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FITTED NON-POLYNOMIAL CUBIC SPLINE METHOD FOR SINGULARLY PERTURBED DELAY CONVECTION-DIFFUSION EQUATIONS

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ABSTRACT: This paper presents a fitted non-polynomial cubic spline method for solving singularly perturbed delay differential equations with left and right end layers for which a small delay parameter is in the convection term. The stability and convergence of the method have been established. To validate the applicability of the proposed method two model examples without exact solution have been considered and solved for different values of the perturbation parameter and mesh sizes. Both theoretical error bounds and numerical rate of convergences have been investigated for the proposed method and observed to be in agreement. The numerical results have been tabulated and further to examine the effect of delay parameter on the boundary layer solution, graphs have been given for different values of delay parameter.

Keywords/ phrases: Singularly perturbed, delay convection-diffusion equation, fitted spline, stability

INTRODUCTION

A differential equation in which the highest order derivative is multiplied by a small positive parameter ε is called perturbed problem and the parameter ε is known as the perturbation parameter (Roos et al., 2008). Any system involving a feedback control will almost involve time delays. This is because a finite time is required to sense information and then to react on it. If we restrict the class of delay differential equations to a class in which the highest derivative is multiplied by a small positive parameter and involving at least one delay term, then it is said to be a singularly perturbed delay differential equations. In this problem typically there are thin transition layers where the solution varies rapidly or jumps abruptly, while away from the layers the solution behaves regularly and varies slowly. Thus, there has been a growing interest in the numerical treatment of such differential equations. This is due to the usefulness of such type of differential equations in the mathematical modeling of various physical and biological phenomena. For example, population ecology, control theory, viscous elasticity, and materials with thermal memory (Elsgolt's, 1973).

Recently, many researchers have been trying to develop different numerical methods for solving singularly perturbed delay differential equations. For example, Awoke Andargie and Reddy (2013), presented parameter fitted scheme to solve singularly perturbed delay differential equations. Gemecchis File et al., (2017) and Gashu Gadisa et al., (2018) presented different fourth order finite difference methods for solving singularly perturbed delay reaction-diffusion equations with layer or oscillatory behaviour. Erdogan, (2009) presented an exponentially fitted method to solve singular perturbed delay differential equations. Cubic spline in compression approximations for singularly perturbed delay differential equation with large delay has been presented by (Chakravarthy et al., 2015). The use of cubic splines for the solution of linear two point boundary value problems was suggested by (Bickley, 1968). A fitted finite difference method using polynomial cubic on uniform mesh for solving singularly perturbed two-point boundary value problems is also presented by (Phaneendra and Prasad, 2015). But, still numerical treatment of singularly perturbed boundary value problems needs improvement. Thus, in this paper we present a fitted non-polynomial cubic spline method for solving

*Author to whom all correspondence should be addressed singularly perturbed delay convection-diffusion equations.

DESCRIPTION OF THE METHOD

Consider singularly perturbed delay convection-diffusion equations with variable coefficients of the form:

$$\varepsilon y''(x)+a(x)y'(x-\delta)+b(x)y(x)=f(x),\quad 0\le x\le 1 \quad (1)$$

with interval and boundary conditions,

$$y(x)=\phi(x),\ -\delta\le x\le 0\ ,\ y(1)=\varphi \quad (2)$$

where ε is small perturbation parameter, $0<\varepsilon<<1$ and δ is delay parameter satisfying $0<(\varepsilon-\delta a(x))<<1$ for all $x\in[0,1]$; $a(x)$, $b(x)$, $f(x)$ and $\phi(x)$ are bounded functions in $[0,1]$ and φ is known constant. Further, when $a(x)>0$, Eqs. (1) - (2) has boundary layer on left end of the interval and when $a(x)<0$ it has boundary layer on right end of the interval.

By using Taylor series expansion on the delay term, we have:

$$y'(x-\delta)\approx y'(x)-\delta y''(x)+o(\delta^2) \quad (3)$$

Substituting Eq. (3) into Eq. (1), we obtain an asymptotically equivalent singularly perturbed boundary value problem of the form:

$$\gamma y''(x)+a(x)y'+b(x)y(x)=f(x),\ \text{for } x\in[0,1], \quad (4)$$

where, $\gamma=\varepsilon-\delta a(x)$ under the boundary conditions,

$$y(0)=\phi_0 \text{ and } y(1)=\varphi$$

Consider a uniform mesh with nodal points x_i on $[0,1]$ such that:

$$0=x_0<x_1<x_2<...<x_{N-1}<x_N=1,\ x_i=x_0+ih,\ i=0,1,...,N,\ \text{ where } h=\frac{1}{N}$$

For each segment $[x_i,x_{i+1}], i=0,1,...,N-1$ the non-polynomial cubic spline $S(x)$ has the following form:

$$S(x)=a_i+(x-x_i)b_i+\left(e^{(x-x_i)w}-e^{-(x-x_i)w}\right)c_i+\left(e^{(x-x_i)w}+e^{-(x-x_i)w}\right)d_i \quad (5)$$

where, a_i,b_i,c_i and d_i are unknown coefficients and w is a free parameter.

To determine the unknown coefficients in Eq. (5), we denote:

$$S(x_i)=y_i,\quad S(x_{i+1})=y_{i+1},\quad S'(x_i)=m_i,$$
$$S''(x_i)=M_i,\ S'(x_{i+1})=m_{i+1},\ S''(x_{i+1})=M_{i+1}$$

The coefficients in Eq. (5) are determined as:

$$a_i=y_i-\frac{M_i}{w^2}\ ,\quad c_i=2M_{i+1}-\frac{M_i\left(e^{\theta}+e^{-\theta}\right)}{2w^2\left(e^{\theta}-e^{-\theta}\right)},$$

$$b_i=\frac{y_{i+1}-y_i}{h}+\frac{M_i-M_{i+1}}{w\theta} \text{ and } d_i=\frac{M_i}{2w^2} \quad (6)$$

where, $wh=\theta$.

