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ON THE EQUATION XY = Z (Y - X²) OVER A UNIQUE FACTORIZATION DOMAIN

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ABSTRACT: The article gives the parametric solutions of $xy = z(y - x^2)$ over any unique factorization domain. An application of the obtained characterization to Diophantine equations is demonstrated.

1. INTRODUCTION

b = 5 and a = 2 are natural numbers such that the decimal representation of b/a is $(a.b)_{ten}$. In an informal discussion, L. Kadosh* asks whether there are other natural numbers b and a such that $b/a = (a.b)_{ten}$.

To fix notations, let a, b and g be natural numbers with $g \geq 2$. Put

$$a = \sum_{i=0}^{k} a_i g^i \text{ and } b = \sum_{j=0}^{n-1} b_j g^j$$

with $0 \leq a_i, b_j < g$, k and n, whole numbers, $n > 0$ and $a_k b_{n-1} \neq 0$. The expression

$$\alpha = a + \frac{b}{g^n} = \sum_{i=0}^{k} a_i g^i + \sum_{j=0}^{n-1} b_j g^{-n+j}$$

* L. Kadosh says that he got the problem from a colleague. A reviewer has pointed out that the problem is due to Michael S. Runge (1991).

is called the representation of the rational number $\alpha = a + \frac{b}{g^n}$ in base g. For simplicity, we denote this representation of α by $(a.b)_g$.

Thus a generalization of the above posed problem is whether

$$b/a = (a.b)_g \tag{1}$$

holds for some other pair of natural numbers (b,a). Still more, the above problem is contained in a question that looks for natural numbers a and b, b not a multiple of a, such that

$$\frac{b}{a} = a + \frac{b}{h} \tag{2}$$

for some natural number h.

Our basic interest is on the solutions of (2) in natural numbers. On the other hand, the arguments developed to solve (2) in natural numbers also hold in unique factorization domains. Evidently (2) is equivalent to

$$ab = h(b - a^2) \tag{3}$$

with $ah \neq 0$, b is not a multiple of a, which is the title of the article with x = a, y = b and z = h over such domains. In the next two sections, we discuss (3) in any unique factorization domain. Theorem 1 gives the parametric solutions of (3). The particulars about natural numbers are indicated in the remarks. The last section gives positive answer to (1). The base ten problem in (1) shall be considered in a forthcoming article.

2. THE PRIMARY RESULTS

In the present article, D denotes a unique factorization domain (UFD) with $1 \neq 0$. If a, b, A, $B \in D \setminus \{0\}$, then a greatest common factor of a and b exists, denoted by (a , b), and is unique up to associates. We write (a, b) = (A, B) to mean that (a, b) and (A, B) are associates. Moreover (a, b) = 1 shall mean that a and b are relatively prime.

Suppose a, b$\in D$, b$\notin aD$ such that

$$\frac{b}{a}=a+\frac{b}{h}^{**} \tag{4}$$

for some h$\in D$. Then h is a non-unit in D and (4) is equivalent to (3) over D. Put c = b - a^2 and let d = (a, b). Then a = da_1 and b = db_1 with a_1, b_1 $\in D$ and (a_1, b_1) = 1. Therefore c = dc_1 with $c_1 = b_1 - da_1^2 \in D$. It is then clear that a_1 is a non-unit and relatively prime to b_1c_1. Unless otherwise stated, a, a_1, b, b_1, c, c_1, d and h refer to the notations defined above. For a prime p$\in D$ and x$\in D\backslash\{0\}$, $\text{ord}_p(x)$ is the largest integer n such that

$$x \equiv 0 \pmod{p^n}.$$

Lemma 1. Suppose a, b and h$\in D$ satisfying (4). Then, with the notations defined above, c_1 divides d^2. If D is the ring of rational integers, then h is odd iff c_1 is even and

$$\text{ord}_2(c_1) = 2\,\text{ord}_2(d).$$

Proof. If a, b, h$\in D$ satisfy (4), then they satisfy (3).

