

## APPLICATION OF NORMAL FAMILY OF $\delta$ -SUBHARMONIC FUNCTIONS TO THE EDREI-FUCHS ELLIPSE THEOREM

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**ABSTRACT:** In this paper we study the asymptotic behaviour of functions extremal for the well known inequality introduced by Edrei-Fuchs (called the Ellipse Theorem) by considering a normal family of  $\delta$ -subharmonic functions. This approach allows us to describe precisely the prototype of all functions extremal for the Edrei-Fuchs ellipse theorem. Indeed, it turns out that the functions which are extremal for the Edrei-Fuchs inequality are also extremal for the inequality introduced by Essèn-Rossi-Shea.

**Key words/phrases:**  $\delta$ -subharmonic, characteristic function, deficiency, lower order, Pólya peaks,  $*$ -function

### INTRODUCTION

Let  $f$  be a meromorphic function in the complex plane  $\mathbb{C}$  of lower order  $\lambda$ ,  $0 < \lambda < 1$ . We assume the reader is familiar with the notations  $T(r, f)$ ,  $N(r, f)$ ,  $\delta(\infty, f)$  involved in the theory of Nevanlinna. We set:

$$a = 1 - \delta(0, f), \quad b = 1 - \delta(\infty, f).$$

The Edrei-Fuchs Ellipse Theorem (Edrei *et al.*, 1960) states that

$$a^2 + b^2 - 2ab \cos \lambda \pi \geq \sin^2 \lambda \pi. \quad (1)$$

Furthermore  $a \leq \cos \lambda \pi$  implies  $b = 1$  and  $b \leq \cos \lambda \pi$  implies  $a = 1$ .

Besides (Edrei *et al.*, 1960) proofs for (1) are given in Edrei (1969) and in (Hayman, 1989, Ch. 9) using the spread theorem due to Baernstein (1973). In addition, the behaviour of the extremal functions, *i.e.*, one for which equality holds in (1) has been discussed by Edrei (1969). In this paper we give a proof of (1) and also describe the

behaviour of the extremal functions using a "normal family" of  $\delta$ -subharmonic functions (functions representable as a difference of subharmonic functions) in the sense of Anderson and Baernstein (1978). The approach has also been used to describe the asymptotic behaviour of a certain class of extremal functions, such as the well known inequality, Paley inequality (Hayman, 1989, Ch. 9).

Let  $u = u_1 - u_2$  be a  $\delta$ -subharmonic function in the complex plane, where  $u_i$  ( $i= 1,2$ ) is a subharmonic function. If  $u = \log|f|$ ,  $f$  meromorphic, then  $u$  is a  $\delta$ -subharmonic function. Any  $u \in C^2(\mathbb{C})$  is  $\delta$ -subharmonic (Arsove, 1953). We write

$$N(r, u) = \frac{1}{2\pi} \int_0^{2\pi} u(re^{i\theta}) d\theta \quad (0 < r < \infty), \quad u^+ = \max(u, 0)$$

and define the Nevanlinna characteristic of  $u$  by

$$T(r) = T(r, u) = N(r, u^+) + N(r, u_2).$$

Clearly,  $T(r, u)$  depends on the particular decomposition  $u_1 - u_2$  as well as on  $u$ , but one can ignore such an ambiguity, for discussion see (Hayman, 1989, Ch. 9).

The lower order  $\lambda$  of  $u$  is defined by

$$\lambda = \liminf_{r \rightarrow \infty} \frac{\log T(r)}{\log r}$$

and the Nevanlinna deficiency of  $u$

$$\delta(\infty, u) = \liminf_{r \rightarrow \infty} \frac{N(r, u^+)}{T(r, u)} = 1 - \liminf_{r \rightarrow \infty} \frac{N(r, u_2)}{T(r, u)}$$

Throughout this paper we shall write

$$\begin{aligned} a &= 1 - \delta(0, u), \text{ where } \delta(0, u) = \delta(\infty, -u) \\ b &= 1 - \delta(\infty, u). \end{aligned} \tag{2}$$

Clearly  $0 \leq 1 \leq a, b \leq 1$ .