Using the continuity condition of the first derivative at x_i, $S'_{\Delta-1}(x_i)=S'_{\Delta}(x_i)$, we have:

$$b_{i-1}+wc_{i-1}\left(e^{\theta}+e^{-\theta}\right)+wd_{i-1}\left(e^{\theta}-e^{\theta}\right)=b_i+2wc_i \quad (7)$$

Reducing indices of Eq. (6) by one and substituting into Eq. (7), we obtain:

$$\frac{y_i-y_{i-1}}{h}+\frac{M_{i-1}-M_i}{w\theta}+w\left(\frac{2M_i-\left(e^{\theta}+e^{-\theta}\right)M_{i-1}}{2w^2\left(e^{\theta}-e^{-\theta}\right)}\right)\left(e^{\theta}+e^{-\theta}\right)+\left(\frac{M_{i-1}}{2w^2}\right)w\left(e^{\theta}-e^{-\theta}\right)$$
$$=\frac{y_{i+1}-y_i}{h}+\frac{M_i-M_{i+1}}{w\theta}+2w\left(\frac{2M_{i+1}-\left(e^{\theta}+e^{-\theta}\right)M_i}{2w^2\left(e^{\theta}-e^{-\theta}\right)}\right)$$

$$\Rightarrow\frac{y_{i-1}-2y_i+y_{i+1}}{h^2}=\alpha M_{i-1}+2\beta M_i+\alpha M_{i+1} \quad (8)$$

where, $\alpha=\frac{1}{\theta^2}\left(1-\frac{2\theta}{\left(e^{\theta}-e^{-\theta}\right)}\right)$ and $\beta=\frac{1}{\theta^2}\left(\frac{\theta\left(e^{\theta}+e^{-\theta}\right)}{e^{\theta}-e^{-\theta}}-1\right)$.

As $\theta\to 0$ in Eq. (8), we get $\alpha+\beta=\frac{1}{2}$.

Using $S''(x_i)=y''_i=M_i$ into Eq. (4), we get:

$$\gamma M_i=f_i-a_iy'_i-b_iy_i,$$
$$\gamma M_{i-1}=f_{i-1}-a_{i-1}y'_{i-1}-b_{i-1}y_{i-1} \quad \text{and}$$
$$\gamma M_{i+1}=f_{i+1}-a_{i+1}y'_{i+1}-b_{i+1}y_{i+1} \quad (9)$$

Using Taylor's series expansions of y_{i-1}, y_{i+1}, y'_{i-1}, y'_{i+1} and simplifying, we have:

$$y'_i=\frac{y_{i+1}-y_{i-1}}{2h}+T_1,$$

$$y'_{i-1}=\frac{-y_{i+1}+4y_i-3y_{i-1}}{2h}+T_2 \quad \text{and}$$

$$y'_{i+1}=\frac{3y_{i+1}-4y_i+y_{i-1}}{2h}+T_2 \quad (10)$$

where, $T_1=-\frac{h^2}{6}y'''(\xi)$ and $T_2=\frac{h^2}{12}y'''(\xi)$, for $\xi\in(x_{i-1},x_i)$.

Using Eq. (10) into Eq. (9), we get:

$$M_i=\frac{1}{\gamma}\left\{f_i-a_i\left(\frac{y_{i+1}-y_{i-1}}{2h}+T_1\right)-b_iy_i\right\}, \quad (11)$$

$$M_{i-1}=\frac{1}{\gamma}\left\{f_{i-1}-a_{i-1}\left(\frac{-y_{i+1}+4y_i-3y_{i-1}}{2h}+T_2\right)-b_{i-1}y_{i-1}\right\}, \quad (12)$$

$$M_{i+1}=\frac{1}{\gamma}\left\{f_{i+1}-a_{i+1}\left(\frac{3y_{i+1}-4y_i-y_{i-1}}{2h}+T_2\right)-b_{i+1}y_{i+1}\right\} \quad (13)$$

Substituting Eqs. (11) - (13) into Eq. (8) and rearranging, we get:

$$\frac{\gamma}{h^2}(y_{i-1}-2y_i+y_{i+1})+\frac{\alpha a_{i-1}}{2h}(-y_{i+1}+4y_i-3y_{i-1})+\frac{2\beta a_i}{2h}(y_{i+1}-y_{i-1})$$
$$+\frac{\alpha a_{i+1}}{2h}(3y_{i+1}-4y_i+y_{i-1})=\alpha(f_{i-1}-b_{i-1}y_{i-1}+f_{i+1}-b_{i+1}y_{i+1})+2\beta(f_i-b_iy_i)+T \quad (14)$$

where, $T=(4\beta a_i-\alpha a_{i-1}-\alpha a_{i+1})\frac{h^2}{12}y'''(\xi)$ is a local truncation error.

From the theory of singular perturbations described in O' Malley, (1974) and the Taylor's series expansion of $a(x)$ about the point '0' in the asymptotic solution of the problem in Eq. (4), we have:

$$y(x_i)\approx y_0(x_i)+(\phi_0-y_0(0))e^{-a(0)\left(\frac{ih}{\gamma}\right)}$$

and letting $\rho=\frac{h}{\gamma}$, we get:

$\lim_{h\to 0}y(ih)\approx y_0(0)+(\phi_0-y_0(0))e^{-a(0)i\rho}$, since $x_i=x_0+ih=ih$.

