

REGULARITY THEOREM FOR FUNCTIONS THAT ARE EXTREMAL TO PALEY INEQUALITY

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ABSTRACT: In this paper we study the asymptotic behavior of functions that are extremal to the inequality introduced by Paley (1932) via a normal family of subharmonic functions.

Key words/phrases: Subharmonic, characteristic function, lower order, Pólya peaks, *-function.

INTRODUCTION

Let u be a subharmonic function in the complex plane. We set

$$B(r, u) = \sup_{|z|=r} u(z), \quad u^+(z) = \max(u(z), 0).$$

$$N(r, u) = \frac{1}{2\pi} \int_0^{2\pi} u(re^{i\theta}) d\theta$$

and define the Nevanlinna characteristic of u by $T(r) = T(r, u) = N(r, u^+)$.

The order ρ of u is by definition

$$\rho = \limsup_{r \rightarrow \infty} \frac{\log T(r, u)}{\log r}$$

Paley's inequality asserts that

$$\liminf \frac{B(r, u)}{T(r, u)} \leq \frac{\pi\rho}{\sin \pi\rho} \tag{1}$$

provided $0 < \rho \leq \frac{1}{2}$.

Inequality (1) was conjectured by Paley (1932) in the case where $u = \log |f(z)|$ for entire function f . Govorov (1969) proved Paley's conjecture. For a proof of (1) for general subharmonic u , see Essen (1975). The function

$$u(re^{i\theta}) = \frac{\pi \rho r^\rho \cos \rho \theta}{\sin \rho \pi}, \quad |\theta| \leq \pi \quad (2)$$

in a subharmonic function which is extremal for (1) with

$$T(r, u) = r^\rho$$

and

$$B(r, u) = \frac{\pi \rho r^\rho}{\sin \pi \rho}.$$

Indeed all subharmonic functions extremal for (1), *i.e.*, for which equality holds in (1) behave asymptotically as (rotations of) the subharmonic function in (2). We shall prove,

Theorem 1: Let u be a subharmonic function of order ρ , $0 < \rho \leq \frac{1}{2}$. If

$$\overline{\lim} \frac{T(r, u)}{B(r, u)} = \frac{\sin \pi \rho}{\pi \rho} \quad (3)$$

Then

- i) $T(r, u) = r^\rho L(r)$
- ii) $B(r, u) \sim \frac{\pi \rho}{\sin \pi \rho} T(r, u), \quad (r \rightarrow \infty, r \in G)$
- iii) $N(r, u) \sim T(r, u), \quad (r \rightarrow \infty, r \in G),$

where $L(r)$ varies slowly in a set G of logarithmic density one, *i.e.*

$$\lim_{\substack{r \rightarrow \infty \\ r \in G}} \frac{L(\sigma r)}{L(r)} = 1, \quad (0 < \sigma < \infty)$$

holds (uniformly for σ in any interval $A^{-1} \leq \sigma \leq A$, $A > 1$) with

$$G = \bigcup_{n=1}^{\infty} [a_n, b_n], \quad \left(a_n \rightarrow \infty, \frac{b_n}{a_n} \rightarrow \infty \right) \quad (4)$$

satisfying

$$\int_{G \cap (1, r)} t^{-1} dt \sim \log r, \quad (r \rightarrow \infty).$$

Moreover, for any sequence $\{r_n\} \subseteq G$ ($r_n \rightarrow \infty$) there is a subsequence of positive integers $I = \{n_k\}$ and a real $\alpha \in [-\pi, \pi]$ such that for almost all z in the set $\{re^{i\theta}, |\theta - \alpha| < \pi\}$ with respect to the Lebesgue measure of the plane we have

$$iv) \quad \lim_{\substack{n \rightarrow \infty \\ n \in I}} \frac{u(r_n z)}{T(r_n, u)} = \frac{\pi \rho r^\rho}{\sin \pi \rho} \cos \rho(\theta - \alpha), \quad |\theta - \alpha| \leq \pi, \quad z = re^{i\theta}$$