STATEMENT OF RESULTS

A sequence  $\{r_n\}$  of positive numbers is a sequence of Pólya peaks for  $T(r,u)$  of order  $\lambda$ ,  $0 \leq \lambda < \infty$ , if there is a sequence  $\{\epsilon_n\}$ ,  $\epsilon_n > 0$ , such that as  $n \rightarrow \infty$ ,  $\epsilon_n \rightarrow 0$  and  $\epsilon_n r_n \rightarrow \infty$  and such that  $\epsilon_n r_n \leq t \leq r_n/\epsilon_n$  implies

$$T(t, u_n) \leq (1 + \epsilon_n) \left( \frac{t}{r_n} \right)^\lambda T(r_n, u).$$

If  $\lambda$  is finite  $T(r,u)$  has a sequence of Pólya peaks of order  $\lambda$  (for a proof see (Edrei, 1965)). Let us state a special case of the “normal family” theorem due to Baernstein (1973).

**Theorem A**

Let  $u = u_1 - u_2$  be a  $\delta$ -subharmonic function of lower order  $\lambda$ ,  $0 \leq \lambda < \infty$ , and let  $\{r_n\}$  be a sequence of Pólya peaks for  $T(r,u)$  of order  $\lambda$ . For each  $n = 1, 2, 3 \dots$  let

$$u_n(z) = \frac{u(r_n z)}{T(r_n, u)} = u_n^1(z) - u_n^2(z) \quad (z \in \mathbb{C}) \tag{3}$$

where  $u_n^i(z) = \frac{u^i(r_n z)}{T(r_n, u)} \quad (i=1,2)$ .

Then there is a  $\delta$ -subharmonic function  $v = v_1 - v_2$  in  $\mathbb{C}$  and a subsequence  $I = (n_k)$  of positive integers such that (as  $n \rightarrow \infty$ ,  $n \in I$ ) the following holds:

- a)  $\lim_{n \rightarrow \infty} N(r, |u_n - v|) = 0 \quad (0 < r < \infty),$
- b)  $\lim_{n \rightarrow \infty} T(r, u_n) = T(r, v) \leq r^\lambda \quad (0 < r < \infty),$
- c)  $\lim_{n \rightarrow \infty} N(r, u_n^1) = N(r, v_1) \leq (1 - \delta(0, u))r^\lambda \quad (0 < r < \infty),$
- d)  $\lim_{n \rightarrow \infty} N(r, u_n^2) = N(r, v_2) \leq (1 - \delta(\infty, u))r^\lambda \quad (0 < r < \infty),$

We notice from (3) and (b) that  $T(1, v) = 1$ . Since a subsequence of  $\{r_n\}$  is a sequence of Pólya peaks for  $T(r, u)$ , we consider the class of all  $\delta$ -subharmonic functions  $v = v_1 - v_2$  such that for some subsequence of  $\{r_n\}$  assertions (a) to (d) of Theorem A hold. We denote this class by  $S(u, \{r_n\})$ . We also write  $|E|$  for the Lebesgue measure of the set  $E$ .

We shall prove

**Theorem 1**

Let  $u = u_1 - u_2$  be a  $\delta$ -subharmonic function of lower order  $\lambda$ ,  $0 < \lambda < 1$ , and  $\{r_n\}$  a sequence of Pólya peaks for  $T(r, u)$  of order  $\lambda$ . Then

$$(i) \quad a^2 + b^2 - 2ab \cos \lambda \pi \geq \sin^2 \lambda \pi.$$

Furthermore  $a \leq \cos \lambda \pi$  implies  $b = 1$  and  $b \leq \cos \lambda \pi$  implies  $a = 1$ , where  $a$  and  $b$  are defined in (2).

(ii) Let  $h$  be any nonnegative function defined for  $r > 0$  and satisfying  $h(r) = O(\log r)$  as  $r \rightarrow \infty$  and consider the sequence:

$$\beta_n = \beta_n(h) = \frac{1}{2} |\{\theta \in [-\pi, \pi] : u(r_n e^{i\theta}) > h(r_n)\}| \quad (n = 1, 2, 3, \dots) \quad (4)$$

If equality holds in (i) with  $a < 1$  and  $b < 1$ , then

$$(a') \quad \beta = \lim_{n \rightarrow \infty} \beta_n = \frac{1}{\lambda} \cos^{-1}(b), \quad (0 \leq \beta \leq \frac{\pi}{2})$$

and we have  $b = \cos \lambda \beta$  and  $a = \cos \lambda(\pi - \beta)$ .

