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A COMPARATIVE STUDY OF A FAMILY OF ESTIMATORS FOR THE COMMON MEAN OF SEVERAL NORMAL POPULATIONS

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ABSTRACT: A new approach of estimating the common mean of several different normal populations is introduced. It is shown that this approach yields the most commonly used estimators as special cases. An empirical comparative study of these estimators and three new ones is made through computer simulation. The results of the study show that for small samples and also when the population variances are very different the performance of the new estimators is better than that of the commonly used estimators.

Key words/phrases: Estimator, mean-squared error, mixed likelihood function, precision, relative efficiency

INTRODUCTION

The problem of making inference about the common mean of several normal populations, when the variances are unknown, was first treated by Bartlett (1936). Since then the problem has attracted the attention of many researchers. Most of the papers on this topic deal with estimation of the common mean: In this paper we shall first present some of the suggested estimators and then discuss an estimation procedure that includes these estimators as special cases. Finally a comparison of some of these estimators and other new ones is made by using the Monte-Carlo approach.

THE PROBLEM AND SOME COMPETING ESTIMATORS

Suppose there are k normal populations with the same mean μ and unknown and possibly different variances σ_i^2, $i=1,2,3, \ldots k$. The objective here is to estimate the common mean μ. For this purpose we take independent random samples of sizes n_i, $i=1,2,3, \ldots k$. We shall denote the sample means and variances by $\bar{X}_i$ and S_i^2; i.e.,

$$\bar{X}_i = \frac{1}{n_i}\sum_{j=1}^{n_i} X_{ij} \qquad S_i^2 = \frac{1}{n_i-1}\sum_{j=1}^{n_i}(X_{ij}-\bar{X}_i)^2 .$$

Some of the competing estimators of μ are listed below.

i) The Unweighted Estimator (UWE): The simplest estimator one can think of is the unweighted estimator which is given by:

$$UWE = (\sum_{i=1}^{k} n_i)^{-1}\sum_{i=1}^{k}\sum_{j=1}^{n_i} X_{ij} = (\sum_{i=1}^{k} n_i)^{-1}\sum_{i=1}^{k} n_i\bar{X}_i$$

ii) The Weighted Estimator (WTE): If the population variances, σ_i^2 $i=1,2,\ldots k$, are known the best estimator for μ is given by

$$\hat{\mu} = (\sum_{i=1}^{k} n_i/\sigma_i^2)^{-1}\sum_{i=1}^{k}\frac{n_i}{\sigma_i^2}\bar{X}_i .$$

If we replace σ_i^2 by its unbiased estimator S_i^2 we obtain the weighted estimator.

$$WTE = (\sum_{i=1}^{k} n_i/S_i^2)^{-1}\sum_{i=1}^{k}\frac{n_i}{S_i^2}\bar{X}_i .$$

iii) The Maximum Likelihood Estimator (MLE): The maximum likelihood estimator is derived by maximizing the likelihood function given below.

$$L = \prod_{i=1}^{k}(2\pi\sigma_i^2)^{-\frac{n_i}{2}}\exp[-\frac{1}{2\sigma_i^2}\sum_{j=1}^{n_i}(X_{ij}-\mu)^2] .$$

The MLE is the solution of the estimating equation

$$\sum_{i=1}^{k}\frac{n_i^2(\bar{X}_i-\hat{\mu})}{(n_i-1)S_i^2 + n_i(\bar{X}_i-\hat{\mu})^2} = 0 .$$

iv) The Neyman-Scott Estimator (NSE): Neyman and Scott (1948) studied a more general version of the above estimating equation and reached the conclusion that the estimator which emerges as the solution of

$$\sum_{i=1}^{k} \frac{(n_i-2)n_i(\bar{X}_i-\hat{\mu})}{(n_i-1)S_i^2 + n_i(\bar{X}_i-\hat{\mu})^2} = 0$$

is generally more precise than the MLE. We shall call this estimator the Neyman-Scott estimator. From the estimating equation of the NSE one can see that samples of size 2 make no contribution to the estimation and this may be an undesirable property of the estimator.

v) The Kalbfleish-Sprott Estimator (KSE): Kalbfleish and Sprott (1970) obtained an estimator for the common μ by using the conditional likelihood approach. The KSE is the solution of the estimating equation

$$\sum_{i=1}^{k} \frac{(n_i-1)n_i(\bar{X}_i-\hat{\mu})}{(n_i-1)S_i^2 + n_i(\bar{X}_i-\hat{\mu})^2} = 0 .$$

This estimator is an improvement over the NSE in that samples of size 2 make a contribution to the estimation. There is a similarity among the estimating equations for the MLE, NSE and KSE, and from this we may expect these estimators to have similar properties.