Hence

$$h = \frac{ab}{c} = \frac{d^2a_1^3}{c_1} + a = a_1\left(\frac{d^2a_1^2}{c_1} + d\right) \tag{5}$$

Since (a_1, c_1) = 1, we conclude that $d_2 \equiv 0 \pmod{c_1}$.

To prove the second part, let a, b, h be rational integers satisfying (3). Suppose h is odd. Then from (5), we conclude that a_1 is odd, hence $\frac{d^2}{c_1}$ and d

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**The fraction notation seems handier for this problem and we continue to use it.

have opposite parity. If $\frac{d^2}{c_1}$ were even, then d would be even. Hence $\frac{d^2}{c_1}$ is necessarily odd, and hence d is even. Therefore $ord_2(c_1) = 2\, ord_2(d) \geq 2$.

Conversely, if c_1 is even and $ord_2(c_1) = 2\, ord_2(d) \geq 2$, then d is even. But then $(a_1, c_1) = 1$, hence a_1 is odd. Therefore

$$h = a_1\left(\frac{d^2}{c_1}a_1^2 + d\right) \text{ is odd.}$$

Assume a, b, $h \in D$ satisfy (4). Then by Lemma 1, $d^2 = qc_1$ for some $q \in D$. Using (5), we have $h = qa_1^3 + a = a_1(qa_1^2 + d)$.

On the other hand, $b = a^2 + c = d^2a_1^2 + dc_1 = c_1(qa_1^2 + d)$.

Conversely, consider the following algorithm.

(2.1) Choose arbitrary but relatively prime $a_1, c_1 \in D\backslash\{0\}$, a_1 non-unit and $d \in D\backslash\{0\}$ such that $d^2 = qc_1$ for some $q \in D$.

(2.2) Put $a = da_1$, $c = dc_1$ so that $d^3 \equiv 0 \pmod c$.

(2.3) Set $b = a^2 + c = c_1(qa^2_1 + d)$
$h = qa_1^3 + a = a_1\,(qa_1^2 + d)$

Then a, b and h satisfy the relation

$$\frac{b}{a} = a + \frac{b}{h}\,.$$

Hence we have proved the following.

Theorem 1. Suppose a, b, $h \in D$ such that $b \notin aD$ and $\frac{b}{a} = a + \frac{b}{h}$. Then $b = c_1(qa_1^2 + d)$ and $h = a_1(qa_1^2 + d)$, where $d^2 = qc_1$. Conversely, the solutions of $\frac{b}{a} = a + \frac{b}{h}$ are obtained by the algorithm **(2.1)** - **(2.3)**.

Remark 1

(1.1) Consider the partial ordering on D given by $x \leq y$ iff $y \in xD$, and $x < y$ iff $x \leq y$ but $x \notin yD$. If c_1 in Theorem 1 is a unit, then, with $u = c_1^{-1}$, $bu = qa^2_1 + d$. Thus the element $h = a_1(qa_1^2 + d) = ua_1b$ is such that $b < h$, in the sense of the partial ordering just indicated. However, if c_1 is not a unit, it is immediate from the Theorem that neither $h \leq b$ nor $b \leq h$ holds.

(1.2) If a, b and h are natural numbers satisfying (4), it follows from Theorem 1 that $b < h$ iff $c_1 < a_1$, where $<$ is the usual (natural) order on the set of rational integers.

(1.3) In the choice of parameters in the above algorithm, it could happen that $h = g^n$ for some $g \in D$ and natural number $n > 1$. If a,b and h are natural numbers, then $h = g^n$, $n > 1$, could happen only if $d = (a, b) > 1$. Indeed, otherwise $d = 1$ and $c | d^2$ gives that $c = 1$. But then $b = a^2 + 1$ and $ab = h = g^n$ gives that $b = A^n$ and $B^n = a$ with natural numbers A, B each greater than 1. Consequently $1 = b - a^2 = A^n - B^{2n}$, which obviously does not hold for $n > 1$.

