SINET: Ethiop. J. Sci., 23(2):181-196, 2000

 ISSN: 0379-2897

CONSISTENCY OF s^2 IN THE LINEAR REGRESSION MODEL WHEN THE DISTURBANCES ARE SPATIALLY CORRELATED

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ABSTRACT: Conditions for the consistency of the estimator S^2 of the variance of the disturbance σ_u^2 under first-order spatial error processes are given.

Key words/ phrases: Consistency, ordinary least squares estimator, spatial correlation, spatial error process

INTRODUCTION

Consider the linear regression model for spatial correlation

$$y = X\beta + u, \qquad u = C\epsilon, \tag{1}$$

where y is a $T \times 1$ observable random vector, X is a $T \times k$ matrix of known constants with full column rank k, ß is a $k \times 1$ vector of unknown parameters, ϵ is a $T \times 1$ random vector with expectation zero and covariance matrix $Cov(\epsilon) = \sigma_\epsilon^2 I$ (I is the T-dimensional identity matrix and σ_ϵ^2 an unknown positive scalar). C denotes a $T \times T$ nonsingular matrix such that the product CC' has identical diagonal elements. The assumption that the diagonal elements of the matrix product CC' are all identical indicates that we consider only homoscidastic disturbances u_i's which are correlated.

The ordinary least squares (OLS) estimator of the unknown parameter ß in model (1) is given by $\hat{\beta} = (X'X)^{-1}X'y$ with the covariance matrix $Cov(\hat{\beta}) = \sigma_\epsilon^2 (X'X)^{-1}X'V_* X(X'X)^{-1}$, where $V_* = CC'$.

The OLS based estimator $s^2 = (y-X\hat{\beta})'(y-X\hat{\beta})/(T-k)$ of the disturbance variance, in the linear regression model with correlated disturbances, is biased and not consistent in general (see Dhrymes 1978, Chapter 3). This means that when the disturbances are correlated, the standard procedures for testing hypothesis and constructing confidence intervals with respect to the regression coefficients lead to incorrect conclusions.

Several papers investigate the behaviour of the bias of s^2 under different correlation structures (Martin, 1974; Neudecker, 1977; 1978; Dufour, 1986; 1988; Krämer, 1991; Kiviet and Krämer, 1992; Fiebig *et al.*, 1992; Song, 1994). In contrast, there are very few published studies on the problem concerning the consistency of the variance estimator in the presence of correlation. Based on the sample variance of the disturbances, Krämer and Berghoff (1991) give a simple sufficient condition for the consistency of s^2. Baltagi and Krämer (1994) deal with the consistency of the estimator in the linear regression model with error component disturbances.

The present paper provides conditions for the consistency of the estimator s^2 when the disturbances follow a first-order spatial error process.

CONSISTENCY OF s^2

Spatial dependence among the disturbance terms can be expressed in a number of ways. In general, an autoregressive or a moving average formulation could be used as is frequently done in time series analysis.

Let the components of u follow a stationary first-order spatial autoregressive (AR(1)) process

$$u_i = \rho \sum_{j=1}^{T} w_{ij} u_j + \epsilon_i$$

or, in matrix form

$$u = \rho W u + \epsilon, \tag{2}$$

where ρ denotes a spatial correlation coefficient for a given area partitioned into T nonoverlapping regions R_i, $i = 1, \ldots, T$. W is a weight matrix with known nonnegative weights defined by Cliff and Ord (1981, pp. 17–19)

$$w_{ij}\begin{cases} >0, & if\ regions\ R_i\ and\ R_j\ are\ neighbours\ (i \neq j) \\ =0, & otherwise. \end{cases}$$

The element w_{ij} of the weight matrix shows the strength of the effect of region R_j on region R_i.

When the components of u are of the pattern

$$u_i = \rho \sum_{j=1}^{T} w_{ij}\epsilon_j + \epsilon_i$$

or, in matrix form

$$u = \rho W\epsilon + \epsilon, \tag{3}$$

then we have another scheme which is known as first-order spatial moving average (MA(1)) process.