Comparative studies of some or all of these estimators have been made by, among others, Levy (1970), Levy and Mantel (1974), Rao (1980), and recently by Gebre-Egziabher Kiros (1990). Levy compared the MLE and WTE and found the MLE to be generally more precise than the WTE. Levy and Mantel studied the relative efficiencies of the UWE, WTE and MLE relative to the best unbiased estimator when the variances are known. Their study suggests that

1) the UWE has a better relative efficiency than the other two when the population variances are nearly equal;
2) the WTE performs better than the MLE when the sample sizes are equal; and
3) the performance of the MLE is superior in all other cases.

Rao investigated empirically the relative efficiencies of the MLE and KSE and other estimators, and concluded that the MLE is less efficient than the KSE. Gebre-Egziabher compared all these estimators when $3 \le k \le 10$ and n_i's are not large using the Monte-Carlo approach. He recommends the use of the WTE because of its computational simplicity and high efficiency unless k is large and/or the n_i's differ considerably.

MIXED (OR WEIGHTED) LIKELIHOOD FUNCTION APPROACH

Since we know that the independent random samples come from k normal populations with the same mean but possibly different variances, we may construct the likelihood function as a mixed (or weighted) likelihood function of k likelihood functions; i.e.,

$$L = \sum_{i=1}^{k} p_i (2\pi\sigma_i^2)^{-\frac{n_i}{2}} \exp[-(2\sigma_i^2)^{-1} \sum_{j=1}^{n_i} (X_{ij} - \mu)^2]$$

where p_i's are the mixing probabilities (or weights) and are independent of μ and σ_i^2's, but could depend on functions of X_{ij}'s. The maximum likelihood estimator of μ is then the solution of the estimating equation

$$\sum_{i=1}^{k} p_i \frac{n_i(\bar{X}_i - \hat{\mu})}{(2\pi e \hat{\sigma}_i^2)^{(n_i/2)+1}} = 0$$

where $\hat{\sigma}_i^2 = \frac{1}{n_i} \sum_{j=1}^{n_i} (X_{ij} - \hat{\mu})^2$

For different choices of p_i we get different estimators. The previous five estimators can be obtained through this estimation procedure by appropriately selecting p_i as can be seen from Table 1. Even though the number of estimators one can obtain through this process is limitless, only three new estimators are studied here. These estimators which are labelled NE1, NE2 and NE3 are selected because of their respective similarities to the MLE and the WTE. A comparative study of these estimators will be discussed in the following sections. The MLE and the NSE were found to have more or less similar behaviour, and because of this the NSE was dropped from further consideration.

Table 1. Some special cases of the estimation procedure.

P_i	Estimator or estimating equation	Notation
$p_i \propto (2\pi e\hat{\sigma}_i^2)^{\frac{n_i}{2}+1}$	$\hat{\mu} = \frac{1}{\sum n_i}\sum_{i=1}^{k} n_i\bar{X}_i$	UWE
$p_i \propto (S_i^2)^{-1}(2\pi e\hat{\sigma}_i^2)^{\frac{n_i}{2}+1}$	$\hat{\mu} = \frac{1}{\sum n_i/S_i^2}\sum_{i=1}^{k}\frac{n_i}{S_i^2}\bar{X}_i$	WTE
$p_i \propto (2\pi e\hat{\sigma}_i^2)^{\frac{n_i}{2}}$	$\sum_{i=1}^{k} n_i^2(\bar{X}_i-\hat{\mu})/\sum_{j=1}^{n_i}(X_{ij}-\hat{\mu})^2 = 0$	MLE
$p_i \propto \frac{(n_i-2)}{n_i}(2\pi e\hat{\sigma}_i^2)^{\frac{n_i}{2}}$	$\sum_{i=1}^{k}\frac{(n_i-2)n_i(\bar{X}_i-\hat{\mu})}{\hat{\sigma}_i^2} = 0$	NSE
$p_i \propto \frac{(n_i-1)}{n_i}(2\pi e\hat{\sigma}_i^2)^{\frac{n_i}{2}}$	$\sum_{i=1}^{k}\frac{(n_i-1)n_i(\bar{X}_i-\hat{\mu})}{\hat{\sigma}_i^2} = 0$	KSE
$p_i \propto (2\pi e\hat{\sigma}_i^2)^{\frac{(n_i+1)}{2}}$	$\sum_{i=1}^{k}\frac{n_i(\bar{X}_i-\hat{\mu})}{\hat{\sigma}_i} = 0$	NE1
$p_i \propto (S_i)^{-1}(2\pi e\hat{\sigma}_i^2)^{\frac{n_i}{2}+1}$	$\hat{\mu} = \frac{1}{\sum n_i/S_i}\sum_{i=1}^{k}\frac{n_i}{S_i}\bar{X}_i$	NE2
$p_i \propto (2\pi e\hat{\sigma}_i^2(S_i^2)^{-1})^{\frac{n_i}{2}+1}$	$\hat{\mu} = \frac{1}{\sum n_i/(S_i^2)^{n_i/2+1}}\sum_{i=1}^{k}\frac{n_i}{(S_i^2)^{n_i/2+1}}\bar{X}_i$	NE3

