then $\mathcal{F}^2 = \mathbf{0}$, or by Remark 6.2, which implies $(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{\varphi}_0) = \mathbf{0}$. This means $\boldsymbol{A}(p, \boldsymbol{v}) = \mathbf{0}$, $\boldsymbol{\varphi}_0 = \mathbf{0}$, hence by Theorem 7.1(iii), $\boldsymbol{v} = \mathbf{0}, p = 0, \boldsymbol{\psi} = \mathbf{0}$. Therefore, $\mathcal{X} = \mathbf{0}$.

Let us denote

$$\tilde{\mathcal{D}}^2 = \begin{bmatrix} I & 0 & -\Pi^s \\ \mathbf{0} & \boldsymbol{I} & -\boldsymbol{V} \\ \mathbf{0} & \mathbf{0} & \frac{1}{2}\boldsymbol{I} \end{bmatrix}$$

Then $\tilde{\mathcal{D}}^2$ is continuous which is bounded. It is invertible due to its triangular structure and invertibility of its diagonal operators $I : L_*^2(\Omega) \longrightarrow L_*^2(\Omega)$ and $\boldsymbol{I} : \boldsymbol{H}_{div}^1(\Omega) \longrightarrow \boldsymbol{H}_{div}^1(\Omega)$. Due to Theorem 3.1 proposition 3.5, the operator

$$\mathcal{D}^2 - \tilde{\mathcal{D}}^2 = \begin{bmatrix} 0 & R^{\bullet} & 0 \\ \mathbf{0} & \boldsymbol{I} + \boldsymbol{R} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{T}^+(R^{\bullet}, \boldsymbol{R}) & \boldsymbol{\mathcal{W}}' \end{bmatrix}$$

is compact, implying that operator (7.5) is Fredholm operator with zero index (see, McLean (2000), Theorem 2.27) and then the injectivity of operator (7.5) implies its invertibility.

To prove the invertibility of the operator (7.6), consider the solution $\mathcal{X} = (\mathcal{D}^2)^{-1}\mathcal{F}^2$. Here $\mathcal{F}^2 \in \boldsymbol{H}_{div,*}^{1,0}(\Omega; \boldsymbol{A}) \times \boldsymbol{H}^{\frac{1}{2}}(\partial\Omega)$ is an arbitrary right hand side and is the inverse of the operator (7.5) which exists. Applying Lemma 5.2 to the first two equations of the system (6.7) - (6.9), we get that $\mathcal{X} \in \boldsymbol{H}_{div,*}^{1,0}(\Omega; \boldsymbol{A}) \times \boldsymbol{H}^{\frac{-1}{2}}(\partial\Omega)$. Consequently, the operator is also the continuous inverse of the operator (7.6).

Appendix: Proof of the relations

The parametrix-based integral operators depending on the variable coefficient, μ, can be expressed in terms of the corresponding integral operators for the constant coefficient case, $\mu = 1$.

Here we prove relations (3.10) -(3.17).

$$[U\rho]_k(y) = \int_\Omega u_j^k(x,y)\rho_j(x)dx = \int_\Omega \frac{1}{\mu(y)} \mathring{u}_j^k(x,y)\rho_j(x)dx = \frac{1}{\mu(y)}\int_\Omega \mathring{u}_j^k(x,y)\rho_j(x)dx$$

$$= \frac{1}{\mu(y)}[\mathring{U}\rho]_k(\boldsymbol{y}),$$ which is relation (3.10).

For $\boldsymbol{y} \in \mathbb{R}^2$,

$$R_k\rho(y) = R_{kj}\rho_j(y) = \int_\Omega R_{kj}(x,y)\rho_j(x)dx = \int_\Omega \frac{1}{\mu(y)}\frac{\partial(\mu(x))}{\partial x_i}\mathring{\sigma}_{ij}(\mathring{q}^k, \mathring{u}^k)(x-y)\rho_j(x)dx$$

$$= \int_\Omega \frac{1}{\mu(y)}\frac{\partial(\mu(x))}{\partial x_i}\left(-\delta_i^j\mathring{q}^k + \left(\frac{\partial\mathring{u}_j^k}{\partial x_i} + \frac{\mathring{u}_i^k}{\partial x_j}\right)\right)\rho_j(x)dx \quad (7.6)$$