Using the specification in (1) equations (2) and (3) can be written as

$$u = (I - \rho W)^{-1}\epsilon \quad \text{and} \quad u = (I + \rho W)\epsilon, \tag{4}$$

respectively, where in AR(1) case the matrix $I - \rho W$ must be nonsingular. From (1) and (4), we get four possible structures of $Cov(u) = \sigma_\epsilon^2 CC' = \sigma_\epsilon^2 V_*$ for a first-order spatial error process:

$$V_* = \begin{cases} (I + \rho W)(I + \rho W') & : MA(1) \\ (I + \rho W) & : MA(1) - conditional \\ (I - \rho W)^{-1}(I - \rho W')^{-1} & : AR(1) \\ (I - \rho W)^{-1} & : AR(1) - conditional. \end{cases} \tag{5}$$

The conditional cases are special cases of the unconditional process (see Bartlett, 1971; Besag, 1974). Note that the possible values of ρ must be identified to ensure that V_* is positive definite (see Horn and Johnson, 1985, p. 301).

According to the assumptions given in model (1) the matrix V_* has identical diagonal elements, and denoting this element by υ, the covariance of u can be expressed as

$$Cov(u) = \sigma_\epsilon^2 V_* = (\upsilon \sigma_\epsilon^2) V = \sigma_u^2 V, \tag{6}$$

where $V = (1/\upsilon)V_*$ and $\sigma_u^2 = \upsilon \sigma_\epsilon^2$ is the variance of the disturbances u_i, $i = 1, \dots, T$. Using the above assumptions under spatial process we can now write model (1) as the general linear regression model:

$$y = X\beta + u, \quad E(u) = 0, \quad Cov(u) = \sigma_u^2 V \tag{7}$$

Let $\mu_i(A)$ be the i-th eigenvalue of a square matrix A, and let $\xrightarrow{p}$ and $\xrightarrow{q.M.}$ denote convergence in probability and in quadratic mean, respectively. Under the assumptions of model (7) Krämer and Berghoff (1991) state that the OLS based estimator $S^2 = (T-k)s^2/T$ of σ_u^2 is weakly consistent if

$$\frac{u'u}{T} \xrightarrow{p} \sigma_u^2 \quad \text{and} \quad \mu_{max}(V) = o(T) \tag{8}$$

where $\mu_{max}(V)$ denotes the maximum eigenvalue of V. In other words, S^2 is weakly consistent if the sample variance of the true disturbances is consistent, and $\mu_{max}(V)/T \longrightarrow 0$ as $T \longrightarrow \infty$.

Whether the above result is operational under spatial error process, depends on the form of the error process and the weight matrix W. Note that the consistency of s^2 is implied by that of S^2 because $(T\text{-}k)/T$ converges to one as T goes to infinity.

In the following, conditions that guarantee the consistency of S^2 in the presence of spatial correlation will be given. For this purpose, the following results are needed.

Definition

An interval (ρ_l, ρ_u), $\rho_l, \rho_u \in [-1, 1]$, where $\rho_l \leq \rho_u$ for a real valued function $f: (\rho_l, \rho_u) \longrightarrow \mathbb{R}$ is said to be suitable if

$$\lim_{(\rho,T)\rightarrow(\rho_u,\infty)} \frac{f(\rho)}{T} = \lim_{(\rho,T)\rightarrow(\rho_l,\infty)} \frac{f(\rho)}{T} = 0, \tag{9}$$

that is, for $\rho \longrightarrow \rho_l$ or $\rho \longrightarrow \rho_u$ we have $f(\rho) = o(T)$. ◇

In this paper, we focus on the positive values of ρ, so the suitable interval in the above definition becomes (ρ_l, ρ_u) with $\rho_l, \rho_u \in (0,1]$. Further, suitable intervals will be assumed in calculating the eigenvalues of the matrix V as a function of ρ.

Lemma 1

Suppose that the weight matrix W is symmetric with row sums equal to unity, and let $V = (1/\upsilon)V_*$, where V_* is as given in (5) with diagonal elements all equal to υ. Then $\mu_{max}(V) = o(T)$ for values of ρ from a suitable interval (ρ_l, ρ_u), $\rho_l > 0$.

Proof:

The asserted result will be proved for $MA(1)$ and conditional $AR(1)$ cases given in (5). Similar arguments can be used for the proofs of $AR(1)$ and conditional $MA(1)$ cases.

