

APPROXIMATING A COMMON SOLUTION OF A FINITE FAMILY OF GENERALIZED EQUILIBRIUM AND FIXED POINT PROBLEMS

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ABSTRACT: In this paper, we introduce and investigate an iterative scheme for finding a common element of the set of common solutions of a finite family of generalized equilibrium problems and the set of fixed points of a Lipschitz and hemiccontractive-type multi-valued mapping. We obtain strong convergence theorems of the proposed iterative process in real Hilbert space settings. Our results improve, generalize and extend most of the recent results that have been proved by many authors in this research area.

Key words/phrases: Continuous monotone mapping, demiclosedness principle, fixed point problem, generalized equilibrium problems, hemiccontractive-type multi-valued mapping

INTRODUCTION

Let C be a nonempty subset of a real Hilbert space H with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$. A single-valued mapping $T: C \rightarrow H$ is said to be *k*-strictly pseudocontractive in the sense of Browder and Petryshyn (1967) if there exists $k \in [0, 1)$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|x - Tx - (y - Ty)\|^2, \quad \forall x, y \in C \dots\dots\dots (1.1)$$

If $k = 1$ in (1.1), then T is called *pseudocontractive* mapping.

A mapping $T: C \rightarrow H$ is called *Lipschitzian* if there exists $L \geq 0$ such that $\|Tx - Ty\| \leq L\|x - y\|, \forall x, y \in C$. If $L = 1$, then T is called *nonexpansive* and if $L \in [0, 1)$, then T is called *contraction*.

Observe that the class of pseudocontractive mappings contains the class of *k*-strictly pseudocontractive mappings and nonexpansive mappings (see Browder and Petryshyn, 1967; Chidume *et al.*, 2013).

A mapping $T: C \rightarrow H$ is said to be *firmly nonexpansive* if $\|Tx - Ty\|^2 \leq \langle Tx - Ty, x - y \rangle$ for all $x, y \in C$. It is known that every firmly

nonexpansive mapping is nonexpansive mapping, but the inclusion is proper (see Mongkolkeha *et al.*, 2013).

A mapping $T: C \rightarrow H$ with $F(T) = \{x \in C: x = Tx\}$ nonempty is said to be *quasi-nonexpansive* if $\|Tx - p\| \leq \|x - p\|$ holds for all $p \in F(T), x \in C$ and T is called *hemiccontractive* if $\|Tx - p\|^2 \leq \|x - p\|^2 + \|x - Tx\|^2$ holds for all $p \in F(T), x \in C$.

We remark that the class of hemiccontractive mappings contains the class of pseudocontractive mappings with $F(T) \neq \emptyset$ and the class of quasi-nonexpansive mappings. The following examples show that the inclusion is proper.

Example 1.1. Let $H = \mathbb{R}$ and $C = [0, 1]$. Let $T: C \rightarrow H$ be defined by $Tx = x^2 \sin\left(\frac{1}{x}\right)$ if $x \neq 0$ and $T0 = 0$. Then, zero is the only fixed point of T and for all $x \in C$, we have

$$\begin{aligned} |Tx - 0|^2 &= \left| x^2 \sin\left(\frac{1}{x}\right) \right|^2 \leq |x|^4 \\ &\leq |x|^2 \leq |x - 0|^2 + |x - Tx|^2. \end{aligned}$$

Hence, T is hemiccontractive mapping. However, if we take $x = \frac{2}{\pi}$ and $y = \frac{1}{\pi}$, then we get that

$$|Tx - Ty|^2 = \left| \frac{4}{\pi^2} \sin \frac{\pi}{2} - \frac{1}{\pi^2} \sin \pi \right|^2 = \frac{16}{\pi^4}.$$

But,

$$\begin{aligned}
 |x - y|^2 + |x - y - (Tx - Ty)|^2 &= \frac{1}{\pi^2} + \left(\frac{\pi - 4}{\pi^2}\right)^2 \\
 &= \frac{2\pi^2 - 8\pi + 16}{\pi^4} \\
 &< \frac{16}{\pi^4} = |Tx - Ty|^2,
 \end{aligned}$$

which shows that T is not pseudocontractive.