$$= \int_\Omega \frac{1}{\mu(y)}\frac{\partial(\mu(x))}{\partial x_i}\left(\frac{\partial\mathring{u}_j^k}{\partial x_i} + \frac{\mathring{u}_i^k}{\partial x_j}\right)\rho_j(x)dx - \int_\Omega \frac{1}{\mu(y)}\frac{\partial(\mu(x))}{\partial x_i}(\delta_i^j\mathring{q}^k)\rho_j(x)dx$$

$$= \int_\Omega \frac{1}{\mu(y)}\frac{\partial(\mu(x))}{\partial x_i}\frac{\partial\mathring{u}_j^k}{\partial x_i}\rho_j(x)dx + \int_\Omega \frac{1}{\mu(y)}\frac{\partial(\mu(x))}{\partial x_i}\frac{\mathring{u}_i^k}{\partial x_j}\rho_j(x)dx$$

$$- \int_\Omega \frac{1}{\mu(y)}\frac{\partial(\delta_i^j\mu(x))}{\partial x_i}\mathring{q}^k\rho_j(x)dx$$

$$= -\frac{1}{\mu(y)}\left[-\int_\Omega \frac{\partial(\mu(x))}{\partial x_i}\frac{\partial\mathring{u}_j^k}{\partial x_i}\rho_j(x)dx - \int_\Omega \frac{\partial(\mu(x))}{\partial x_i}\frac{\mathring{u}_i^k}{\partial x_j}\rho_j(x)dx + \int_\Omega \frac{\partial(\mu(x))}{\partial x_j}\mathring{q}^k\rho_j(x)dx\right]$$

$$= -\frac{1}{\mu(y)}\left[\int_\Omega \frac{\partial(\mu(x))}{\partial y_i}\frac{\partial\mathring{u}_j^k}{\partial x_i}\rho_j(x)dx + \int_\Omega \frac{\partial(\mu(x))}{\partial x_i}\frac{\mathring{u}_i^k}{\partial y_j}\rho_j(x)dx + \int_\Omega \frac{\partial(\mu(x))}{\partial x_j}\mathring{q}^k\rho_j(x)dx\right]$$

$$= -\frac{1}{\mu(y)}\left[\int_\Omega \frac{\partial\mathring{u}_j^k}{\partial y_i}(\rho_j\partial_i\mu)(x)dx + \int_\Omega \frac{\mathring{u}_i^k}{\partial y_j}(\rho_j\partial_i\mu)(x)dx + \int_\Omega \mathring{q}^k(\rho_j\partial_j\mu)(x)dx\right]$$

$$= -\frac{1}{\mu(y)}\left[\frac{\partial}{\partial y_i}\int_\Omega \mathring{u}_j^k(\rho_j\partial_i\mu)(x)dx + \frac{\partial}{\partial y_j}\int_\Omega \mathring{u}_i^k(\rho_j\partial_i\mu)(x)dx + \int_\Omega \mathring{q}^k(\rho_j\partial_j\mu)(x)dx\right]$$

$$= -\frac{1}{\mu(y)}\left[\frac{\partial}{\partial y_i}\mathring{U}_{kj}(\rho_j\partial_i\mu)(y) + \frac{\partial}{\partial y_j}\mathring{U}_{ki}(\rho_j\partial_i\mu)(y) - \mathring{Q}_k(\rho_j\partial_j\mu)(y)\right],$$

that is,

$$R_k\rho(y) = \frac{-1}{\mu(y)}\left[\frac{\partial}{\partial y_i}\mathring{U}_{kj}(\rho_j\partial_i\mu)(y) + \frac{\partial}{\partial y_j}\mathring{U}_{ki}(\rho_j\partial_i\mu)(y) - \mathring{Q}_k(\rho_j\partial_j\mu)(y)\right].$$

$$[Q\rho]_j(y) = Q_j\rho(y) = -\int_\Omega q^j(x,y)\rho(x)dx = -\int_\Omega \frac{\mu(x)}{\mu(y)} \mathring{q}^{\;j}(x,y)\rho(x)dx = \frac{1}{\mu(y)}[\mathring{Q}\;(\mu\rho)]_j(y).$$