SOME FACTS

Let u be a subharmonic function in \mathbb{C} of order ρ , $0 \leq \rho < \infty$. By a sequence of Pólya peaks for $T(r, u)$ of order ρ we mean a sequence $\{r_n\}$ of positive

numbers such that $T(t, u) \leq (1 + \varepsilon_n) \left(\frac{1}{r_n}\right)^\rho T(r_n, u)$ holds for some sequence

$\varepsilon_n \rightarrow 0$, $r_n \rightarrow \infty$ and for all t , such that $\varepsilon_n r_n \leq t \leq \frac{r_n}{\varepsilon_n}$. If ρ is finite $T(r, u)$ has a sequence of Pólya peaks of order ρ (for a proof see Edrei (1965)). We also need the u^* -function of u introduced by Baernstein (1974) which is defined by

$$u^*(re^{i\theta}) = \frac{1}{2\pi} \sup_{|E| = 2\theta} \int_E u(re^{i\phi}) d\phi, \quad (0 < r < \infty, 0 \leq \theta \leq \pi), \tag{5}$$

where $E \subseteq [-\pi, \pi]$ and $|E| =$ Lebesgue measure of the set E . It is shown that u^* is subharmonic in the upper half plane, π^+ and is continuous in the closure of π^+ except possibly at the origin. It is also proved that if the function $\theta \rightarrow \tilde{u}(re^{i\theta})$, for fixed $r > 0$ and $|\theta| \leq \pi$, is the symmetric decreasing rearrangement of $u(re^{i\theta})$, then

$$u^*(re^{i\theta}) = \frac{1}{2\pi} \int_{-\theta}^{\theta} \tilde{u}(re^{i\phi}) d\phi \quad (0 \leq \theta \leq \pi).$$

u^* satisfies (see also Hayman (1989), chap. 9, especially p. 712).

$$T(r, u) = \max_{0 \leq \theta \leq \pi} u^*(re^{i\theta}), \quad u^*(r) = 0 \tag{6}$$

and

$$B(r, u) = \frac{\pi \hat{\partial} u^*}{\partial \theta} (re^{i\theta})|_{\theta=0} = \tilde{u}'(r).$$

Proof of Theorem 1: First we shall establish assertions (i) and (ii) of Theorem 1. We use the well-known result due to Petrenko (1969) given below by Lemma 1.

Lemma 1. Suppose u is subharmonic in C . Fix γ , $0 < \gamma \leq 1$, and let

$$k(t, \gamma) = \frac{\gamma^{-2} t^{\gamma}}{(t^{\gamma} + 1)^2}.$$

Then

$$B(r, u) \leq \int_0^R u(te^{i\pi\gamma}) k\left(\frac{r}{t}, \gamma\right) \frac{dt}{t} + c \left(\frac{r}{R}\right)^{\gamma} T(2R, u), \quad (0 < r < \frac{R}{2}), \quad (7)$$

for an absolute constant c .

Besides Petrenko's (1969) proofs for Lemma 1 are given in Essen (1975) and in Edrei and Fuchs (1976, Lemma 11.1) where it was shown that

$$\hat{k}(s, \gamma) = \int_0^{\infty} k(t, \gamma) \frac{dt}{t^{1+s}} = \frac{\pi s}{\sin(\pi\gamma s)}, \quad (0 < s < \frac{1}{\gamma}).$$

We apply Lemma 1 with $\gamma = 1$ together with the hypotheses of Theorem 1 to obtain (on letting $R \rightarrow \infty$) in (7)

$$\begin{aligned} T(r, u) &\leq \left(\frac{\sin \pi\rho}{\pi\rho} + o(1) \right) \int_0^{\infty} T(t, u) k(r/t, \gamma) \frac{dt}{t}, \quad (r \rightarrow \infty) \\ &= \left(\frac{\sin \pi\rho}{\pi\rho} + o(1) \right) T(r, u) * k(r, \gamma). \end{aligned} \quad (8)$$

Since $\hat{k}(\rho, \gamma) = \frac{\pi\rho}{\sin \pi\rho}$ and $T(r, u)$ is increasing it follows from the Drasin and Shea (1976) Tauberian theorem that there is a set G of the form (4) such that

$$T(r, u) = r^{\rho} L(r) \quad (9)$$

where $L(r)$ varies slowly in G , and further

$$T(r, u) * k(r, \gamma) = \left(\frac{\pi\rho}{\sin \pi\rho} + o(1) \right) T(r, u), \quad (r \rightarrow \infty \in G). \quad (10)$$