Under first-order spatial moving average process the matrix V is given by $V = (1/\upsilon)(I + \rho W)(I + \rho W')$. Using the assumption that the matrix W is symmetric we can express the eigenvalues of V in terms of the eigenvalues of W as

$$\mu_i(V) = \frac{1}{\upsilon}(1 + \rho\,\mu_i(W))^2, \qquad \upsilon, \rho > 0.$$

Denoting the largest eigenvalue of the weight matrix W by $\mu_{max}(W)$, and assuming that the eigenvalues of W and V are in ascending order of size for positive values of ρ we have

$$\mu_i(V) \leq \frac{1}{\upsilon}(1+\rho\,\mu_{max}(W))^2.$$

If the row sums of W are all equal to one, then the absolute value of $\mu_i(W)$ is less than or equal to one for all i (see Graybill, 1983, p. 98). This implies that $\mu_{max}(W) \leq 1$ and

$$\mu_i(V) \leq \frac{1}{\upsilon}(1+\rho)^2, \qquad \rho>0.$$

From this we get $\mu_{max}(V) = o(T)$.

For the conditional $AR(1)$ case, the matrix V is given as $V = (\upsilon(I-\rho W))^{-1}$, and

$$\mu_i(V) = \frac{1}{\upsilon(1-\rho\,\mu_i(W))}.$$

Analogous to the $MA(1)$ case we get, for positive values of ρ,

$$\mu_i(V) \leq \frac{1}{\upsilon(1-\rho)}. \tag{10}$$

Using (10) we obtain $\mu_{max}(V) = o(T)$. ◇

Lemma 2

Assume that the weight matrix W is symmetric with row sums equal to unity. When the components of u follow a first-order spatial $MA(1)$ or $AR(1)$ process, then for values of ρ from a suitable interval (ρ_l, ρ_u), $\rho_l > 0$,

$$\frac{u' P_X u}{T} \xrightarrow{P} 0,$$

where $P_X = X(X'X)^{-1}X'$.

Proof:

Let $tr(A)$ denote the trace of a square matrix A. For the expectation of $u'P_X u/T$ we have (see *e.g.*, Magnus and Neudecker, 1988, p. 247)

$$E(\frac{u'P_X u}{T}) = \frac{1}{T}\left(tr\left(P_X Cov(u)\right) + E(u)'P_X E(u)\right) = \frac{\sigma_u^2}{T} tr(P_X V) . \tag{11}$$

The trace of the matrix product $P_x V$ can be expressed as

$$tr(P_X V) = tr(Z'VZ) = \sum_{i=1}^{k} \mu_i(Z'VZ),$$

where $Z = X(X'X)^{-1/2}$. This implies

$$E(\frac{u'P_X u}{T}) = \frac{\sigma_u^2}{T}\sum_{i=1}^{k} \mu_i(Z'VZ).$$

From Poincaré separation theorem (see Horn and Johnson, 1985, p. 190) it follows that all eigenvalues of $Z'VZ$ are less than or equal to $\mu_{max}(V)$. Using this fact gives

$$E(\frac{u'P_X u}{T}) \leq \frac{\sigma_u^2}{T} k \mu_{max}(V) . \tag{12}$$

By applying Lemma 1 we get $\mu_{max}(V) = o(T)$, and from (12) it is clear that

$$E(\frac{u'P_X u}{T}) \longrightarrow 0 \qquad (T \longrightarrow \infty) .$$

Since P_X is symmetric and idempotent, $u'P_X u \geq 0$. Furthermore, for $\epsilon^* > 0$ we have (see Davidson, 1994, p. 132: Markov-Inequality)

$$P(\frac{u'P_X u}{T} > \epsilon^*) \leq E(\frac{u'P_X u}{\epsilon^* T}) \longrightarrow 0 \qquad (T \longrightarrow \infty) .$$

This means, by definition, $(u'P_X u)/T \xrightarrow{p} 0$. ◇

Now, given model (1), suppose that the error vector ϵ has the following finite moments:

$$E(\epsilon \otimes \epsilon\epsilon') = \Phi \quad \text{and} \quad E(\epsilon\epsilon' \otimes \epsilon\epsilon') = \psi, \tag{13}$$

where $\otimes$ denotes the Kronecker-product.

The following theorem provides a sufficient condition for the consistency of S^2 under first-order spatial error processes that can be verified in practice. In what follows let $C_{i.}$ denote the i-th row of the matrix C in model (1).