$$R^{\bullet}\rho(y) = R_j^{\bullet}\rho_j(y) = 2v.p.\int_\Omega \frac{\partial \mathring{q}^{\;j}(x,y)}{\partial x_i}\frac{\partial\mu(x)}{\partial x_i}\rho_j(x)dx - 2\rho_j(y)\frac{\partial\mu}{\partial y_i}(y),(y\in\Omega),$$

$$= -2\int_\Omega \frac{\partial \mathring{q}^{\;j}(x,y)}{\partial y_i}\frac{\partial\mu(x)}{\partial x_i}\rho_j(x)dx - 2\rho_j(y)\frac{\partial\mu}{\partial y_i}(y)$$

$$= -2\frac{\partial}{\partial y_i}\int_\Omega \mathring{q}^{\;j}(x,y)(\partial_i\mu(x)\rho_j)(x)dx - 2\rho_j(y)\frac{\partial\mu}{\partial y_i}(y)$$

$$= -2\frac{\partial}{\partial y_i}\mathring{Q}_j(\rho_j\partial_i\mu)(y) - 2\rho_j(y)\frac{\partial\mu}{\partial y_i}(y).$$

$$[V\rho]_k(y) = -\int_{\partial\Omega} u_j^k(x,y)\rho_j(x)dS_x = \frac{1}{\mu(y)}\left[-\int_{\partial\Omega}\mathring{u}_j^{\;k}(x,y)\rho_j(x)dS_x\right]$$

$$= \frac{1}{\mu(y)}\left[-\int_{\partial\Omega}\mathring{u}_j^{\;k}(x,y)\rho_j(x)dS_x\right] = \frac{1}{\mu(y)}[\mathring{V}\;\rho]_k(y)$$

$$[W\rho]_k(y) := -\int_{\partial\Omega} T_j^{+}(x;q^k,u^k)(x,y)\rho_j(x)dS_x = -\int_{\partial\Omega}\frac{\mu(x)}{\mu(y)}\mathring{T}_j^{+}(x;\mathring{q}^{\;k},\mathring{u}^{\;k})(x,y)\rho_j(x)dS_x$$

$$= -\frac{1}{\mu(y)}\int_{\partial\Omega}\mathring{T}_j^{+}(x;\mathring{q}^{\;k},\mathring{u}^{\;k})(x,y)\mu(x)\rho_j(x)dS_x = \frac{1}{\mu(y)}[\mathring{W}\;\rho]_k(y)$$

$$[V\rho]_k(y) = -\int_{\partial\Omega} u_j^k(x,y)\rho_j(x)dS_x = \frac{1}{\mu(y)}\left[-\int_{\partial\Omega}\mathring{u}_j^{\;k}(x,y)\rho_j(x)dS_x\right]$$

$$= \frac{1}{\mu(y)}\left[-\int_{\partial\Omega}\mathring{u}_j^{\;k}(x,y)\rho_j(x)dS_x\right] = \frac{1}{\mu(y)}[\mathring{V}\;\rho]_k(y)$$

$$[W\rho]_k(y) := -\int_{\partial\Omega} T_j^{+}(x;q^k,u^k)(x,y)\rho_j(x)dS_x = -\int_{\partial\Omega}\frac{\mu(x)}{\mu(y)}\mathring{T}_j^{+}(x;\mathring{q}^{\;k},\mathring{u}^{\;k})(x,y)\rho_j(x)dS_x$$

$$= -\frac{1}{\mu(y)}\int_{\partial\Omega}\mathring{T}_j^{+}(x;\mathring{q}^{\;k},\mathring{u}^{\;k})(x,y)\mu(x)\rho_j(x)dS_x = \frac{1}{\mu(y)}[\mathring{W}\;\rho]_k(y)$$

$$\Pi^s\rho(y) = \Pi_j^s\rho_j(y) := \int_{\partial\Omega}\mathring{q}^{\;j}(x,y)\rho_j(x)dS_x = \mathring{\Pi}_j^{\;s}\rho_j(y) = \mathring{\Pi}^{\;s}\rho(y), \therefore \Pi^s\rho = \mathring{\Pi}^{\;s}\rho.$$