(1.4) If D is the ring of rational integers and c_1 in the above algorithm is odd, then by Lemma 1, h is necessarily even. This in particular holds when $d = c$. On the other hand, if c_1 is even such that $ord_2(c_1) = 2\, ord_2(d)$, then h is necessarily odd. For instance with $c_1 = 4$, $a_1 = 7$, $d = 2(5) = 10$, we have $a = 70$, $c = 40$, $b = 4940$, $h = 8645$ and

$$\frac{4940}{70} = 70 + \frac{4940}{8645}.$$

Corollary 1. If c_1, a_1, q and d are natural numbers with $c_1 < a_1$ and $d^2 = qc_1$, then $a = da_1$, $b = c_1(qa_1^2 + d)$ and $h = a_1(qa_1^2 + d)$ satisfy the relation

$$\frac{b}{a} = a + \frac{b}{h} = (a.b)_h$$

Proof. By Theorem 1 and Remark (1.2),

$b = c_1 (qa_1^2 + d) < a_1 (qa_1^2 + d) = h$

satisfy (4). Thus a and b are one digit natural numbers in base h, and the result follows.

3. THE CASE $h = g^n$

In this section for a, b, h satisfying (4), we formulate necessary and sufficient conditions for $h = g^n$ for some $g \in D$ and natural number $n > 1$. We need the following Lemma.

Lemma 2. Suppose e, $f \in D\backslash\{0\}$ are relatively prime and for some $g \in D$ and natural number $n > 1$, $ef = g^n$. Then

$e = uB^n$, $f = u^{-1}A^n$ and $g = AB$

for some u, A, $B \in D$, with unit u and $(A, B) = 1$.

Proof. The assertion is obvious if both e and f are units. Hence assume, without loss of generality, that e is a non-unit. Then g is non-unit. Since D is a UFD,

$$g = wp_1^{\alpha_1} p_2^{\alpha_2} \dots p_k^{\alpha_k}$$

with primes $p_1, p_2, \dots p_k$ in D, pairwise relatively prime, $\alpha_i > 0$ and unit $w \in D$. If need be by reindexing, let $p_1, p_2, \dots, p_t$ be the only prime factors of e, up to associates. If $t = k$, then f is a unit and we can set $A = 1$ and $B = g$. Otherwise $t < k$. In any of the cases, put

$$B = wp_1^{\alpha_1} \dots p_t^{\alpha_t} \text{ and } A = p_{t+1}^{\alpha_{t+1}} \dots p_k^{\alpha_k}$$

so that $AB = g$ and $(A, B) = 1$. It follows that

$ef = B^n A^n$.

Since $(e, A) = 1 = (B, f)$, it is clear from the last equation that

$e = uB^n$ and $f = u^{-1}A^n$

for some unit $u \in D$.

3.1 The case (a, b) is an associate of b - a^2

Suppose d = (a, b) is an associate of c = b - a^2. Then c = dw for some unit w $\in$ *D*. But then

$$hc = ab = a(a^2 + c) = da_1(d^2a_1^2 + dw) \text{ implies that } hw = da_1(da_1^2 + w).$$

Therefore $g^n = h = w^{-1}da_1\,(da_1^2 + w)$. Since $(w^{-1}da_1,\ da_1^2 + w) = 1$, by Lemma 2, we get $w^{-1}da_1 = uB^n$ and $da_1^2 + w = u^{-1}A^n$ for some unit u $\in$ *D*, g = AB, A, B $\in$ *D* and (A, B) = 1. Multiplying the first equation by w and the second by d, we respectively get

$$a = da_1 = wuB^n \text{ and } b = (da_1)^2 + dw = u^{-1}dA^n \tag{6}$$

with units u, w $\in$ *D*, A, B $\in$ *D*, (A, B) = 1, dw = c = b - a^2, d = (a, b), a_1 non-unit in *D* and h = $(AB)^n$.

3.2 The case (a, b) is not an associate of b - a^2

Suppose (a, b) is not an associate of b - a^2. Then c_1 is not a unit in *D*. Recall that

$$\frac{c_1}{a_1} = \frac{c}{a} = \frac{b-a^2}{a} = \frac{b}{a} - a = \frac{b}{h} = \frac{db_1}{h} \tag{7}$$

As $b_1 - da_1^2 = c_1$ and $(a_1, b_1) = 1$, it is clear that c_1 divides d iff c_1 divides b_1. We now consider two cases.