Theorem 1

Let the weight matrix W be symmetric with row sums equal to unity. Suppose that the components of ϵ in model (1) are independent and identically distributed, and the components of u follow a first-order spatial AR or MA process. Then S^2 is weakly consistent for σ_u^2 if for positive values of ρ from a suitable interval (ρ_l, ρ_u), $\rho_l > 0$, and two neighbouring regions R_i and R_j

$$tr(C_{i.}'C_{j.}) = o(T) . \tag{14}$$

__Proof:__

The OLS based estimator S^2 can be expressed as

$$S^2 = \frac{u'M_X u}{T} = \frac{u'u}{T} - \frac{u'P_X u}{T} .$$

From Lemma 2 we have

$$\frac{u'P_X u}{T} \xrightarrow{p} 0$$

and therefore it suffices to show, under condition (14), that

$$\frac{u'u}{T} \xrightarrow{p} \sigma_u^2 .$$

The proof of the theorem is apparent if, for $T \longrightarrow \infty$, we can show

$$E\left(\frac{u'u}{T}\right) \longrightarrow \sigma_u^2 \text{ and } Var\left(\frac{u'u}{T}\right) \longrightarrow 0 \; . \tag{15}$$

For the disturbance vector $u = C\epsilon$, as defined in (1), the following holds:

$$E(u'u) = E(\epsilon' C' C \epsilon) = tr(C' C \sigma_\epsilon^2 I) = \sigma_\epsilon^2 tr(CC').$$

Since the matrix $V_* = CC'$ has diagonal elements which are all equal to υ,

$$E(u'u) = \sigma_\epsilon^2 tr(V_*) = \sigma_\epsilon^2 T \upsilon \; ,$$

and from the expression $Cov(u) = \sigma_u^2 V = \sigma_\epsilon^2 V_* = \upsilon \sigma_\epsilon^2 V$, it follows that

$$E\left(\frac{u'u}{T}\right) = \upsilon \sigma_\epsilon^2 = \sigma_u^2 \; ,$$

showing the first part of (15). Now, to prove the second part of (15) which states

$$Var\left(\frac{u'u}{T}\right) = E\left(\frac{u'u}{T}\right)^2 - (\sigma_u^2)^2 \longrightarrow 0 \qquad (T \longrightarrow \infty) \; ,$$

it suffices to show that $E((u'u)/T)^2$ converges to $(\sigma_u^2)^2$. In order to get the result we will first consider $E(u'u)^2$. Since W is symmetric, we obtain $C = C'$ implying $u'u = \epsilon' CC' \epsilon = \epsilon' V_* \epsilon$, and

$$E(u'u)^2 = E(\epsilon' V_* \epsilon \epsilon' V_* \epsilon) \; . \tag{16}$$

Using the result of Rao and Kleffe (1988, p. 32) we get

$$\begin{aligned} E(\epsilon' V_* \epsilon \epsilon' V_* \epsilon) &= E(tr(V_* \epsilon \epsilon' V_* \epsilon \epsilon')) \\ &= tr((V_* \otimes V_*) \psi) \; , \end{aligned} \tag{17}$$

where $\psi = E(\epsilon\epsilon' \otimes \epsilon\epsilon')$.

When the components of ϵ are independent and identically distributed, then

$$E(\epsilon_i \epsilon_j \epsilon_{i*}) = \begin{cases} \varphi^*, & i = j = i^* \\ 0, & otherwise \end{cases}$$

and

$$E(\epsilon_i \epsilon_j \epsilon_{i*} \epsilon_{j*}) = \begin{cases} (\sigma_\epsilon^2)^2, & \textit{pairwise equal} \\ \varphi, & i=j=i^*=j^* \\ 0, & \textit{otherwise.} \end{cases}$$

where $\varphi^* = E(\epsilon_i)^3$ and $\varphi = E(\epsilon_i)^4$.

Let ψ_{ij} be a $T \times T$ symmetric matrix with elements

$$\psi_{ij}(i^*, l) = \psi_{ij}(l, i^*) = \begin{cases} \sigma_\epsilon^4, & i=l, j=i^* \\ 0, & \textit{otherwise.} \end{cases}$$

Further, let $\psi_1, \psi_2, \ldots, \psi_T$ be $T \times T$ diagonal matrices with diagonal elements equal to φ or σ_ϵ^4 such that

$$\psi_j(i,i) = \begin{cases} \varphi, . & i=j \\ \sigma_\epsilon^4, & \textit{otherwise.} \end{cases}$$

For the expectation of the Kronecker-product ψ we obtain

$$\psi = \begin{pmatrix} \psi_1 & \psi_{12} & \cdots & \psi_{1T} \\ \psi_{21} & \psi_2 & \cdots & \psi_{2T} \\ \vdots & \vdots & \ddots & \vdots \\ \psi_{T1} & \cdots & \psi_{TT-1} & \psi_T \end{pmatrix}$$