Furthermore for each  $v = v_1 - v_2 \in S(u, \{r_n\})$  we have

$$(b') \quad T(r, v) = r^\lambda, \quad N(r, v_1) = ar^\lambda, \quad N(r, v_2) = br^\lambda, \quad \text{and}$$

$$(c') \quad v(re^{i\theta}) = \pi \lambda r^\lambda \sin \lambda(\beta - |\theta - \alpha|) \quad \text{for some } \alpha \in [-\pi, \pi] \text{ and } |\theta - \alpha| \leq \pi.$$

All the assertions of Theorem 1 except part (c') are well known for  $u = \log |f|$ , where  $f$  is a meromorphic function (see Theorem 1, Edrei, 1969). However, the Ellipse Theorem, that is assertion (i) of Theorem 1 is already known for  $\delta$ -subharmonic function. Thus Theorem 1, apart from generalising Edrei's result to arbitrary  $\delta$ -subharmonic functions it describes precisely the functions which are extremal for the Ellipse inequality. The function  $v$  in (c') of Theorem 1 is a  $\delta$ -subharmonic function (usually called a harmonic spline (see Hayman, 1989, p. 739) extremal for (i). Indeed the function  $v$  in (c') is a typical example extremal for the inequality introduced by Essén, Rossi and Shea (see Theorem 1, Essén *et al*, 1992). Thus, in some sense the functions extremal for (i) of Theorem 1 are also extremal for the inequality of Essen-Rossi-Shea (Theorem 1, Essén *et al*, 1992). Also Theorem A (a) and Theorem 1 (c') imply that there is a subsequence  $I = \{n_k\}$  of positive integers and a real  $\alpha \in [-\pi, \pi]$  (depending on  $v$ ) such that for almost all  $z (z = re^{i\theta})$ ,  $u_n(re^{i\theta}) \rightarrow v(re^{i\theta})$  as  $n \rightarrow \infty, n \in I$ .

### SOME FACTS

One of our main tools to prove Theorem 1 is the \*-function introduced by Baernstein (1974). Let  $u = u_1 - u_2$  be a  $\delta$ -subharmonic function in  $\mathbb{C}$ . The \*-function of  $u$ , denoted by  $u^*$ , is defined

$$u^*(re^{i\theta}) = \frac{1}{2\pi} \sup \int_E u(re^{i\phi}) d\phi + N(r, u_2) \quad (0 \leq \theta \leq \pi, r > 0)$$

where the supremum is taken over all sets  $E \subset [-\pi, \pi]$  with  $|E| = 2\theta$ . In (Baernstein, 1974) it is proved that  $u^*$  is subharmonic in the upper half plane,  $\pi^+$  and continuous in the closure of  $\pi^+$  except possibly at  $z = 0$ . Also in (Baernstein, 1974) it is proved that for each  $r > 0$  and  $0 \leq \theta \leq \pi$  there is a set  $A \subset [-\pi, \pi]$  with  $|A| = 2\theta$  such that

$$\sup_{|E|=2\theta} \int_E u(re^{i\phi}) d\phi = \int_A u(re^{i\phi}) d\phi = \int_{-\theta}^{\theta} \tilde{u}(re^{i\phi}) d\phi,$$

where  $\theta \rightarrow \tilde{u}(re^{i\theta})$  is a symmetric decreasing rearrangement of  $u(re^{i\theta})$  on  $[-\pi, \pi]$ .

Further properties of  $u^*$  (see also Hayman, 1989, Ch. 9) needed here:

$$T(r, u) = \max_{\theta} u^*(re^{i\theta}), \quad u^*(r) = N(r, u_2), \quad u^*(-r) = N(r, u_1), \quad (5)$$

$$B(r, u) = \sup_{|z|=r} u(z) = \pi \frac{\partial}{\partial \theta} u^*(re^{i\theta})|_{\theta=0} = 0.$$

### *Proof of Theorem 1*

#### **Lemma 1**

Let  $u = u_1 - u_2$  be a  $\delta$ -subharmonic function of lower order  $\lambda$ ,  $0 < \lambda < \infty$  and let  $\{r_n\}$  be a sequence of Pólya peaks for  $T(r, u)$  of order  $\lambda$ . If  $\beta$  is any accumulation point of  $\{\beta_n\}$ , where  $\beta_n$  is defined in (4), then there is a  $\delta$ -subharmonic function  $v = v_1 - v_2 \in s(u, \{r_n\})$  such that  $v^*(e^{i\beta}) = 1$ . Conversely if  $v = v_1 - v_2 \in s(u, \{r_n\})$  then  $v^*(e^{i\beta}) = 1$  for some accumulation point  $\beta$  of  $\{\beta_n\}$ .