Case 1: Suppose c_1 divides d. Then d = c_1d_1 and (7) reduces to

$$\frac{1}{a_1} = \frac{d_1b_1}{h}\ . \tag{8}$$

Since *D* is a UFD, set $d_1 = d_{11}d_{12}$ such that every prime factor of d_{11} is a factor of a_1 and $(a_1, d_{12}) = 1$. Thus (8) reduces to

$$g^n = h = (a_1d_{11})\,(d_{12}b_1).$$

Since $(a_1d_{11},\ d_{12}b_1) = 1$, by Lemma 2, we have

$$a_1d_{11} = uB^n \text{ and } d_{12}b_1 = u^{-1}A^n \tag{9}$$

with $g = AB$, $(A, B) = 1$ and unit $u \in D$. Multiplying the first equation in equ. (9) by c_1d_{12} and the second by c_1d_{11}, we respectively obtain

$$a = da_1 = c_1d_{12}uB^n \text{ and } b = u^{-1}c_1d_{11}A^n \tag{10}$$

with $(A, B) = 1$, u unit, $c_1d_{11}d_{12} = d = (a, b)$, a_1 and c_1 non-units $dc_1 = c = b - a^2$ and $h = (AB)^n$.

Case 2: Suppose c_1 does not divide d. Then, as remarked in the first paragraph of the section, c_1 does not divide b_1 too. However from (7), since $(a_1, c_1) = 1$, c_1 divides db_1. Then put $c_1=c_{11}c_{12}$ so that $b_1=c_{11}b_2$ and $d=c_{12}d_2$ with $d_2 \notin c_{11}D$ and $b_2 \notin c_{12}D$. Hence (7) reduces to

$$\frac{1}{a_1} = \frac{d_2b_2}{h} . \tag{11}$$

Since b_2 is a factor of b_1 and $(a_1, b_1)=1$, we have $(b_2, a_1)=1$. Express $d_2=d_{21}d_{22}$ such that every prime factor of d_{21} is a factor of a_1 and $(d_{22}, a_1)=1$. Thus equ. (11) is equivalent to $g^n = h = (a_1d_{21})(b_2d_{22})$ with $(a_1d_{21}, b_2d_{22}) = 1$. By Lemma 2, we have

$$a_1d_{21} = uB^n \text{ and } d_{22}b_2 = u^{-1}A^n \tag{12}$$

for some u, A, $B \in D$ unit u, $g = AB$ and $(A, B) = 1$. Multiplying the first in equ. (12) by $c_{12}d_{22}$ and the second by $c_{11}c_{12}d_{21}$, we respectively obtain
$a_1c_{12}(d_{21}d_{22}) = c_{12}d_{22}\, uB^n$ and $(b_2c_{11})(d_{21}d_{22})c_{12} = (c_{11}c_{12})d_{21}u^{-1}A^n$.

But $d_{21}d_{22} = d_2$, $b_2c_{11} = b_1$, $d_2c_{12} = d$, $da_1 = a$, $db_1 = b$ and $c_{11}c_{12} = c_1$. Thus the above equations respectively reduce to

$$a = c_{12}d_{22}\, uB^n \text{ and } b = c_1d_{21}\, u^{-1}A^n \tag{13}$$

for some unit u, $(a, b)=d=c_{12}d_{21}d_{22}$, $c_1=c_{11}c_{12}$, $d_2=d_{21}d_{22}\notin c_{11}D$, $\dfrac{b}{dc_{11}} = b_2 \notin c_{12}D$ and $b-a^2=c=dc_1$. Thus we have almost justified the following.

Theorem 2. Suppose a, b, h $\in D$ satisfy (4). Then $h = g^n$ for some $g \in D$ and natural number $n > 1$ iff the following respective cases hold with $g = AB$, $(A, B) = 1$, $A, B \in D$.

(2.1) When c is an associate of d, a and b are given by the relations in (6) satisfying the condition $c = b - a^2$.