This matrix can be split into

$$\psi = \sigma_\epsilon^4 I_{T^2} + (\varphi - \sigma_\epsilon^4) I^* + \sigma_\epsilon^4 \psi^*, \tag{18}$$

where I_{T^2} denotes the $T^2 \times T^2$ identity matrix. I^* and ψ^* denote $T^2 \times T^2$ matrices given as

$$I^*_{ij} = \begin{cases} 1, & i = j = (i^*-1)T + i^*,\ i^* = 1,\ldots,T \\ 0, & otherwise. \end{cases} \tag{19}$$

$$\psi^* = \begin{pmatrix} \psi_0 & \psi^*_{12} & \cdots & \psi^*_{1T} \\ \psi^*_{21} & \psi_0 & \cdots & \psi^*_{2T} \\ \vdots & \vdots & \ddots & \vdots \\ \psi^*_{T1} & \cdots & \psi^*_{TT-1} & \psi_0 \end{pmatrix}$$

with $\psi_0 := \psi^*_{ii} = O_{T\times T}$, where $O_{T\times T}$ denotes the $T\times T$ matrix whose elements are all equal to zero. The $T\times T$ matrix ψ^*_{ij} is given by

$$\psi^*_{ij}(i^*, l) = \psi_{ij}(l, i^*) = \begin{cases} 1, & i = l,\ j = i^* \\ 0, & otherwise. \end{cases}$$

and is symmetric according to the definition.

From (16), (17) and (18), we get

$$\begin{aligned} E\,(u'u)^2 &= tr((V_*\otimes V_*)\psi) \\ &= tr((V_*\otimes V_*)(\sigma^4_\epsilon I_{T^2} + (\varphi - \sigma^4_\epsilon)I^* + \sigma^4_\epsilon \psi^*)) \\ &= tr((V_*\otimes V_*)\sigma^4_\epsilon I_{T^2}) + tr((V_*\otimes V_*)(\varphi - \sigma^4_\epsilon)I^*) \\ &\quad + tr((V_*\otimes V_*)\sigma^4_\epsilon \psi^*). \end{aligned} \tag{20}$$

The first term of the right hand side of equation (20) can be expressed as

$$tr((V_*\otimes V_*)\sigma^4_\epsilon I_{T^2}) = \sigma^4_\epsilon tr((V_*\otimes V_*) = \sigma^4_\epsilon tr(V_*)tr(V_*) = \sigma^4_\epsilon \upsilon^2 T^2, \tag{21}$$

because $tr(V_*) = \upsilon T$ (see Magnus and Neudecker, 1988, p. 28).

By the assumption in model (1) all diagonal elements of $V_*\otimes V_*$ are equal to υ^2, and the matrix I^* has exactly T diagonal elements which are equal to unity (zero otherwise). Thus for the second term we have

$$tr((V_*\otimes V_*)(\varphi - \sigma^4_\epsilon)I^*) = (\varphi - \sigma^4_\epsilon)T\upsilon^2\ . \tag{22}$$

Since V_* is symmetric, we can write the third term as (see Magnus and Neudecker, 1988, p. 30)

$$\sigma_\epsilon^4 tr((V_* \otimes V_*)\psi^*) = \sigma_\epsilon^4 (vec(V_* \otimes V_*))' vec(\psi^*) . \tag{23}$$

Suppose a_j stands for the j-th column of an $m \times n$ matrix A. Then $vec(A)$ is a vector of length mn with a_1 as its first m elements, a_2 its second m elements and so on.

For R_i and R_j being neighbours, $E(u_i u_j) = \sigma_\epsilon^2 V_*(i,j)$, and by successive calculation we get

$$\sigma_\epsilon^4 (vec(V_* \otimes V_*))' vec(\psi^*) = 2 \sum_i^T \sum_j^{gi} (E(u_i u_j))^2 , \tag{24}$$

where g_i denotes the number of neighbours for the i-th region R_i. Furthermore, $u_i = C_{i.} \epsilon = \epsilon' C_{i.}'$ and $u_j = C_{j.} \epsilon$. From this we obtain

$$E(u_i u_j) = E(\epsilon' C_{i.}' C_{j.} \epsilon) = E(tr(C_{i.}' C_{j.} \epsilon \epsilon')) = \sigma_\epsilon^2 tr(C_{i.}' C_{j.}) . \tag{25}$$

From (23), (24) and (25) follows

$$\sigma_\epsilon^4 tr((V_* \otimes V_*)\psi^*) = 2 \sigma_\epsilon^4 \sum_i^T \sum_j^{gi} (tr(C_{i.}' C_{j.}))^2 . \tag{26}$$