#### *Proof*

Consider the  $\delta$ -subharmonic function

$$\begin{aligned} w_n(re^{i\theta}) &= u_n^1(re^{i\theta}) - [u_n^2(re^{i\theta}) + \frac{h(r_n)}{T(r_n, u)}], \quad (n = 1, 2, 3, \dots) \\ &= w_n^1(re^{i\theta}) - w_n^2(re^{i\theta}), \end{aligned}$$

where  $u_n^i (i = 1, 2)$  is defined in (3),  $h$  is as in Theorem 1, and  $w_n^1 = u_n^1$

and  $w_n^2 = u_n^2 + \frac{h(r_n)}{T(r_n, u)}$ .

It is easy to check that for each  $v = v_1 - v_2 \in S(u, \{r_n\})$  there is a subsequence of  $\{w_n\}$  such that all assertions of Theorem A hold with  $w_n$  instead of  $u_n$ . Let  $\beta$  be any accumulation point of  $\{\beta_n\}$ . Assume  $\beta_n \rightarrow \beta$ . Let  $v = v_1 - v_2 \in S(u, \{r_n\})$ . Then there is subsequence  $I = \{n_k\}$  of positive integers such that the function  $v$  and the subsequence  $\{w_n\}$ ,  $n \in I$ , satisfy all assertions of Theorem A. The definition of  $\beta_n$

and  $w_n$  shows that  $w_n^*(e^{i\beta_n})=1$  for all  $n$ . Let  $A_n \subset [-\pi, \pi]$  with  $|A_n| = 2\beta_n$  and such that

$$1 = \frac{1}{2\pi} \int_{A_n} w_n(e^{i\theta})d\theta + N(1, w_n^2).$$

Now applying Theorem A we obtain

$$\begin{aligned} 1 &= \frac{1}{2\pi} \int_{A_n} v(e^{i\theta})d\theta + N(1, v_2) + o(1) \quad (n \rightarrow \infty, n \in I) \\ &\leq v^*(e^{i\beta_n}) + o(1) \quad (n \rightarrow \infty, n \in I) \end{aligned}$$

The continuity of  $v^*$  implies  $1 \leq v^*(e^{i\beta})$ . Since  $v^*(e^{i\beta}) \leq T(1, v) = 1$  we conclude that  $v^*(e^{i\beta}) = 1$ . The converse is similarly proved.

**Lemma 2**

Let  $u = u_1 - u_2$  be a  $\delta$ -subharmonic function of lower order  $\lambda$ ,  $0 < \lambda < 1$ , and  $\{r_n\}$  a sequence of Pólya peaks of order  $\lambda$  for  $T(r, u)$ . If  $\beta$  is any accumulation point of  $\{\beta_n\}$  where  $\beta_n$  is defined in (4) of Theorem 1, then

$$\sin\lambda\pi \leq a\sin\lambda\beta + b\sin\lambda(\pi - \beta) \tag{6}$$

where the numbers  $a$  and  $b$  are defined in (2).

*Proof*

Let  $I = \{n_k\}$  be a sequence of positive integers such that

$$\lim_{n \rightarrow \infty} \beta_n = \beta, \quad (n \in I).$$

Lemma 1 shows that for any  $v = v_1 - v_2 \in S(r, \{r_n\})$ ,  $n \in I$ ,  $v^*(e^{i\beta}) = 1$ . Put

$$S(\theta) = \frac{a\sin\lambda\theta + b\sin\lambda(\pi - \theta)}{\sin\lambda\pi} \quad (0 \leq \theta \leq \pi, 0 < \lambda < 1), \tag{7}$$

and define

$$w(re^{i\theta}) = v^*(re^{i\theta}) - r^\lambda S(\theta) \quad (0 \leq \theta \leq \pi, r > 0).$$

The function  $w$  is subharmonic on the upper half plane,  $\pi^+$ . Using Theorem A and (5) we obtain  $w(r) \leq 0$ ,  $w(-r) \leq 0$  and  $w(re^{i\theta}) \leq r^\lambda$ . Since  $\lambda\pi < \lambda$  we conclude by Phragmén-Lindelöf principle that  $w(re^{i\theta}) \leq 0$ . Thus

$$v^*(re^{i\theta}) \leq r^\lambda S(\theta) \quad (0 \leq \theta \leq \pi, r > 0). \quad (8)$$

The Lemma is now proved by setting  $r = 1$  and  $\theta = \beta$  in (8).