METHOD OF COMPARISON

The comparison of the seven estimators analytically is very difficult and because of this the comparison was made by using the Monte-Carlo approach. The common mean μ was set at zero in order to simplify the computations involved and this has no effect on the conclusions reached. To compute values of the estimators a computer program having several sub-programs was written in Pascal. The sub-programs and their functions are given in Table 2.

Table 2. Sub-programs and their functions.

Name of sub-program	Function
Uniform	Generates uniform random variables by using the Wichmann-Hill algorithm. (See Gebre-Egziabher Kiros, 1992).
Generate	Converts the uniform random variables to normal random variables by using the Polar-Marsagalia method.
Statistics	Computes sample sums, means, sum of squares and variances.
Func1, Func2, Func3 and Regula Falsi	These are used to estimate the MLE, KSE and NE1.
Main Program	Reads the number of populations, population variances and sample sizes, estimates UWE, WTE, MLE, KSE, NE1, NE2 and NE3, and writes them in that order. It also writes the number of times each iteration failed to converge.

The performance of an estimator depends on the number of populations under consideration (k), the population variances (σ_i^2), the sample sizes (n_i) and the pattern in which the n_i's and σ_i^2's are combined. Because real life comparisons may not involve more than 12 populations, three values of k (i.e., 3, 7, 12) were selected. The ratios σ_i^2/σ_j^2 $i \neq j$ rather than the actual magnitudes of the variances affect the relative performance of the estimators, and because of this the σ_i^2's were selected in such a way that $\sum_{i=1}^{k} \sigma_i^2 = 1$ and $\delta = \max\ \delta_i^2/\min\ \delta_i^2$ equals 4/3, 2, 5, 10, 50, 100 and 1000. The quantity δ was used as a measure of heterogeneity in the σ_i^2's. The sample sizes selected were of three types; viz. small ($3 \leq n_i < 10$), medium ($10 \leq n_i \leq 30$) and large ($n_i > 30$). The actual sample sizes for the different values of k are given in Table 3. Two different combination of sample sizes and population variances were used:

1) n_i's and σ_i^2's having the same rank order;

2) n_i's and σ_i^2's having the reverse rank order.

For each combination of k, n_i and σ_i^2 the program was run 1000 times and 1000 estimates were computed for each estimator.

Table 3. Selected sample sizes.

Type	K		
	3	7	12
Small	4,7,9	3,4,5,6,7,8,9	3,4,4,5,5,6,6,7,7,8,8,9
Medium	15,19,26	13,15,17,19,24,26,28	11,13,15,16,17,19,21,23,24,25,27,29
Large	31,38,47	33,35,37,39,44,46,48	31,33,35,37,39,41,43,45,47,49,51,53

To obtain the estimates for the MLE, KSE and NE1 the Newton-Raphson method was firstly tried. However, on several occasions it was observed that the iterative method failed to converge for the MLE and KSE, especially when the sample sizes were small. To see if this problem could be overcome and also because of its better effectivity index (see Froberg, 1970) the method was changed to Regula Falsi. Even with this iterative procedure the problem could not be overcome, and in the program a value of 99.99 was given to the estimator on such an occasion. This has the effect of exaggerating the estimates for the mean and the mean-squared error.

A second program, also in Pascal, was written to make the comparative study. The mean- squared error of the estimates about the true common mean zero was used as a measure of precision. The comparison was made by computing these mean-squared errors.