Example 1.2. Let $H = \mathbb{R}$ and $C = [0,1]$. Let $T: C \rightarrow H$ be defined by

$$Tx = \begin{cases} \frac{1}{2}, & \text{if } x \in \left[0, \frac{1}{2}\right], \\ 0, & \text{if } x \in \left(\frac{1}{2}, 1\right]. \end{cases}$$

Then, $\frac{1}{2}$ is the only fixed point of T . If $x \in \left[0, \frac{1}{2}\right]$, we have

$$\left|Tx - T\left(\frac{1}{2}\right)\right|^2 = 0 \leq \left|x - \frac{1}{2}\right|^2 + |x - Tx|^2.$$

And if $x \in \left(\frac{1}{2}, 1\right]$, we get

$$\left|Tx - T\left(\frac{1}{2}\right)\right|^2 = \frac{1}{4} < x^2 \leq \left|x - \frac{1}{2}\right|^2 + |x - Tx|^2.$$

Thus, T is hemicontractive mapping. However, T is not quasi-nonexpansive mapping. In fact, for $x = \frac{3}{5}$ we have

$$\left|Tx - T\left(\frac{1}{2}\right)\right| = \frac{1}{2} > \frac{1}{10} = \frac{3}{5} - \frac{1}{2} = \left|x - \frac{1}{2}\right|.$$

Let $CB(C)$ denote the family of nonempty, closed and bounded subsets of C . The Pompeiu-Hausdorff metric (see Berinde and Păcurar, 2013) on $CB(C)$ is defined by

$$D(A, B) = \max\{sup_{x \in A} d(x, B), sup_{y \in B} d(y, A)\},$$

for all $A, B \in CB(C)$, where $d(x, B) = \inf\{\|x - b\| : b \in B\}$.

A multi-valued mapping $T: C \rightarrow CB(C)$ is called Lipschitzian if there exists $L \geq 0$ such that

$$D(Tx, Ty) \leq L\|x - y\|, \forall x, y \in C \dots\dots\dots (1.2)$$

If $L = 1$ in (1.2), then T is called *nonexpansive* and if $L \in [0,1)$, then T is called *contraction* mapping.

A multi-valued mapping $T: C \rightarrow CB(C)$ is said to be *k-strictly pseudocontractive* in the sense of Chidume et al. (2013) if there exists a constant $k \in [0,1)$ such that

$$D^2(Tx, Ty) \leq \|x - y\|^2 + k\|(x - u) - (y - v)\|^2, \dots (1.3)$$

for all $x, y \in C$ and $u \in Tx, v \in Ty$, where $D^2(Tx, Ty) = (D(Tx, Ty))^2$. If $k = 1$ in (1.3), then T is said to be *pseudocontractive* mapping.

Let $T: C \rightarrow CB(C)$ be a multi-valued mapping, then an element $x \in C$ is called *fixed point* of T if $x \in Tx$. We denote the set of fixed points of a mapping T by $F(T)$. We also write weak convergence and strong convergence of a sequence $\{x_n\}$ to x in H as $x_n \rightharpoonup x$ and $x_n \rightarrow x$, respectively.

A multi-valued mapping $T: C \rightarrow CB(C)$ with nonempty set of fixed points is called:

- i) *Quasi-nonexpansive* if for all $p \in F(T), x \in C$, we have $D(Tx, Tp) \leq \|x - p\|$.
- ii) *Hemicontractive-type* in the sense of Sebsibe Teferi et al. (2015) if for all $p \in F(T), x \in C$

$$\begin{aligned}
 D^2(Tx, Tp) &\leq \|x - p\|^2 + \|x - u\|^2 \\
 &\text{holds for all } u \in Tx. \dots\dots\dots (1.4)
 \end{aligned}$$

We observe that every nonexpansive mapping T with $F(T) \neq \emptyset$ is quasi-nonexpansive mapping, and every pseudocontractive mapping T with $F(T) \neq \emptyset$ and $T(p) = \{p\}, \forall p \in F(T)$ is hemicontractive-type mapping (Habtu Zegeye et al., 2017).