Introducing a fitting factor $\sigma(\rho)$ into Eq. (14), we get:

$$\frac{\sigma(\rho)\gamma}{h^2}(y_{i-1}-2y_i+y_{i+1})+\frac{\alpha a_{i-1}}{2h}(-y_{i+1}+4y_i-3y_{i-1})+\frac{2\beta a_i}{2h}(y_{i+1}-y_{i-1})$$
$$+\frac{\alpha a_{i+1}}{2h}(3y_{i+1}-4y_i+y_{i-1})=\alpha(f_{i-1}-b_{i-1}y_{i-1}+f_{i+1}-b_{i+1}y_{i+1})+2\beta(f_i-b_iy_i) \quad (15)$$

Multiplying Eq. (15) by h and taking a limit as $h\to 0$, we get:

$$\frac{\sigma}{\rho}\lim_{h\to 0}(y_{i-1}-2y_i+y_{i+1})+\frac{\alpha a(0)}{2}\lim_{h\to 0}(-y_{i+1}+4y_i-3y_{i-1})$$
$$+\beta a(0)\lim_{h\to 0}(y_{i+1}-y_{i-1})+\frac{\alpha a(0)}{2}\lim_{h\to 0}(3y_{i+1}-4y_i+y_{i-1})=0 \quad (16)$$

Thus, we consider two cases of the boundary layers.

Case I: For $a(x)>0$ (Left-end boundary layer), we have:

$$\lim_{h\to 0}(y_{i-1}-2y_i+y_{i+1})=(\phi_0-y_0(0))e^{-a(0)i\rho}\left(e^{a(0)\rho}+e^{-a(0)\rho}-2\right) \quad (17)$$

$$\lim_{h\to 0}(-3y_{i-1}+4y_i-y_{i+1})=(\phi_0-y_0(0))e^{-a(0)i\rho}\left(-3e^{a(0)\rho}-e^{-a(0)\rho}+4\right) \quad (18)$$

$$\lim_{h\to 0}(y_{i-1}-4y_i+3y_{i+1})=(\phi_0-y_0(0))e^{-a(0)i\rho}\left(e^{a(0)\rho}+3e^{-a(0)\rho}-4\right) \quad (19)$$

$$\lim_{h\to 0}(y_{i+1}-y_{i-1})=(\phi_0-y_0(0))e^{-a(0)i\rho}\left(e^{-a(0)\rho}-e^{a(0)\rho}\right) \quad (20)$$

Using Eqs. (17) - (20) into Eq. (16) and simplifying, we get:

$$\sigma_0=\rho a(0)(\alpha+\beta)\frac{e^{a(0)\rho}-e^{-a(0)\rho}}{e^{a(0)\rho}+e^{-a(0)\rho}-2}$$
$$=\rho a(0)(\alpha+\beta)\coth\left(\frac{a(0)\rho}{2}\right)$$

Case II: For $a(x)<0$ (Right-end boundary layer), we have:

$$\lim_{h\to 0}(y_{i-1}-2y_i+y_{i+1})=(\varphi-y_0(1))e^{-a(1)i\rho}\left(e^{a(1)\rho}+e^{-a(1)\rho}-2\right) \quad (21)$$

$$\lim_{h\to 0}(-3y_{i-1}+4y_i-y_{i+1})=(\varphi-y_0(1))e^{-a(1)i\rho}\left(-3e^{a(1)\rho}-e^{-a(1)\rho}+4\right) \quad (22)$$

$$\lim_{h\to 0}(-y_{i-1}-4y_i+3y_{i+1})=(\varphi-y_0(1))e^{-a(1)i\rho}\left(-e^{a(1)\rho}+3e^{-a(1)\rho}-4\right) \quad (23)$$

$$\lim_{h\to 0}(y_{i+1}-y_{i-1})=(\varphi-y_0(1))e^{-a(1)i\rho}\left(e^{-a(1)\rho}-e^{a(1)\rho}\right) \quad (24)$$

Using Eqs. (21) - (24) into Eq. (16) and simplifying, we get:

$$\sigma_N=\rho a(1)(\alpha+\beta)\frac{e^{a(1)\rho}-e^{-a(1)\rho}}{e^{a(1)\rho}+e^{-a(1)\rho}-2}=\rho a(1)(\alpha+\beta)\coth\left(\frac{a(1)\rho}{2}\right)$$

In general, we can take a variable fitting factor as:

$$\sigma(\rho_i)=\rho_i a(x_i)(\alpha+\beta)\coth\left(\frac{a(x_i)\rho_i}{2}\right) \quad (25)$$

where, $\rho_i=\frac{h}{\gamma}$.

Thus, Eq. (15) can be rewritten as:

$$\left\{\frac{\gamma_i\sigma_i}{h^2}-\frac{3\alpha a_{i-1}}{2h}+\alpha b_{i-1}-\frac{\beta a_i}{h}+\frac{\alpha a_{i+1}}{2h}\right\}y_{i-1}-\left\{\frac{2\gamma_i\sigma_i}{h^2}-\frac{2\alpha a_{i-1}}{h}-2\beta b_i+\frac{2\alpha a_{i+1}}{h}\right\}y_i$$
$$+\left\{\frac{\gamma_i\sigma_i}{h^2}-\frac{\alpha a_{i-1}}{2h}+\frac{\beta a_i}{h}+\frac{3\alpha a_{i+1}}{2h}+\alpha b_{i+1}\right\}y_{i+1}=\alpha(f_{i-1}+f_{i+1})+2\beta f_i \quad (26)$$