(2.2) When $d \in c_1D$, c_1 non-unit, a and b are given by the relations in (10) satisfying the condition $c = b - a^2$.

(2.3) When $d \notin c_1D$, a and b are given by the relations in (13) satisfying the condition $c = b - a^2$.

Proof. The forward implications have already been shown in the subsections (3.1), (3.2) case 1 and case 2, respectively.

The converses are immediate. For instance, if a and b satisfy (6) with $c=b-a^2$, then

$$\frac{b}{a} - a = \frac{u^{-1}dA^n - w^2u^2B^{2n}}{a} = \frac{c}{uwB^n} = \frac{(cw^{-1})u^{-1}A^n}{B^nA^n} = \frac{du^{-1}A^n}{h} = \frac{b}{h}.$$

The other cases are similarly shown.

Of course the equations arising from (6), (10) and (13) with the condition $c=b-a^2$ are not manageable in their generalities. One may set some additional conditions on the parameters.

One such condition is to set $u = w = 1$, which necessarily holds if a, b and h are natural numbers. So, if we set $u = w = 1$, the equations in (6), (10) and (13), respectively reduce to

$$a = da_1 = B^n \text{ and } b = (da_1)^2 + d = dA^n \tag{14}$$

$$a = da_1 = c_1d_{12}B^n \text{ and } b = d_{11}c_1A^n \tag{15}$$

and

$$a = da_1 = c_{12}d_{22}B^n \text{ and } b = c_1d_{21}A^n \quad (16)$$

From the pair of equations in (14), we get

$$A^n - a_1B^n = 1 \quad (17)$$

with $B^n \in a_1D$, which we shall use in the next section to exhibit solutions for (1).

From the pair of equations in (15) and parameters defined under case 1 in (3.2), $c = b - a^2$ gives

$$d = c_1d_{11}d_{12} = d_{11}A^n - c_1d_{12}^2 B^{2n} \quad (18)$$

and from the pair of equations in (16) and parameters defined under case 2 in (3.2), $c = b - a^2$ gives

$$(c_{11}c_{12})d_2 = c_{11}c_{12}d_{21}d_{22} = c_{11}d_{21}A^n - c_{12}d_{22}^2 B^{2n} \quad (19)$$

Lemma 3. Consider the equation $fe = k_1k_2A^n - fB^{2n}$ for any natural number n and indeterminates A, B, e, f, k_1, k_2 over a commutative ring **R** with unity. For any e, B $\in$ **R**, the equation has a solution in **R**.

Proof. Choose any two elements e, B $\in$ **R** and natural number n. Put

$$B^{2n} + e = k_1A^m$$

for some k_1, A $\in$ **R** and natural number $m \leq n$. Then with $f = k_2 A^{n-m}$ for some $k_2 \in$ **R**, we have

$$k_1k_2A^n = (k_1A^m)(k_2A^{n-m}) = (B^{2n} + e)f = fB^{2n} + ef.$$

Thus

$$k_1k_2A^n - fB^{2n} = ef,$$

completing the proof of the Lemma.

Corollary 2

(2.1) Suppose $d_{11} = d_{12} = 1$ in (18). Then $d = c_1$ and the equation has a solution for any $B \in D$ and natural number n. In particular, for any non-unit $B \in D\backslash\{0\}$ and natural number $n > 1$ such that $B^{2n} + 1 = A^m$ for some $A \in D$ and natural number $m < n$,

$a = B^n A^{n-m}$, $b = A^{2n-m}$ and $h = (AB)^n$ satisfy (4).

(2.2) Suppose $d_{21} = d_{22} = 1$ and $c_{12} = qc_{11}$ for some $q \in D\backslash\{0\}$ in (19). Then the equation has a solution for any c_{11}, $B \in D$ and natural number n. In particular, for any non-units c_{11}, $B \in D\backslash\{0\}$, $(c_{11}, B)=1$ and natural number $n > 1$ such that $B^{2n} + c_{11} = A^m$ for some $A \in D$ and natural number $m < n$,

$a = A^{n-m}B^n(A^m-B^{2n})$, $b = A^{2n-m}(A^m-B^{2n})^2$ and $h = (AB)^n$ satisfy (4).