$$\Pi^d\rho(y) = \Pi_j^d\rho_j(y) := -2\int_{\partial\Omega}\frac{\partial \mathring{q}_j(x,y)}{\partial n(x)}\mu(x)\rho_j(x)dS_x = -2\int_{\partial\Omega}\frac{\partial \mathring{q}_j(x,y)}{\partial n(x)}(\mu\rho_j)(x)dS_x = \mathring{\Pi}_j^{\;d}(\mu\rho)(y).$$

$$[W'\rho]_k(y) = -\int_{\partial\Omega} T_j^{+}(y;q^k,u^k)(x,y)\rho_j(x)dS_x = -\int_{\partial\Omega}\sigma_{ij}(y;q^k,u^k)(x,y)n_i(y)\rho_j(x)dS_x$$

$$= -\int_{\partial\Omega}\left(-\delta_i^j\mathring{q}^{\;k} + \mu(y)\left(\frac{\partial u_i^k}{\partial y_j} + \frac{\partial u_j^k}{\partial y_i}\right)\right)n_i(y)\rho_j(x)dS_x$$

$$= -\int_{\partial\Omega}\left(-\delta_i^j\mathring{q}^{\;k}\right)n_i(y)\rho_j(x)dS_x - \int_{\partial\Omega}\left(\mu(y)\left(\frac{\partial}{\partial y_j}\left(\frac{1}{\mu(y)}\mathring{u}_i^{\;k}\right) + \frac{\partial}{\partial y_i}\left(\frac{1}{\mu(y)}\mathring{u}_j^{\;k}\right)\right)\right)n_i(y)\rho_j(x)dS_x$$

$$= -\int_{\partial\Omega}\Bigg(-\delta_i^j\mathring{q}^{\;k} + \mu(y)\left(\frac{\partial}{\partial y_j}\left(\frac{1}{\mu(y)}\right)\mathring{u}_i^{\;k} + \frac{\partial}{\partial y_j}(\mathring{u}_i^{\;k})\frac{1}{\mu(y)} + \frac{\partial}{\partial y_i}\left(\frac{1}{\mu(y)}\right)\mathring{u}_j^{\;k}\frac{\partial}{\partial y_i}(\mathring{u}_j^{\;k})\frac{1}{\mu(y)}\right)\Bigg)n_i(y)\rho_j(x)dS_x$$

$$-\int_{\partial\Omega}\left(-\delta_i^j q^k + \frac{\partial \mathring{u}_i^{\;k}}{\partial y_j} + \frac{\partial \mathring{u}_j^{\;k}}{\partial y_i}\right)n_i(y)\rho_j(x)dS_x$$

$$-\int_{\partial\Omega}\left(\mu(y)\frac{\partial}{\partial y_j}\left(\frac{1}{\mu(y)}\right)\mathring{u}_i^{\;k} + \mu(y)\frac{\partial}{\partial y_i}\left(\frac{1}{\mu(y)}\right)\mathring{u}_j^{\;k}\right)n_i(y)\rho_j(x)dS_x$$

$$= [\mathring{W}\,'\rho]_k(y) - \left(\frac{1}{\mu(y)}\frac{\partial\mu(y)}{y_j}\left(-\int_{\partial\Omega}\mathring{u}_i^{\;k}\rho_j(x)dS_x\right)\right) + \frac{1}{\mu(y)}\frac{\partial\mu(y)}{y_i}\left(-\int_{\partial\Omega}\mathring{u}_j^{\;k}\rho_j(x)dS_x\right)\Big)$$

$$-\sum_{m=1}^{2}\alpha\delta_i^j\frac{1}{\mu(y)}\frac{\partial\mu(y)}{y_m}\left(-\int_{\partial\Omega}u_m^k\rho_j(x)dS_x\right)\Big)n_i(y)$$

$$= [\mathring{W}\,'\rho]_k(y) - \left(\frac{1}{\mu(y)}\frac{\partial\mu(y)}{y_j}[\mathring{V}\;\rho]_k + \frac{1}{\mu(y)}\frac{\partial\mu(y)}{y_i}[\mathring{V}\;\rho]_i\right)n_i(y).$$

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