Further, Eq. (26) can be rewritten as a three term recurrence relation of the form:

$E_iy_{i-1}-F_iy_i+G_iy_{i+1}=H_i$, for $i=1, 2,..., N$-1. (27)

where,

$$E_i=\frac{\gamma_i\sigma_i}{h^2}-\frac{3\alpha a_{i-1}}{2h}+\alpha b_{i-1}-\frac{\beta a_i}{h}+\frac{\alpha a_{i+1}}{2h},$$

$$F_i=\frac{2\gamma_i\sigma_i}{h^2}-\frac{2\alpha a_{i-1}}{h}-2\beta b_i+\frac{2\alpha a_{i+1}}{h}$$

$$G_i=\frac{\gamma_i\sigma_i}{h^2}-\frac{\alpha a_{i-1}}{2h}+\frac{\beta a_i}{h}+\frac{3\alpha a_{i+1}}{2h}+\alpha b_{i+1} \quad \text{and}$$

$$H_i=\alpha(f_{i-1}+f_{i+1})+2\beta f_i$$

The tri-diagonal system in Eq. (26) can be easily solved by the method of Discrete Invariant Imbedding Algorithm.

Stability and Convergence Analysis

Theorem 1: (Stability)

Let B be a coefficient matrix of the tri-diagonal system, Eq. (26). Then, for all $\varepsilon > 0$ and sufficiently small h, the matrix B is an irreducible and diagonally dominant matrix and hence the scheme is stable.

Proof: Substituting Eq. (25) in Eq. (26) and multiplying both sides of the equation by h, we get the equivalent tri-diagonal scheme:

$$\left\{\frac{a_i}{2}\coth\left(\frac{a_i\rho_i}{2}\right)-\frac{3\alpha a_{i-1}}{2}+h\alpha b_{i-1}-\beta a_i+\frac{\alpha a_{i+1}}{2}\right\}y_{i-1}-\left\{a_i\coth\left(\frac{a_i\rho_i}{2}\right)-2\alpha a_{i-1}-2h\beta b_i+2\alpha a_{i+1}\right\}y_i$$
$$+\left\{\frac{a_i}{2}\coth\left(\frac{a_i\rho_i}{2}\right)-\frac{\alpha a_{i-1}}{2}+\beta a_i+\frac{3\alpha a_{i+1}}{2}+h\alpha b_{i+1}\right\}y_{i+1}=h\left(\alpha\left(f_{i-1}+f_{i+1}\right)+2\beta f_i\right) \tag{28}$$

This can be rewritten as

$$E_i^* y_{i-1} - F_i^* y_i + G_i^* y_{i+1} = H_i^* \tag{29}$$

where,

$$E_i^* = \frac{a_i}{2}\coth\left(\frac{a_i\rho_i}{2}\right)-\frac{3\alpha a_{i-1}}{2}+h\alpha b_{i-1}-\beta a_i+\frac{\alpha a_{i+1}}{2},$$

$$F_i^* = a_i\coth\left(\frac{a_i\rho_i}{2}\right)-2\alpha a_{i-1}-2h\beta b_i+2\alpha a_{i+1}$$

$$G_i^* = \frac{a_i}{2}\coth\left(\frac{a_i\rho_i}{2}\right)-\frac{\alpha a_{i-1}}{2}+\beta a_i+\frac{3\alpha a_{i+1}}{2}+h\alpha b_{i+1}$$

and $H_i^* = h\left(\alpha\left(f_{i-1}+f_{i+1}\right)+2\beta f_i\right)$

Rewriting Eq. (29) in a matrix vector form, we obtain:

$BY = C$ where, B is a coefficient matrix, $Y = \left(y_1, y_2, \cdots, y_{N-1}\right)^T$ and

$C = \left(H_1^* - E_1^*\phi_0,\ H_2^*, \cdots,\ H_{N-1}^* - G_{N-1}^*\varphi\right)^T$.

The matrix B is tri-diagonal matrix and its off-diagonal elements are E_i^* and G_i^*.

Now,

$$\left|E_i^*+G_i^*\right|=\left|a_i\coth\left(\frac{a_i\rho_i}{2}\right)+\alpha\left(a_{i+1}-a_{i-1}\right)\right| < \left|a_i\coth\left(\frac{a_i\rho_i}{2}\right)+2\alpha\left(a_{i+1}-a_{i-1}\right)\right|=\left|F_i^*\right|$$

This implies that for each row of B, the sum of the two off-diagonal elements is less than the modulus of the diagonal element. Therefore, B is diagonally dominant.

Further, for sufficiently small h $(i.e,\ h \to 0)$, we have: $E_i^* \neq 0$ and $G_i^* \neq 0$, $\forall i = 1, 2, \ldots, N-1$. Hence, B is irreducible (Varga, 2000). Therefore, from these two conditions, the scheme in Eq. (27) is stable (Kadalbajoo and Reddy, 1989).

Theorem 2: (Convergence)

Let $y(x)$ be the analytical solution of the problem in Eq. (4) and (5), and y^N be the numerical solution of the discretized problem of Eq. (27). Then, $\left\|y - y^N\right\| \leq ch^2$ for sufficiently small h and c is positive constant.