Thus from (7), (10) and the hypotheses of Theorem 1 we conclude that

$$\lim_{\substack{r \rightarrow \infty \\ r \in G}} \frac{B(r, u)}{T(r, u)} = \frac{\pi\rho}{\sin \pi\rho}. \quad (11)$$

This proves assertions (i) and (ii) of Theorem 1. It also follows from (9) that any sequence $\{r_n\} \subseteq G$, $r_n \rightarrow \infty$, is a sequence of Polya peaks for $T(r,u)$ of order ρ (see Edrei (1969) Lemma 4). We proceed to prove Theorem 1.

Let $\{r_n\}$ be any sequence of Polya peaks for $T(r,u)$ of order ρ , $0 \leq \rho < \infty$. Then the sequence

$$u_n(z) = \frac{u(r_n z)}{T(r_n u)} \quad (z \in \mathbb{C}, \quad n = 1, 2, 3, \dots) \tag{12}$$

forms a normal family of subharmonic functions in the sense of Anderson and Baernstein (1978), i.e., there is a subharmonic function v in \mathbb{C} and a subsequence $I = \{n_k\}$ of positive integers such that

- a) $\lim_{\substack{n \rightarrow \infty \\ n \in I}} \int_0^{2\pi} |u_n(re^{i\theta}) - v(re^{i\theta})| d\theta = 0, \quad (0 < r < \infty)$ (13)
- b) $\lim_{\substack{n \rightarrow \infty \\ n \in I}} T(r, u_n) = \lim_{\substack{n \rightarrow \infty \\ n \in I}} \frac{T(r r_n, u)}{T(r_n, u)} = T(r, v) \leq r^\rho, \quad (0 < r < \infty).$

Thus if u satisfies the hypotheses of Theorem 1 then from (9) and (13) we have

$$T(r, v) = r^\rho. \tag{14}$$

Indeed we prove,

Theorem 2: Let u be a subharmonic function that satisfies the hypotheses of Theorem 1 and $\{r_n\}$ a sequence of Polya peaks of $T(r,u)$ of order ρ , $0 < \rho \leq \frac{1}{2}$. If v is a subharmonic function that satisfies (13) corresponding to u and the sequence $\{r_n\}$, then

- a) $T(r, v) = r^\rho = N(r, v)$
- b) $v(re^{i\theta}) = \frac{\pi \rho r^\rho}{\sin \pi \rho} \cos \rho(\theta - \alpha), \quad |\theta - \alpha| \leq \pi$ for some $\alpha \in [-\pi, \pi]$.

It is clear that assertions (iii) and (iv) of Theorem 1 follow from Theorem 2 and (13). We first establish,

Lemma 2. Suppose u satisfies the hypotheses of Theorem 1 and $\{r_n\}$ is a sequence of Pólya peaks of $T(r,u)$ of order ρ , $0 < \rho \leq \frac{1}{2}$. If v is a subharmonic function that satisfies (13) corresponding to u and the sequence $\{r_n\}$ then v^* is harmonic in the upper half plane and

$$v^*(re^{i\theta}) = r^\rho \frac{\sin \rho^\theta}{\sin \pi\rho} \quad (0 \leq \theta \leq \pi)$$

Proof. First we prove $B(r,v) = \frac{\pi\rho r^\rho}{\sin \pi\rho}$. Since $v^*(re^{i\theta}) \leq T(r,v) = r^\rho$ and $v^*(r) = 0$ by (6) and (14), we have by Phragme'n - Lindeöf principle

$$v^*(re^{i\theta}) \leq \frac{r^\rho \sin \rho^\theta}{\sin \pi\rho} \quad (0 \leq \theta \leq \pi) \quad (15)$$

which implies $\frac{\partial v^*}{\partial \theta}(re^{i\theta})|_{\theta=0} \leq \frac{\rho r^\rho}{\sin \pi\rho}$. Thus it follows from (6) that

$$B(r,v) \leq \frac{\pi\rho r^\rho}{\sin \pi\rho}$$

To prove the reverse inequality we let $r > 0$ and choose $\alpha_n \in [-\pi, \pi)$, ($n \in I = \{n_k\}$) such that $B(r, u_n) = u_n(re^{i\alpha_n})$, where u_n is defined by (12).