In recent years, the existence and approximation of fixed points for multi-valued (including hemicontractive-type) mappings in various spaces under different assumptions has been studied by several authors; (see, for example, Nadler, 1969; Panyanak, 2007; Shahzad and Habtu Zegeye, 2008; Yu et al., 2012; Chidume et al., 2013; Isiogugu and Osilike, 2014; and references therein).

Sebsibe Teferi et al. (2015) proved the following result:

Theorem WSZ. Let C be a nonempty, closed and convex subset of a real Hilbert space H . Let $T_i: C \rightarrow CB(C), i = 1, 2, \dots, N$, be a finite family of Lipschitz hemicontractive-type mappings with Lipschitz constants $L_i, i = 1, 2, \dots, N$, respectively. Assume that $(I - T_i), i = 1, 2, \dots, N$, are demi-closed at zero and $\mathcal{F} = \bigcap_{i=1}^N F(T_i)$ is nonempty, closed and convex with $T_i(p) = \{p\}, \forall p \in \mathcal{F}$. Let $\{x_n\}$ be the sequence generated from an arbitrary $x_1, w \in C$ by

$$\begin{cases} y_n = (1 - \beta_n)x_n + \beta_n u_n, & u_n \in T_n x_n, \\ z_n = (1 - \gamma_n)x_n + \gamma_n w_n, & w_n \in T_n y_n, \dots\dots\dots (1.5) \\ x_{n+1} = \alpha_n w + (1 - \alpha_n)z_n, & \forall n \geq 1, \end{cases}$$

where $T_n := T_{n \bmod (N)+1}$ and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subset (0,1)$ satisfy the following conditions:

- i) $0 < \alpha_n \leq c < 1, \forall n \geq 1$ such that $\lim_{n \rightarrow \infty} \alpha_n \rightarrow 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$;
- ii) $0 < \alpha \leq \gamma_n \leq \beta_n \leq \beta < \frac{1}{\sqrt{4L^2+1}}, \forall n \geq 1,$
for $L := \max\{L_i: i = 1, 2, \dots, N\}$.

Then, $\{x_n\}$ converges strongly to some point $p \in \mathcal{F}$ nearest to w .

Recall that a mapping $A: C \rightarrow H$ is called *monotone* if $\langle Ax - Ay, x - y \rangle \geq 0, \forall x, y \in C$

A is called α -*inverse strongly monotone* if there exists a positive real number α such that $\langle Ax - Ay, x - y \rangle \geq \alpha \|Ax - Ay\|^2, \forall x, y \in C$.

We note that the class of α -inverse strongly monotone mappings is properly contained in the class of monotone mappings (see Habtu Zegeye *et al.*, 2017).

Let C be a nonempty, closed and convex subset of a real Hilbert space H . Let $F: C \times C \rightarrow \mathbb{R}$ be a bifunction and $A: C \rightarrow H$ be a nonlinear mapping. Takahashi and Takahashi (2008) considered the following generalized equilibrium problem:

Finding a point $z \in C$ such that

$$F(z, y) + \langle Az, y - z \rangle \geq 0, \forall y \in C. \dots\dots\dots(1.6)$$

In this paper, we denote the set of solutions of problem (1.6) by $EP(F, A)$, *i.e.*,

$$EP(F, A) = \{z \in C : F(z, y) + \langle Az, y - z \rangle \geq 0, \forall y \in C\}.$$

If in (1.6) we have $A \equiv 0$, then problem (1.6) reduces to the equilibrium problem of finding an element $z \in C$ such that

$$F(z, y) \geq 0, \forall y \in C, \dots\dots\dots(1.7)$$

which was studied by Blum and Oettli (1994) and many others (Combettes and Hirstoaga, 2005; Takahashi and Takahashi, 2007; Wang *et al.*, 2007; Ali, 2009; Cholamjiak *et al.*, 2015). The set of solutions of problem (1.7) is denoted by $EP(F)$.

