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Short communication

A NOTE ON SOLUTION OF ORDINARY DIFFERENTIAL EQUATION OF THE SECOND-ORDER

Teshome Abay

Department of Physics, Faculty of Science, Addis Ababa University
PO Box 1176, Addis Ababa, Ethiopia

ABSTRACT: For a continuous function f(x) in a closed interval [-a,a], solution of the differential equation $w''(x) - f(x)\,w(x) = 0$ is presented and the result is then applied to specific non-analytic function $f(x) = -|x|$.

INTRODUCTION

This work deals in the main, with the equation of the type

$$w''(x) - f(x)\,w(x) = 0, \tag{1}$$

where f(x) is, throughout the closed interval [-a,a] where $a > 0$, continuous function of the variable x where $x \in [-a,a]$. It is to be recalled that solution of equation (1) has been obtained (Copson, 1974) by the method of power series expansion; on the assumption that f is analytic at some point in [-a, a]. The application of the result thus obtained however has been widely discussed (Heading, 1962; Arfken, 1966; Olver, 1974). Therefore it is important, from the point of view of physical application such as wave equation, to analyze solution of this equation, that does not depend on the analyticity of f(x).

We first state solution of the equation and provide its proof in section 2. Section 3 is devoted to the application of the result for non-analytic function $f(x) = -|x|$.

PROPOSITION

Let f(x) be continuous in the closed interval [-a,a], a > 0 and $|f(x)| \leq m$. Define recursively the functions $S_n(y)$ and $K_n(y)$ as follows:

$$S_0(y) = 1.0,\ K_0(y) = y,$$

and

$$S_{n+1}(y) = \int_0^y \int_0^t S_n(z) f(z) dt dz, \tag{a}$$

and

$$K_{n+1}(y) = \int_0^y \int_0^t f(z) K_n(z) dt dz \tag{b}$$

where t and y are in [-a,a].

Then

(i) the series $\displaystyle\sum_{n=0}^{\infty} f(y) S_n(y)$ $\qquad (a_1)$

and $\displaystyle\sum_{n=0}^{\infty} f(y) K_n(y)$ $\qquad (b_1)$

converge absolutely in [-a, a] and

(ii) the functions $\displaystyle P(x) = 1 + \int_0^x \int_0^y \sum_{n=0}^{\infty} f(t) S_n(t) dt dy$ $\qquad$ (I)

and $\displaystyle Q(X) = X + \int_0^x \int_0^y \sum_{n=0}^{\infty} f(t) K_n(t) dt dy$ $\qquad$ (II)

are linearly independent solutions of the equation

$$w''(x) - f(x)\, w(x) = 0.$$

Proof

Proof (i)

From equation (a1) one has the relation

$$\sum_{n=0}^{\infty} |f(y)S_n(y)| \leq \sum_{n=0}^{\infty} m|S_n(y)| \tag{c_1}$$

but from equation (a) one obtains

$$\sum_{n=0}^{\infty} |S_n(y)| \leq \cosh(\sqrt{m}y) \tag{c_2}$$

therefore

$$\sum_{n=0}^{\infty} |f(y)S_n(y)| \leq m\cosh(\sqrt{m}y) \tag{c_3}$$

similarly from equation (b_1) one obtains

$$\sum_{n=0}^{\infty} |f(y)K_n(y)| \leq \sum_{n=0}^{\infty} m|K_n(y)| \tag{d_1}$$

and equation (b) gives

$$\sum_{n=0}^{\infty} |f(y)K_n(y)| \leq m\sinh(\sqrt{m}y) \tag{d_2}$$

Therefore, since the left hand side of equations (c_3) and (d_2) are bounded in [-a,a], they will converge for finite m.

Proof (ii)

Differentiating twice, equations (I) and (II) and using equations (a) and (b), respectively, one obtains:

$$P''(x) - f(x)P(x) = 0 \tag{2}$$

and $\quad Q''(x) - f(x)Q(x) = 0.$ (3)

Since 0 is an interior point of [-a,a] and by the assumption of the proposition $P(0)=1$, $P'(0)=0$, $Q(0)=0$ and $Q'(0)=1$ and consequently, the Wronskian of the two functions evaluated at $x = 0$ is

$$\Delta(P, Q) = 1.0. \tag{4}$$

Hence, by equations (2), (3) and using the Wronskian property (Ferra, 1958) in equation (4), (ii) is proved.

Results of solution for $f(x)=-|x|$

Consider the function $\quad f(x)=-|x|,$ (5)

which is not analytic at $x = 0$. Since f is continuous in R, one therefore can use equation (5) in equations (I) and (II), respectively, and integrate the resulting equations in [-x,0] and [0,x] as a result of which one obtains:

$$P(x)=1+\sum_{n=1}^{\infty}\left\{(-1)^n|x|^{3n}\prod_{l=0}^{(n-1)}[(3l+1)/(3n)!]\right\}, \tag{6}$$

and

$$Q(x)=x+\sum_{n=1}^{\infty}\left\{(-1)^n|x|^{3n+1}\prod_{l=1}^{n}[(3l-1)/(3n+1)!]\right\}, \tag{7}$$

respectively.

One therefore observes, from equations (6) and (7), that P(x) has even parity and Q(x) has no definite parity. Moreover, by the ratio test, one finds that both P(x) and Q(x) converge for $x:|x|<1$. These results therefore, directly demonstrate the method of using equation (I) and (II) in order to obtain solution of equation (1) in the case when f is non-analytic. One also notices that $x=0$ is a singular point which is not regular sine $x|x|$ is not analytic at $x=0$.

Therefore, the usual power series method cannot be applied to solve this differential equation. Thus, the method being more general, could be applied to all classes of functions f, that are continuous in a closed interval [-a,a] regardless of the analytic property. In addition, solution of Airy equation (Olver, 1974) is easily obtainable from these results in different regions of the real axis.

Remark

In the case when direct integration is not possible one has to use numerical methods. The lower limits of integration in equations (I) and (II) has been set to zero in order to simplify the result. Its convergence however depends on the property of the function f.

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