Assume $\alpha_n \rightarrow \alpha_0$ as $n \rightarrow \infty$ ($n \in I$). Then for any $s > r$, we have

$$B(r, u_n) \leq \frac{1}{2\pi} \int_0^{2\pi} u_n(se^{i\theta}) p_{r/s}(\theta - \alpha_n) d\theta. \quad (16)$$

where $p_r(\theta)$ is the Poisson kernel. By dominated convergence theorem and by (13) it follows (letting $n \rightarrow \infty$, $n \in I$ in (16)) that

$$\frac{\pi\rho r^\rho}{\sin \pi\rho} \leq \frac{1}{2\pi} \int_0^{2\pi} v(se^{i\theta}) p_{r/s}(\theta - \alpha_n) d\theta \leq B(s,v). \quad (17)$$

Since (17) holds for any $s > r$ and $B(s,v)$ is a continuous function of s , we have, letting $s \rightarrow r$ in (17), $\frac{\pi\rho r^\rho}{\sin \pi\rho} \leq B(r,v)$. Thus $B(r,v) = \frac{\pi\rho r^\rho}{\sin \pi\rho}$. To prove

the lemma we apply (7) to v with $\gamma = \frac{\alpha}{\pi}$ ($0 < \alpha < \pi$), and using (16) we get

$$\frac{\pi \rho r^\rho}{\sin \pi \rho} = B(r, v) \leq \int_0^\infty v^*(te^{i\alpha}) k(r/t, \gamma) \frac{dt}{t} \leq \frac{\sin \rho \alpha}{\sin \pi \rho} \int_0^\infty t^\rho k\left(\frac{r}{t}, \gamma\right) \frac{dt}{t} = \frac{\pi \rho r^\rho}{\sin \pi \rho}$$

which implies $v^*(te^{i\alpha}) = t^\rho \frac{\sin \rho \alpha}{\sin \pi \rho}$. Thus, applying the maximum principle

to (15) we conclude that $v^*(re^{i\theta}) = \frac{r^\rho \sin \rho \theta}{\sin \pi \rho}$, $(0 \leq \theta \leq \pi)$.

To complete the proof of Theorem 2, let \tilde{v} be the symmetric decreasing rearrangement of v , so that

$$v^*(re^{i\theta}) = \frac{1}{2\pi} \int_{-\theta}^{\theta} \tilde{v}(re^{i\phi}) d\phi, \quad (0 \leq \theta \leq \pi). \tag{18}$$

Since by Lemma 2 $v^*(z)$ is harmonic in $\{z: 0 < \arg(z) < \pi\}$, it follows from (18) that

$$0 = \Delta v^*(re^{i\theta}) = \frac{1}{2\pi} \int_{-\theta}^{\theta} \Delta \tilde{v}(re^{i\phi}) d\phi, \quad (0 < \theta < \pi).$$

Thus \tilde{v} is harmonic in $\{re^{i\theta} : r > 0, |\theta| < \pi\}$. Let $r_1 \geq 0$ and $\alpha \in [-\pi, \pi]$ be such that $v(r_1 e^{i\alpha}) = B(r_1, v) = \tilde{v}(r_1)$. Then a comparison of v^* with the subharmonic function

$$v_\alpha(re^{i\theta}) = \frac{1}{2\pi} \int_{-\theta+\alpha}^{\theta+\alpha} v(re^{i\phi}) d\phi$$

shows that (Essén and Shea, 1978/79)

$$v(ze^{i\alpha}) = \tilde{v}(z), \quad (|\arg(z)| < \pi). \tag{19}$$

By Lemma 2, (18) and (19) we get

$$v(re^{i\theta}) = \frac{\pi \rho r^\rho}{\sin \pi \rho} \cos \rho(\theta - \alpha), \quad (|\theta - \alpha| \leq \pi). \tag{20}$$

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