Proof: Multiplying both sides of Eq. (26) by $\dfrac{-h^2}{\gamma_i\sigma_i}$ and simplifying, we obtain:

$$\left(-1+u_i\right)y_{i-1}+\left(2+v_i\right)y_i+\left(-1+w_i\right)y_{i+1}+g_i+T_i=0 \tag{30}$$

where,

$$u_i=\frac{1}{\gamma_i\sigma_i}\left(\frac{3\alpha h a_{i-1}}{2}-\alpha h^2 b_{i-1}+\beta a_i h-\frac{\alpha h a_{i+1}}{2}\right),$$

$$v_i=\frac{2}{\gamma_i\sigma_i}\left(\alpha h a_{i+1}-\alpha h a_{i-1}-\beta h^2 b_i\right)$$

$$w_i=\frac{1}{\gamma_i\sigma_i}\left(\frac{\alpha h a_{i-1}}{2}-\beta h a_i-\frac{3\alpha h a_{i+1}}{2}-\alpha h^2 b_{i+1}\right)$$

, $g_i=\dfrac{-h^2}{\gamma_i\sigma_i}\left\{\alpha\left(f_{i-1}+f_{i+1}\right)+2\beta f_i\right\}$ and

$T_i(h)=\dfrac{\alpha\left(a_{i-1}+a_{i+1}\right)-4\beta a_i}{12\gamma_i\sigma_i}h^4 y'''(\xi)$ is a local truncation error for $i = 1, 2, \cdots, N\text{-}1$.

Incorporating the boundary condition $y_0 = \phi(x_0) = \phi_0$, $y_N = y(1) = \varphi$ in Eq. (30), we get the system of equation of the form:

$$\left(D+P\right)y+M+T(h)=\bar{0} \tag{31}$$

where, $D=\begin{pmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & \cdots & 0 \\ 0 & - & - & & \\ \vdots & & & & -1 \\ 0 & - & - & -1 & 2 \end{pmatrix}$,

$P=\begin{pmatrix} v_1 & w_1 & 0 & \cdots & 0 \\ u_2 & v_2 & w_2 & \cdots & 0 \\ 0 & - & - & & \\ \vdots & & & & w_{N-2} \\ 0 & - & - & u_{N-1} & v_{N-1} \end{pmatrix}$ are tri-diagonal matrices of order $N-1$,

$M=\left[\left(g_1+(-1+u_1)\phi_0\right),g_2,g_3,\ldots,\left(g_{N-1}+(-1+w_{N-1})\varphi\right)\right]^T$, $T(h)=o(h^4)$ and $y=[y_1,y_2,\ldots,y_{N-1}]^T$, $T(h)=[T_1,T_2,\ldots,T_{N-1}]^T$, $\bar{0}=[0,0,0,\ldots,0]^T$ are associated vectors of Eq. (31).

Let $y^N=\left[y_1^{\,N},y_2^{\,N}\ldots y_{N-1}^{\,N}\right]^T\cong y$ be the solution which satisfies the Eq. (31), we have:

$$(D+P)y^N+M=\bar{0} \tag{32}$$

Let $e_i=y_i-y_i^{\,N}$, for $i=1,\ 2,\ \ldots,N\text{-}1$ be the discretization error then,

$y-y^N=[e_1,e_2,\ldots,e_{N-1}]^T$.

Subtracting Eq. (31) from Eq. (32), we get:

$$(D+P)(y^N-y)=T(h) \tag{33}$$

Let

$|a_{i-1}|\le c_1,\ |a_i|\le c_2,\ |a_{i+1}|\le c_3,\ |b_{i-1}|\le k_1,\ |b_i|\le k_2$ and $|b_{i+1}|\le k_3$

Let $t_{i,j}$ be the $(i,j)^{th}$ element of the matrix P, then:

$$\left|t_{i,i+1}\right|=|w_i|\le\frac{h}{\gamma_i\sigma_i}\left(\frac{3\alpha c_1}{2}+\alpha hc_2+\frac{3\alpha c_3}{2}+\alpha k_3\right)$$

$,\ i=\ 1,\ 2,\ldots,N-2$

$$\left|t_{i,i-1}\right|=|u_i|\le\frac{h}{\gamma_i\sigma_i}\left(\frac{3\alpha c_1}{2}+\alpha hk_1+\beta c_2+\frac{\alpha c_3}{2}\right),$$

$i=\ 2,3,\ldots,N-1$

Thus, for sufficiently small h, we have:

$-1+\left|t_{i,i+1}\right|\ne0,\qquad i=\ 1,\ 2\ldots,N-2,$

$-1+\left|t_{i,i-1}\right|\ne0,\ i=\ 2,3,\ldots,N-1.$

Hence, the matrix (D+P) is irreducible, (Varga, 2000).

Let A_i be the sum of the elements of the i^{th} row of the matrix $(D+P)$, then:

$$A_i=1+v_i+w_i=1+\frac{2h}{\gamma_i\sigma_i}\left(\alpha a_{i+1}-\alpha a_{i-1}+\frac{\alpha a_{i-1}}{4}-\frac{\beta a_i}{2}-\frac{3\alpha a_{i+1}}{4}\right)+o(h^2),\ \text{for } i=1$$

$$A_i=u_i+v_i+w_i=\frac{h^2}{\gamma_i\sigma_i}\left(-\alpha b_{i-1}-2\beta b_i-\alpha b_{i+1}\right),\ \text{for } i=2,3,\ldots,N-2$$

$$A_i=1+u_i+v_i=1+\frac{2h}{\gamma_i\sigma_i}\left(\frac{3\alpha a_{i+1}}{4}-\frac{\alpha a_{i-1}}{4}+\frac{\beta a_i}{2}\right)+o(h^2),\text{for } i=N-1$$

Let $\quad d_1=\min\limits_{1\le i\le N-1}\dfrac{1}{\gamma_i\sigma_i}\left(-\alpha b_{i-1}-2\beta b_i-\alpha b_{i+1}\right)$

and

$d_2=\max\limits_{1\le i\le N-1}\dfrac{1}{\gamma_i\sigma_i}\left(-\alpha b_{i-1}-2\beta b_i-\alpha b_{i+1}\right)$ then,

$0<d_1\le d_2$.