If in (1.6) we have $F \equiv 0$, then the generalized equilibrium problem (1.6) reduced to finding a point $z \in C$ such that

$$\langle Az, y - z \rangle \geq 0, \forall y \in C, \dots\dots\dots(1.8)$$

which is called the classical variational inequality problem. The set of solutions of problem (1.8)

is denoted by $VI(C, A)$. Problem (1.8) has been considered by many authors (see, for instance, Ali, 2009; Habtu Zegeye and Shahzad, 2011a; 2012; Tesfalem Hadush *et al.*, 2016) and references therein.

We note that if a point $z \in VI(C, A) \cap EP(F)$, then $z \in EP(F, A)$, however, the converse is not true (see Habtu Zegeye *et al.*, 2017).

Assumption 1.1. Let C be a nonempty, closed and convex subset of a real Hilbert space H . In the sequel, let F be a bifunction of $C \times C$ into \mathbb{R} satisfying the following assumptions:

- (A1) $F(x, x) = 0, \forall x \in C$;
- (A2) F is monotone, *i.e.*,
 $F(x, y) + F(y, x) \leq 0, \forall x, y \in C$;
- (A3) $\lim_{t \downarrow 0} F(tz + (1-t)x, y) \leq F(x, y), \forall x, y, z \in C$;
- (A4) For each $x \in C, y \mapsto F(x, y)$ is *convex* and *lower semicontinuous*.

For instance, the bifunction $F: [0, \infty) \times [0, \infty) \rightarrow \mathbb{R}$ given by $F(x, y) = y - x$ satisfies Assumption 1.1.

Remark 1.2. Let C be a nonempty, closed and convex subset of a real Hilbert space H . Let F be a bifunction from $C \times C$ into \mathbb{R} satisfying Assumption 1.1 and let $A: C \rightarrow H$ be a continuous monotone mapping. Define $G: C \times C \rightarrow \mathbb{R}$ by $G(x, y) = F(x, y) + \langle Ax, y - x \rangle$, then it is easy to see that the bifunction G satisfies Assumption 1.1. Thus, the generalized equilibrium problem (1.6) is equivalent to the equilibrium problem of finding a point $z \in C$ such that $G(z, y) \geq 0$, for all $y \in C$.

Generalized equilibrium problem is more general in the sense that it includes, as special case, equilibrium problems and hence variational inequality, optimization problems, Nash equilibrium problems, fixed point problems, *etc.* Consequently, many authors have shown their interest in constructing an iterative algorithms for approximating common solution of generalized equilibrium and fixed point problems (see, for example, Hao, 2011; Kamraksa and Wangkeeree, 2011; Razani and Yazdi, 2012; Zhang and Hao, 2016 and references cited therein).

Takahashi and Takahashi (2008) introduced and considered the following iterative algorithm for finding a common point of the set of solu-

tions of problem (1.6) and the set of fixed points of nonexpansive single-valued mapping T and then they obtained a strong convergence theorem in Hilbert space settings.

$$\begin{cases} F(z_n, y) + \langle Ax_n, y - z_n \rangle + \frac{1}{\lambda_n} \langle y - z_n, z_n - x_n \rangle \geq 0, \forall y \in C, \\ y_n = \alpha_n u + (1 - \alpha_n) z_n, \\ x_{n+1} = \beta_n x_n + (1 - \beta_n) T y_n, \quad \forall n \geq 1, \end{cases}$$

where $u, x_1 \in C$ are arbitrary, $F: C \times C \rightarrow \mathbb{R}$ is a bifunction satisfying Assumption 1.1 and A is an α -inverse strongly monotone mapping from C into H , and $\{\alpha_n\}, \{\beta_n\} \subset [0, 1]$ and $\{\lambda_n\} \subset [0, 2\alpha]$ satisfy some appropriate control conditions.

Recently, Huang and Ma (2014) extended the results of Takahashi and Takahashi (2008) from nonexpansive mapping to k -strictly pseudocontractive mapping. In fact, they proved the following weak convergence theorem.