RESULTS AND DISCUSSION

For the unweighted estimator it can be shown that

$$Var(UWE) = \frac{1}{(\Sigma n_i)^2}\sum_{i=1}^{k} n_i\sigma_i^2$$

and expressions for the asymptotic variances of WTE, MLE, NSE and KSE are available. The above variance for the UWE is an exact result and may be used to check the validity of the simulation program. Before running the entire program ratios of the empirical and theoretical variances were computed for different combinations of k, n_i's and σ_i^2's and were found to be close to 1. This was taken as a validation of the simulation process.

Table 4 gives the precision of the different estimators relative to the estimator with the smallest MSE For each value of k the first seven rows refer to combinations in which n_i's and σ_i^2's have the same rank order, and the next seven rows to combinations in which n_i's and σ_i^2's have the reverse rank order.

One can observe from the table that the precision of these estimators depends, to a large extent, on the n_i's and δ. For δ =4/3 and small and medium sample sizes the best estimator was found to be the UWE for all values of k. This is not surprising when one notes that for $\delta = 1$ (i.e., the variances are all equal) the unweighted estimator is the best estimator. The result is also in conformity with suggestion (1) of Levy and Mantel. However, for large sample sizes and δ =4/3 NE1 and NE2 were found to be relatively more efficient than the UWE. For δ =2 and small samples the UWE and NE1 were found to be superior to the other estimators, but for medium and large samples the performance of NE1 and NE2 were found to be relatively better.

Table 4. Precision of the estimators relative to the best among the set.

K	Est.	Small							Medium							Large						
		δ							δ							δ						
		4/3	2	5	10	50	100	1000	4/3	2	5	10	50	100	1000	4/3	2	5	10	50	100	1000
3	UWE	100✓	101	137	171	662	1273	11936	100✓	106	164	219	789	1500	13419	101	107	150	250	792	1405	14918
	WTE	134	127	113	111	110	119	105	107	103	102	101	101	101	100	102	100	100✓	100	100✓	100	100✓
	MLE	127	*	*	*	*	*	*	107	103	100	100	100	100✓	100✓	102	100	100	100	100	100	100✓
	KSE	*	*	*	*	*	*	*	107	103	100✓	100✓	100✓	100	100✓	102	100	100	100✓	100	100✓	100✓
	NE1	106	100✓	100✓	100✓	138	164	239	101	100✓	114	118	150	174	207	100✓	100✓	106	124	143	163	224
	NE2	111	106	107	113	160	196	264	101	100	115	120	153	175	211	100	100	107	125	143	164	223
	NE3	208	175	113	106	100✓	100✓	100✓	198	177	120	119	109	103	100	217	182	139	134	116	105	101
	UWE	100✓	100	120	155	550	994	9531	100✓	105	139	210	712	1405	12703	101	106	141	197	740	1365	13440
	WTE	136	122	106	100✓	100✓	100✓	100✓	109	104	100	100✓	100✓	100✓	100✓	103	101	100	100	100✓	100	100
	MLE	*	*	*	*	*	*	*	109	104	100✓	100	100	100	*	103	101	100	100✓	100	100✓	100✓
	KSE	*	*	*	*	*	*	*	109	104	100	100	100	100	*	103	101	100✓	100✓	100	100	100✓
	NE1	106	100✓	103	105	190	288	1927	102	100✓	108	123	169	227	488	100✓	100	109	117	166	202	346
	NE2	109	101	100✓	100	149	191	359	102	100	107	120	159	203	317	100	100✓	109	116	163	193	307
	NE3	228	206	181	152	144	145	108	243	223	168	150	123	109	101	238	226	159	152	120	108	100
7	UWE	100✓	100✓	114	124	284	490	5640	100✓	107	126	139	367	636	5108	100✓	109	127	153	351	667	5763
	WTE	213	189	160	169	197	186	124	109	105	102	101	101	101	101	104	100	100✓	100	100✓	100✓	100✓
	MLE	145	144	*	117	*	*	*	107	104	100	100	100	100	100	104	100	100	100	100	100	100✓
	KSE	*	137	*	*	*	*	*	107	104	100✓	100✓	100✓	100✓	100✓	104	100✓	100	100✓	100	100	100✓
	NE1	107	103	100✓	100✓	129	138	594	101	100✓	102	101	154	172	371	101	100	103	108	149	177	380
	NE2	121	117	114	119	173	183	480	102	101	104	103	155	172	333	101	100	103	108	148	175	350
	NE3	393	323	234	220	100✓	100✓	100✓	373	355	248	249	125	116	100	429	408	282	251	138	127	104
	UWE	100✓	102	111	133	176	312	1914	100✓	106	117	159	297	544	3871	100	108	123	153	324	629	4867
	WTE	195	179	158	145	102	100✓	100✓	108	107	103	101	100✓	100✓	100✓	103	101	100✓	100	100	100	100✓
	MLE	*	*	*	*	*	*	*	107	107	102	100	100	100	100	103	101	101	100	100✓	100✓	100
	KSE	*	*	*	*	*	*	*	107	107	102	100✓	100	101	100	103	101	101	100✓	100✓	100	100
	NE1	108	100✓	100✓	100✓	107	150	829	101	100	100	105	154	200	728	100	100	102	107	151	196	530
	NE2	118	110	104	103	100✓	124	324	101	100✓	100✓	104	147	186	446	100✓	100✓	102	106	150	192	429
	NE3	424	389	394	377	304	317	398	435	482	479	512	180	142	103	568	476	389	368	145	134	103