For sufficiently small h, (D+P) is monotone, (Varga, 2000) and (Young, 1971).

Hence, $(D+P)^{-1}$ exists and $(D+P)^{-1}\ge0$.

From the error Eq. (33), we have:

$$\left\|y-y^N\right\|\le\left\|(D+P)^{-1}\right\|\left\|T(h)\right\| \tag{34}$$

For sufficiently small h, we have:

$A_i>h^2d_1$ for $i=1,2,\ldots,N-1$.

where,

$$d_1=\min_{1\le i\le N-1}\left(\frac{1}{\gamma_i\sigma_i}\left(-\alpha b_{i-1}-2\beta b_i-\alpha b_{i+1}\right)\right).$$

Let $(D+P)_{i,k}^{-1}$ be $(i,k)^{th}$ element of $(D+P)^{-1}$ and we define,

$\left\|(D+P)^{-1}\right\|=\max\limits_{1\le i\le N-1}\sum\limits_{K=1}^{N-1}(D+P)_{i,k}^{-1}$ and

$$\left\|T(h)\right\|=\max_{1\le i\le N-1}|T_i| \tag{35}$$

Since $(D+P)_{i,k}^{-1}\ge0$, from the theory of matrices, we have:

$\sum\limits_{k=1}^{N-1}(D+P)_{i,k}^{-1}.A_K=1$, for $i=\ 1,2,\ldots,N-1$

$$\text{Hence, }\sum_{k=1}^{N-1}(D+P)^{-1}_{\ \ i,k}\le\frac{1}{\min\limits_{1\le i\le N-1}A_k}\le\frac{1}{h^2d_1} \tag{36}$$

Now, from Eqs. (34) - (36), we get:

$$\left\|y-y^N\right\|\le\frac{1}{h^2d_1}\left|\left(\frac{\alpha\left(a_{i-1}+a_{i+1}\right)-4\beta a_i}{\gamma_i\sigma_i}\right)\frac{1}{12}h^4y'''(\xi)\right|$$

$$\le\left(\frac{y'''(\xi)\left(4\beta a_i+\alpha\left(a_{i-1}+a_{i+1}\right)\right)}{12d_1\gamma_i\sigma_i}\right)h^2=ch^2$$

where, $c=\dfrac{\left(4\beta a_i+\alpha\left(a_{i-1}+a_{i+1}\right)\right)}{12d_1\gamma_i\sigma_i}y'''(\xi)$

which is independent of mesh size h.

This establishes that the method is of second order convergent.

Numerical Examples

To demonstrate the applicability of the method, two model examples one each for right and left layers have been considered. The numerical results are presented for $\alpha = 1/12$ and $\beta = 5/12$. Since both examples (Examples 1 and 2) have no exact solution, the numerical solutions are computed using double mesh principle. The maximum absolute errors are computed using double-mesh principle given by:

$$\|E\| = \max_i \left| y_i^h - y_i^{h/2} \right|, i = 1, 2, ..., N-1 \qquad (37)$$

where y_i^h is the numerical solution on the mesh $\{x_i\}^{N-1}$ at the nodal point x_i and $x_i = x_0 + ih,\ i = 1, 2, ..., N-1,$ $y^{h/2}$ the numerical solution on a mesh, obtained by bisecting the original mesh with N number of mesh intervals, Doolan et al (1980).

Example 1: Consider the following singularly perturbed delay convection-diffusion problem,

$$\varepsilon y''(x) - e^x y'(x-\delta) - xy(x) = 0$$

subject to the interval and boundary conditions,

$$y(x) = 1,\ -\delta \le x \le 0,\quad y(1) = 1.$$

The maximum absolute errors are presented for the present method in Tables 1 and 2 for $\varepsilon = 0.1$ and different values of δ, in Table 5 for different values of ε and $\delta = 0.5\varepsilon$, and the rate of convergence is presented in Table 6 for $\varepsilon = 0.1$ and $\delta = 0.03$.

Example 2: Consider the following singularly perturbed delay convection-diffusion problem,

$$\varepsilon y''(x) + e^{-0.5x} y'(x-\delta) - y(x) = 0$$

subject to the interval and boundary conditions,

$$y(x) = 1,\ -\delta \le x \le 0,\quad y(1) = 1.$$

The maximum absolute errors are presented for the present method in Tables 3 and 4 for $\varepsilon = 0.1$ and different values of δ, in Table 5 for different values of ε, and $\delta = 0.5\varepsilon$, and the rate of convergence is presented in Table 6 for $\varepsilon = 0.1$ and $\delta = 0.03$.

Numerical Results

Table 1: **Maximum Absolute errors of Example 1, for different values of δ and ε = 0.1.**

$\delta\downarrow$ $\underrightarrow{N}$	10^2	10^3	10^4
Our Method			
0.01	2.6064e-05	2.5938e-07	2.6486e-09
0.03	2.1154e-05	2.1130e-07	2.1233e-09
0.06	1.6255e-05	1.6241e-07	1.6528e-09
0.08	1.4022e-05	1.4019e-07	1.4440e-09
Reddy et al., 2012			
0.01	5.75975e-03	5.0842e-04	5.02478e-05
0.03	3.93277e-03	3.6132e-04	3.58384e-05
0.06	2.70257e-03	2.5507e-04	2.53643e-05
0.08	2.24689e-03	2.1413e-04	2.13134e-05

Table 2: **Maximum Absolute errors of Example 1, for different values of δ and N = 100.**