Proof. If $d_{11} = d_{12} = 1$ in (18), then $d_1 = d_{11}d_{12} = 1$, $d = c_1d_1 = c_1$. The equation may be written as

$$A^n - dB^{2n} = d \tag{20}$$

With $f = d$, $e = 1 = k_1 = k_2$ in Lemma 3, for arbitrary natural number n and $B \in D$, (20) has a solution in D.

If we choose a non-unit $B \in D\backslash\{0\}$ and $n > 1$ such that $B^{2n} + 1 = A^m$ for some $A \in D$ and natural number $m < n$, which is always a possible construction, and set $d = A^{n-m}$, then $A^n - dB^{2n} = d$. For these choices, using (10), we have

$$\begin{aligned} c &= dc_1 = d^2 = A^{2(n-m)} \\ a &= da_1 = c_1d_{12}uB^n = A^{n-m}B^n \\ b &= c_1d_{11}u^{-1}A^n = A^{n-m}A^n = A^{2n-m} \\ h &= (AB)^n \end{aligned} \tag{21}$$

with $(A, B) = 1$. Hence a, b, h given by (21) satisfy (4).

To prove the second part, suppose $d_{21} = 1 = d_{22}$ and $c_{12} = qc_{11}$ for some $q \in D$ holds in (19). Then the equation reduces to

$$qc_{11} = A^n - qB^{2n} \tag{22}$$

Then, with $e = c_{11}$, $f = q$, $k_1 = k_2 = 1$ in Lemma 3, the last equation has a solution for any c_{11}, $B \in D$ and natural number n.

If we choose non-units $c_{11}, B \in D\backslash\{0\}$, $(c_{11}, B) = 1$ and natural number $n > 1$ such that $B^{2n} + c_{11} = A^m$, $A \in D$ and natural number $m < n$, again always possible, and set $q = A^{n-m}$, then (22) holds. For these choices, using (13),

$$\begin{aligned} c_{12} &= qc_{11} = A^{n-m}(A^m - B^{2n}) \\ c_1 &= c_{11}c_{12} = qc_{11}^2 = A^{n-m}(A^m - B^{2n})^2 \\ a &= c_{12}d_{22}uB^n = A^{n-m}(A^m - B^{2n})B^n \\ b &= c_1d_{21}u^{-1}A^n = A^{2n-m}(A^m - B^{2n})^2 \\ h &= (AB)^n \end{aligned} \tag{23}$$

with $(A, B) = 1$. Hence a, b and h given by (23) satisfy the relation in (4). This completes the proof of the Corollary.

4. APPLICATION

One may impose additional conditions on the parameters in (17), (18) or (19). Some of the resulting equations arising from such a process are challenging Diophantine equations. This will be elaborated more in a forthcoming article. In this section, we demonstrate two different situations.

Assume, henceforth, that a, b and h are natural numbers satisfying (2) and $h = g^n$ for some natural number $g > 1$ and $n > 1$. In view of Remark (1.3), we necessarily have $(a, b) = d > 1$. Moreover, $u = 1 = w$ necessarily holds. Consequently the equations in (6), (10) and (13) with $c = b - a^2$ are respectively equivalent to the equations in (17), (18) and (19). Besides, it is clear that the parameters in these equations represent natural numbers.

Corollary 3. For natural numbers a, b, h as above, suppose in (18), $d_{11} = d_{12} = 1$ and $c_1 = A^\beta$ for some natural number β. Then $\beta = n - 1$ and $A = B^{2n} + 1$.

Proof. Since $d_{11} = d_{12} = 1$ and $c_1 = A^{\beta}$ the equation in (18) reduces to

$$A^{n-\beta} - B^{2n} = 1 \qquad (24)$$

If $n - \beta > 1$, it is evident that $n - \beta$ is odd, hence $n - \beta \geq 3$. Moreover B is even, hence $B \geq 2$. But then a result of V.A. Lebesgue (see Mordell, 1969) shows that the Diophantine equation in (24) has no solution. Hence $n - \beta \leq 1$. Therefore $n = \beta + 1$ and $A = B^{2n} + 1$.