We proceed to prove Theorem 1. Using Cauchy - Schwarz inequality (6) implies

$$\begin{aligned} \sin\lambda\pi &\leq (a - b\cos\lambda\pi)\sin\lambda\beta + b\sin\lambda\pi \cos\lambda\beta \\ &\leq [(a - b\cos\lambda\pi)^2 + b^2\sin^2\lambda\pi]^{1/2} \\ &= [a^2 + b^2 - 2ab\cos\lambda\pi]^{1/2}. \end{aligned} \quad (9)$$

This proves assertion (i) of Theorem 1. For the proof of the implications or  $a \leq \cos\lambda\pi$  or  $b \leq \cos\lambda\beta\pi$  and assertion ii (a') of Theorem 1 see section 5 and 6 of Theorem 1 (Edrei, 1969).

*Proof of assertions ii (b') and ii (c') of Theorem 1*

The assumptions  $a < 1$  and  $b < 1$  together with Lemma 2 imply  $\beta \neq 0, \pi$ , where  $\beta$  is any accumulation point of  $\{\beta_n\}$  defined in (4). The hypotheses that equality holds in (i) of Theorem 1 implies equality in (9) and hence in (6). Thus  $v^*(e^{i\beta}) = 1 = S(\beta)$ , for any  $v \in S(u, \{r_n\})$ . Since  $0 < \beta < \pi$ , the maximum principle for subharmonic function and (8) imply

$$v^*(re^{i\theta}) = r^\lambda s(\theta) \quad (0 \leq \theta \leq \pi). \quad (10)$$

Consequently  $v^*$  is harmonic in  $\pi^+$  and by (4) we have

$$N(r, v_1) = v^*(re^{i\theta}) = ar^\lambda, \quad N(r, v_2) = v^*(r) = br^\lambda \text{ and } T(r, v) = r^\lambda. \quad (11)$$

This proves assertion ii (b') of Theorem 1. The relations in (11) allow us to define  $v_1(0) = 0 = v_2(0)$ . We proceed to find the extremal functions. We prove Lemma 3.



**Lemma 3**

Let  $u = u_1 - u_2$  be a  $\delta$ -subharmonic function in  $\mathbb{C}$  of order  $\lambda$ ,  $0 < \lambda < 1$  and  $u_1(0) = 0 = u_2(0)$ . if  $u^*$  is harmonic in the upper half plane then

$$u(z) = \int_0^\infty \log \left| 1 + \frac{ze^{i\alpha}}{t} \right| dn_1(t) - \int_0^\infty \log \left| 1 - \frac{ze^{i\alpha}}{t} \right| dn_2(t) \tag{12}$$

for some  $\alpha \in [-\pi, \pi]$ , where  $n_i(t) = \mu_i(|z| \leq t)$  and  $\mu_i$  is the Riesz measure of  $u_i$  ( $i = 1, 2$ ).

*Proof*

Since  $0 < \lambda < 1$  we have the representation

$$u(z) = \int_{\mathbb{C}} \log \left| 1 - \frac{z}{\zeta} \right| d\mu_1(\zeta) - \int_{\mathbb{C}} \log \left| 1 - \frac{z}{\zeta} \right| d\mu_2(\zeta). \tag{13}$$

Consider the  $\delta$ -subharmonic function

$$\begin{aligned} w(z) &= \int_{\mathbb{C}} \log \left| 1 + \frac{z}{|\zeta|} \right| d\mu_1(\zeta) - \int_{\mathbb{C}} \log \left| 1 - \frac{z}{|\zeta|} \right| d\mu_2(\zeta) \\ &= w_1(z) - w_2(z). \end{aligned} \tag{14}$$

It is easy to check that  $w^*$  is harmonic in  $\pi^+$  and that  $N(r, u_i) = N(r, w_i)$ , ( $i = 1, 2$ ). Thus using (4) and the Phragmén-Lindelöf principle,  $\lambda\pi < \pi$  we conclude that  $w^*(re^{i\theta}) = u^*(re^{i\theta})$ ,  $0 \leq \theta \leq \pi, r > 0$ . Again by (4) and (14) we have  $B(r, u) = B(r, w) = w(r)$ . Let  $B(r, u) = u(re^{i\alpha})$  for some  $\alpha \in [-\pi, \pi]$ . Now the Lemma follows using (12), (13) together with  $u(re^{i\alpha}) = w(r)$ .

To prove ii (c') of Theorem 1, we let  $v = v_1 - v_2 \in S(u, \{r_n\})$ , where  $u$  is extremal for (i) of Theorem 1. Then by (10)  $v$  satisfies the hypotheses of Lemma 3. From Jensen's formula and (11) we have  $n_1(t) = a\lambda t^\lambda$  and  $n_2(t) = b\lambda t^\lambda$ . Thus using (12) and applying the residue theorem of calculus yields assertion ii (c').

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