$\varepsilon = 0.1$			$\varepsilon = 0.01$		
$\delta\downarrow$	Awoke and Reddy, 2013	Our Method	$\delta\downarrow$	Awoke and Reddy, 2013	Our Method
0.04	1.05e-03	1.9248e-05	0.002	1.05e-02	3.0712e-04
0.06	8.43e-04	1.6255e-05	0.005	8.79e-03	2.0784e-04
0.07	6.93e-04	1.5062e-05	0.007	7.52e-03	1.6509e-04
0.08	4.75e-04	1.4022e-05	0.008	6.95e-03	1.5229e-04
0.09	3.35e-04	1.3114e-05	0.009	6.42e-03	1.4383e-04

Table 3: **Maximum Absolute errors of Example 2, for different values of δ and ε = 0.1.**

$\delta\downarrow$ $N\rightarrow$	10^2	10^3	10^4
Our Method			
0.01	2.0717e-05	2.0725e-07	2.0660e-09
0.03	2.7350e-05	2.7363e-07	2.7496e-09
0.06	4.4987e-05	4.5034e-07	4.5055e-09
0.08	9.9507e-05	1.0389e-06	1.0389e-08
Reddy et al., 2012			
0.01	6.32996e-03	6.74268e-04	6.78713e-05
0.03	8.15917e-03	8.82563e-04	9.89869e-05
0.06	1.38476e-02	1.57973e-03	1.60200e-04
0.08	2.47716e-02	3.17323e-03	3.26028e-04

Table 4**: Maximum Absolute errors of Example 2, for different values of δ and N = 100.**

$\varepsilon = 0.1$			$\varepsilon = 0.01$		
$\delta\downarrow$	Awoke and Reddy, 2013	Our Method	$\delta\downarrow$	Awoke and Reddy, 2013	Our Method
0.04	6.29e-04	3.1822e-05	0.002	2.69e-04	8.6399e-05
0.05	1.26e-03	3.7493e-05	0.004	2.00e-04	1.1103e-04
0.06	1.55e-03	4.4987e-05	0.006	5.41e-04	1.5349e-04
0.07	2.00e-03	5.5771e-05	0.007	7.48e-04	1.8883e-04
0.08	2.77e-03	9.9507e-05	0.008	1.17e-03	2.4367e-04

Table 5: **Maximum absolute errors for different values of ε and δ = 0.5ε.**

$\varepsilon\downarrow$ $N\rightarrow$	200	400	800	1600	3200

Example 1					
2^{-8}	1.4113e-04	3.1324e-05	7.6013e-06	1.8967e-06	4.7324e-07
2^{-10}	4.1860e-04	1.3923e-04	3.5373e-05	7.8568e-06	1.9068e-06
2^{-12}	5.0318e-04	2.4577e-04	1.0479e-04	3.4837e-05	8.8490e-06
2^{-14}	5.0346e-04	2.5201e-04	1.2601e-04	6.1491e-05	2.6207e-05
2^{-16}	5.0346e-04	2.5201e-04	1.2608e-04	6.3056e-05	3.1515e-05
2^{-20}	5.0346e-04	2.5201e-04	1.2608e-04	6.3057e-05	3.1533e-05
2^{-30}	5.0346e-04	2.5201e-04	1.2608e-04	6.3057e-05	3.1533e-05
2^{-36}	5.0346e-04	2.5201e-04	1.2608e-04	6.3057e-05	3.1533e-05
Example 2					
2^{-8}	4.3004e-04	2.1557e-04	1.9983e-05	5.0105e-06	1.2535e-06
2^{-10}	7.6794e-04	2.6726e-04	7.6210e-05	1.9874e-05	5.0266e-06
2^{-12}	8.7303e-04	4.3400e-04	1.9306e-04	6.7004e-05	1.9087e-05
2^{-14}	8.7309e-04	4.3827e-04	2.1955e-04	1.0882e-04	4.8331e-05
2^{-16}	8.7309e-04	4.3827e-04	2.1957e-04	1.0989e-04	5.4970e-05
2^{-20}	8.7309e-04	4.3827e-04	2.1957e-04	1.0989e-04	5.4974e-05
2^{-30}	8.7309e-04	4.3827e-04	2.1957e-04	1.0989e-04	5.4974e-05
2^{-36}	8.7309e-04	4.3827e-04	2.1957e-04	1.0989e-04	5.4974e-05

Table 6: **Rate of convergence for Examples 1 and 2 when $\varepsilon = 0.1$ and $\delta = 0.03$.**

h	1/100	1/200	1/300	1/400	1/500
Example 1	2.0023	2.0006	2.0003	2.0001	2.0001
Example 2	1.9995	1.9999	1.9999	2.0000	2.0000

The Effect of Delay Term on the Solution Profile

To analyze the effect of the delay term on the solution profile of the problem, the numerical solution of the problem for different values of the delay parameters have been given by the following graphs.