Under the assumptions of Corollary 3, since $c_1 = d$, equation (10) yields $a=A^{n-1}B^n$, $b = A^{2n-1}$, $c = dc_1 = A^{2(n-1)}$ and $h = (AB)^n$ satisfying (2). However, since $B^n < B^{2n} + 1 = A \leq A^{n-1}$, it is clear that $h < b$, for this case.

Remark 2. It was indicated in Remark (1.2) that for natural numbers satisfying (2), $b < h$ iff $c_1 < a_1$. Specially when (a, b) and $b - a^2$ are associates, hence $d = c$, we have $c_1 = 1$ and thus $c_1 = 1 < 2 \leq a_1$. Thus all the solutions of (2) under this case satisfy the relation $c < a < b < h$. Notice that these solutions are related by the equations in (6) which are equivalent to (17) with the condition $c = b-a^2$.

For natural numbers a, b, h satisfying (2), consider the equation in (17)

$$\text{i.e } A^n - a_1B^n = 1 \qquad (25)$$

with $B^n \equiv 0 \pmod{a_1}$. In particular $2 \leq a_1 \leq B^n$. Hence positive solutions of (25) satisfy $c_1 = 1 < a_1 \leq B^n$.

First of all the condition $B^n \equiv 0 \pmod{a_1}$ in (25) is quite restrictive. For instance $(x,y) = (3,2)$ is a solution of

$$x^4 - 5y^4 = 1 \qquad (26)$$

with $5 < 2^4$. However the equation has no solution in natural numbers x and y with $y \equiv 0 \pmod 5$. Indeed, Cohn (1965) has shown that $(x, y) = (\pm 1, 0)$, $(\pm 3, \pm 2)$ are the only solutions of the Diophantine equation in (26).

Assume now that a_1 in (25) is not a perfect square and $n = 2$. Adopting standard techniques to construct the set of units of the ring of integers with norm 1 in the number field $Q(\sqrt{a_1})$ (Borevich and Shafarevich, 1966), or using the explicit result in Carmichael (1959) the Diophantine equation

$$x^2 - a_1y^2 = 1 \tag{27}$$

has a solution with natural numbers x and y.

Let $\alpha + \beta\sqrt{a_1}$ be such a solution of (27). For each natural number m, let $\alpha_m + \beta_m\sqrt{a_1} = (\alpha + \beta\sqrt{a_1})^{ma_1}$ with natural numbers α_m and β_m. Evidently β_m is a multiple of a_1 and hence $a_1 \leq \beta_m$. Besides $(x, y) = (\alpha_m, \beta_m)$ satisfies the equation in (25) with n = 2.

For each natural number m, let

$$d = c = \beta_m^2/a_1 > 1 .$$

Then

$$a = da_1 = \beta_m^2, \quad b_1 = da_1^2 + 1 = \beta_m^2 a_1 + 1 = \alpha_m^2 .$$

Consequently, the natural numbers

$$a = \beta_m^2, \quad b = db_1 = \beta_m^4 + c, \quad h = (\alpha_m \beta_m)^2$$

satisfy the relation

$$\frac{b}{a} = a + \frac{b}{h} .$$

In fact, since $a_1 \leq \beta_m < b < (\alpha_m \beta_m)^2 = g^2 = h$.

Therefore,

$$b = b_1 g + b_0,$$

with $0 < b_0, b_1 < g$. Hence

$$\frac{b}{a} = (a.b_1b_0)_g = (a.b)_g \ .$$

The following Theorem is now immediate from the above illustration, thus giving a stronger positive answer to (1).

Theorem 3. There are infinitely many triples of natural numbers (b,a,g) with (a, b) = $b\text{-}a^2 > 1$, b not a multiple of a, and $h = g^2$, g a natural number, $g < b < g^2$ such that

$$a + \frac{b}{h} = \frac{b}{a} = (a.b)_g \ .$$

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