Using equations (20) to (22) and (26) for values of ρ from a suitable interval (ρ_l, ρ_u) we obtain

$$\lim_{T \to \infty} E(\frac{u'u}{T})^2 = \lim_{T \to \infty} \sigma_\epsilon^4 \upsilon^2 + \lim_{T \to \infty} (\varphi - \sigma_\epsilon^4) \frac{\upsilon^2}{T} + \lim_{T \to \infty} \frac{2\sigma_\epsilon^4}{T^2} \sum_i^T \sum_j^{gi} (tr(C_{i.}' C_{j.}))^2$$

$$= \sigma_\epsilon^4 \upsilon^2 = (\sigma_u^2)^2 . \tag{27}$$

The last expression holds because of the assumption $tr(C'_{i.}\ C_{j.}) = o(T)$. ◇

Example

Let the elements of the weight matrix W be of the form

$$\begin{cases} w_{1,T} = w_{2,T-1} = w_{i,T-i+1} = 1, & i = 3,4,\dots,T \\ w_{i,j} = 0, & otherwise. \end{cases} \tag{28}$$

Furthermore, let the components of ϵ in model (1) be independent and identically distributed. If the components of u follow a first-order spatial MA process, then S^2 is weakly consistent for σ_u^2.

This can be proved by showing that, for ρ from a suitable interval (ρ_l, ρ_u), condition (14) is fulfilled. Under a spatial MA(1) process we have $V_* = (I+\rho W)(I+\rho W')$, and this means $C = I + \rho W$. If the weight matrix W is of the form (28), then C is symmetric, and the regions R_i and R_j with $j = T\text{-}i+1$ are neighbours. Denoting a T-dimensional vector whose i-th element is equal to unity (zero otherwise) by $\vec{e}_i$, we get

$$C'_{i.} = \vec{e}_i + \rho\vec{e}_j .$$

Using this yields

$$C_{i.}C_{j.} = (\vec{e}_i + \rho\vec{e}_j)((\vec{e}_j)' + \rho(\vec{e}_i)') ,$$

implying

$$\begin{aligned} tr(C_{i.}C_{j.}) &= tr(\vec{e}_i(\vec{e}_j)') + tr(\rho\vec{e}_j(\vec{e}_j)') + tr(\rho\vec{e}_i(\vec{e}_i)') + tr(\rho^2\vec{e}_j(\vec{e}_i)') \\ &= 2\rho, \end{aligned} \tag{29}$$

because $tr(\vec{e}_i(\vec{e}_j)') = 0$ and $tr(\vec{e}_i(\vec{e}_i)') = tr(\vec{e}_j(\vec{e}_j)') = 1$. From (29) it is clear that, for ρ from a suitable interval (ρ_l, ρ_u), $tr(C_{i.}\ C_{j.}) = o(T)$, and the weak consistency of S^2 for σ_u^2 follows from Theorem 1. ◇

The next result gives necessary and sufficient condition for the consistency of S^2 under first-order spatial error process.

Theorem 2

Let the weight matrix W be symmetric with row sums equal to unity, and suppose that the components of u follow a first-order spatial MA or AR process. Then S^2 is weakly consistent for σ_u^2 if and only if, for values of ρ from a suitable interval (ρ_l, ρ_u), $\rho_l > 0$,

$$\frac{u'u}{T} \xrightarrow{p} \sigma_u^2. \tag{30}$$

Proof:
(*sufficiency*)

Consider the OLS based estimator

$$S^2 = \frac{u'M_X u}{T} = \frac{u'u}{T} - \frac{u'P_X u}{T}.$$

From Lemma 2 we have

$$\frac{u'P_X u}{T} \xrightarrow{p} 0,$$

and $S^2 \xrightarrow{p} \sigma_u^2$ follows from the assumption $u'u/T \xrightarrow{p} \sigma_u^2$.

(*necessity*)

If S^2 is weakly consistent, then $S^2 \xrightarrow{p} \sigma_u^2$. This means

$$\frac{u'u}{T} - \frac{u'P_X u}{T} \xrightarrow{p} \sigma_u^2 .$$

From Lemma 2 it holds $u'P_X u/T \xrightarrow{p} \sigma_u^2$. So, the statement that $S^2 \xrightarrow{p} \sigma_u^2$ is valid if and only if $u'u/T \xrightarrow{p} \sigma_u^2$. ◇

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