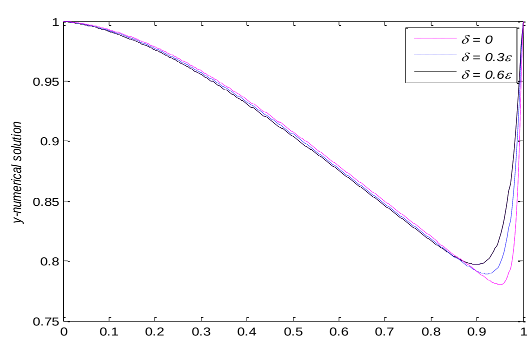

Fig. 1: **The numerical solution of Example 1 with $\varepsilon = 2^{-5}$ and N = 400.**

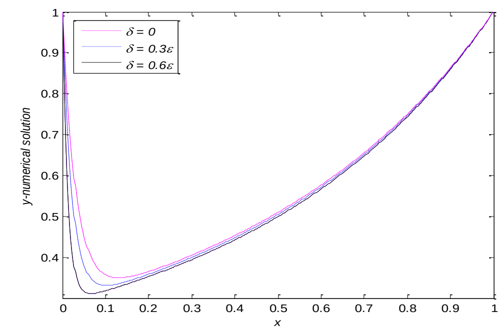

Fig. 2: **The numerical solution of Example 2 with $\varepsilon = 2^{-5}$ and N = 400.**

DISCUSSION AND CONCLUSION

Fitted non-polynomial cubic spline method for solving singularly perturbed delay convection diffusion equations has been presented. The stability and convergence of the method have been investigated. The study is implemented on two examples without exact solutions by taking different values for the perturbation parameter ε and delay parameter δ . The numerical results have been presented in Tables (1 - 5) for different values of the perturbation parameter ε, delay parameter δ and number of mesh points *N*. The results obtained by the present method are compared with results of (Reddy et al, (2012 and Awoke Andargie and Reddy, (2013)) and observed that the present method improved the results. Further, it can also be observed from the tables that the accuracy of the method increases as the resolution of the grid increases which is in agreement with the findings of (Kadalbajoo and Ramesh, 2007), *i.e.*, it is the maximum absolute error decreases rapidly as N increases. As perturbation parameter ε is sufficiently small (*i.e.* for $\varepsilon << h$), some researchers Doolan et al., (1980), Kadalbajoo and Sharma, (2004) and Roos et al., (2008) state that there is a challenge to get more accurate solutions for singularly perturbed boundary value problems. However, in the present method gives good result for ε is sufficiently small an $\varepsilon < h$ (Table 5). The results presented confirmed that computational rate of convergence (Table 6) as well as theoretical estimates indicates that the proposed non-polynomial cubic spline method is a second order convergent.

To demonstrate the effect of delay on the left and right boundary layers solution, graphs for different values of delay parameter δ , mesh size h and perturbation parameter ε are plotted in Figs. 1 and 2; Accordingly, depending on the sign of coefficient of delay term one can see that, from Fig. 2 as δ increases the width of the left boundary layer decreases while the width of the right boundary layer increases Fig. 1.

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REFERENCES

1. Awoke Andargie and Reddy, Y. N. (2013). Parameter fitted scheme for singularly perturbed delay differential equations. *International Journal of Applied Science and Engineering.* 11 (4): 361-373.
2. Bickley, W. G. (1968). Piecewise cubic interpolation and two point boundary value problems. *Comput. J. 11: 206–208.*
3. Chakravarthy, P.P., Kumar, S.D., Rao, R.N. and Ghate, D. P. (2015). A fitted numerical scheme for second order singularly perturbed delay differential equations via cubic spline in compression. *Advances in Difference Equations.* 2015 (300): 1-14.
4. Doolan, E.R. Miller, J.J.H. and Schilders W.H.A. (1980). *Uniform Numerical Methods for Problems with Initial and Boundary Layers*. Boole Press, Dublin.
5. Elsgolts, L. E. and Norkin, S.B. (1973). *Introduction to the Theory and Applications of Differential Equations with Deviating Arguments*. Academic Press, New York.
6. Erdogan, F. (2009). An exponentially fitted method for singularly perturbed delay differential equation. *Advances in Difference Equations.* 781579: 1-9.
7. Gemechis File, Gashu Gadisa, Tesfaye Aga and Reddy, Y. N. (2017). Numerical solution of singularly perturbed delay reaction-diffusion equations with layer or oscillatory behaviour. *American Journal of Numerical Analysis*. 5 (1): 1-10.
8. Gashu Gadisa, Gemechis File and Tesfaye Aga. (2018). Fourth order numerical method for singularly perturbed delay differential equations. *International Journal of Applied Science and Engineering. 15 (1): 17-32.*
9. Kadalbajoo, K. and Ramesh, V.P. (2007). Numerical methods on Shishkin mesh for singularly perturbed delay differential equations with a grid adaptation strategy. *Applied Mathematics and Computation.* 188: 1816–1831.
10. Kadalbajoo, M. K. and Reddy, Y.N. (1989). Asymptotic and numerical analysis of singular perturbation problems. *Applied Mathematics and Computation.* 30: 223-259.
11. Kadalbajoo, M. K. and Sharma, K.K. (2004). Numerical analysis of singularly perturbed delay differential equations with layer behavior. *Appl. Math. Comput.* 157:11–28.
12. O'Malley, R. E. (1974). *Introduction to Singular Perturbations*. Academic Press. New York. pp 22-26.
13. Phaneendra, K. and Prasad, E.S. (2015). Fitted finite difference method for singularly perturbed two-point boundary value problems using polynomial cubic spline. *International Journal on Recent and Innovation Trends in Computing and Communication.* 3(2): 56-60.
14. Reddy, Y.N., Soujanya GBSL, and Phaneendra, K. (2012). Numerical integration method for singularly perturbed delay differential equations. *International Journal of applied science and Engineering.* 10 (3): 249-261.
15. Roos, H.G, Stynes, M. and Tobiiska, L. (2008). *Robust Numerical Methods for Singularly Perturbed Differential Equations*. Springer Series in Computational Mathematics, Berlin.
16. Varga, R.S, (2000). Matrix Iterative analysis. Second Revised and Expanded Edition, Springer Heidelberg Dordrecht London New York.
17. Young, D.M. (1971). *Iterative Solution of Large Linear Systems*. Academic Press, Inc. New York.