Table 4. (Contd.)

	UWE	100√	100√	106	155	328	400	4126	100√	100	115	188	354	680	4948	100	103	118	175	397	667	4852
	WTE	235	261	186	160	252	113	407	112	106	102	102	101	101	101	104	102	101	100	100	101	100√
	MLE	*	144	128	*	100√	*	*	112	106	100	100	100√	100	100√	103	102	100	100√	100√	100	100√
	KSE	*	*	125	105	*	*	*	112	106	100√	100√	100	100√	100√	103	102	100√	100	100√	100√	100√
	NE1	108	105	100√	100√	146	100√	329	103	100√	100	115	145	166	318	100√	100√	103	112	159	170	320
	NE2	122	132	114	118	194	120	489	102	100	102	117	147	168	306	100	100	104	113	159	171	308
	NE3	552	604	301	244	139	114	100√	660	608	287	363	162	159	123	795	714	376	393	172	172	130
12	UWE	100√	100√	105	133	176	307	2028	100√	102	108	163	272	505	3606	101	103	115	177	329	554	3503
	WTE	211	207	178	147	105	100√	100√	112	108	109	101	100	100√	100	103	102	100	101	100	100√	100
	MLE	*	153	*	*	*	*	*	110	107	106	100√	100√	100	100√	103	102	100√	100	100√	100	100√
	KSE	*	*	*	*	*	*	*	110	107	105	100	100	100	100	103	102	100	100√	100	100	100√
	NE1	106	107	100√	100√	111	124	501	101	100	100	113	152	178	596	100	100	103	116	164	169	401
	NE2	117	115	108	103	100√	110	288	101	100√	100√	112	148	171	456	100√	100√	103	115	162	167	370
	NE3	676	751	667	600	532	399	287	767	801	871	607	581	354	194	797	835	883	474	191	238	133

√, Indicates the estimator with the smallest MSE. *, Indicates non-convergence.

When the sample sizes are small and $2<\delta\leq 10$ NE1 seemed to perform the best. The iterative procedure for obtaining the MLE and KSE failed to converge for small sample sizes on several occasions. A count was made of such events and it was observed that it could go to as high as 419 in 1000 runs. For medium and large samples the study indicated that the performance of the KSE was better than that of the rest.

For $50\leq\delta\leq 1000$ and small sample sizes the WTE and NE3 performed better than the other estimators. For medium and large samples the KSE, WTE and MLE had relatively higher precision than the others.

Even though there were indications that increasing the values of k favoured the KSE and MLE, the effect of k on precision was not marked. This could be because of the selected values of k which were all small and, therefore, could not clearly show the effect of k on precision.

CONCLUSION

The results of the study suggest that:

1) The relative precision of the estimators depends, to a large extent, on the sizes of the samples and δ, the measure of heterogeneity in σ_i^2's.
2) For δ near 1 and small and medium samples the UWE is relatively most efficient.
3) For $\delta=2$ and medium and large samples and for $2<\delta\leq 10$ and small samples the performance of NE1 seems to be the best.
4) For $2<\delta\leq 10$ and medium and large samples the KSE is relatively better than the rest.
5) For $50\leq\delta\leq 1000$ and small samples NE3 seems to perform well, but for medium and large samples the WTE is the best because of its relative precision and computational simplicity.

The indiscriminate use of the MLE is to some extent supported by theory. For large samples and under certain regulatory conditions the MLE has optimum properties of being approximately unbiased, consistent and efficient. But as the results of this study and also of other authors (for example Neyman and Scott and Rao) show there are circumstances when the MLE is less efficient than competing estimators. Therefore, when confronted with a new problem, one should carefully examine if the conditions are satisfied before applying the MLE.

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