Theorem 1.3. Let C be a nonempty, closed and convex subset of a real Hilbert space H . Let $A: C \rightarrow H$ be an α -inverse strongly monotone mapping and F be a bifunction from $C \times C$ into \mathbb{R} which satisfies Assumption 1.1. Let $T: C \rightarrow C$ be a k -strictly pseudocontractive mapping such that $F := EP(F, A) \cap F(T)$ is nonempty and let $\{e_n\}$ be a bounded sequence in C . Let $\{x_n\}$ be a sequence generated by

$$\begin{cases} F(u_n, u) + \langle Ax_n, y - u_n \rangle + \frac{1}{r_n} \langle u - u_n, u_n - x_n \rangle \geq 0, \forall u \in C, \\ x_{n+1} = \alpha_n x_n + \beta_n (\delta_n u_n + (1 - \delta_n) T u_n) + \gamma_n e_n, \quad \forall n \in \mathbb{N}, \\ \dots\dots\dots(1.9) \end{cases}$$

where the sequences $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\delta_n\} \subset (0, 1)$ and $\{r_n\} \subset [0, 2\alpha]$ satisfy some mild restrictions. Then the sequence $\{x_n\}$, generated by (1.9), converges weakly to a point $p \in F$, where $p = \lim_{n \rightarrow \infty} P_F x_n$.

In this paper, motivated and inspired by the results surveyed above, we introduce an iterative algorithm for finding a common element of the common solution set of a finite family of generalized equilibrium problems (1.6) and the fixed point set of a multi-valued Lipschitz hemiccontractive-type mapping. The results presented in this paper generalize, improve and extend the corresponding results of Huang and Ma (2014), Tesfalem Hadush *et al.* (2016), Habtu Zegeye *et al.* (2017), Takahashi and Takahashi (2008), Ceng *et al.* (2010) and Zhang

and Hao (2016) and some other recent results that have been obtained previously in this research area.

PRELIMINARIES

Throughout this section unless otherwise stated, C denotes a nonempty, closed and convex subset of a real Hilbert space H . For every point $x \in H$, there exists a unique nearest point in C , denoted by $P_C x$, such that $\|x - P_C x\| = \inf\{\|x - y\| : y \in C\}$. P_C is called the metric projection of H onto C . The following characterizes the metric projection P_C : for given $x \in H$ and $z \in C$,

$$z = P_C x \Leftrightarrow \langle x - z, z - y \rangle \geq 0, \quad \forall y \in C \dots\dots\dots (2.1)$$

Definition 2.1. Let $\{x_n\}$ be a sequence in C such that $x_n \rightarrow x$ and let $T: C \rightarrow CB(C)$ be a multi-valued mapping. Then, $(I - T)$ is said to be demiclosed at zero if $\lim_{n \rightarrow \infty} d(x_n, T x_n) = 0$ implies $x \in T x$, where I is the identity mapping on C .

We note that if the mapping $T: C \rightarrow C$ in Definition 2.1 is a single-valued nonexpansive mapping, then $(I - T)$ is demiclosed at zero (see Agrawal *et al.*, 2009).

In the proof of our main result, we also need the following lemmas.

Lemma 2.2. (Habtu Zegeye and Shahzad, 2011b). Let H be a real Hilbert space and $\{x_i\}_{i=1}^n \subset H$. Then, for $\alpha_i \in [0, 1], i = 1, 2, \dots, n$, such that $\alpha_1 + \alpha_2 + \dots + \alpha_n = 1$, we have the following identity:

$$\begin{aligned} & \|\alpha_1 x_1 + \alpha_2 x_2 + \dots + \alpha_n x_n\|^2 \\ &= \sum_{i=1}^n \alpha_i \|x_i\|^2 - \sum_{1 \leq i < j \leq n} \alpha_i \alpha_j \|x_i - x_j\|^2. \end{aligned}$$

Lemma 2.3. (Agrawal *et al.*, 2009). Let H be a real Hilbert space. Then, for every $x, y \in H$, we have the following:

- i) $\|x - y\|^2 = \|x\|^2 + \|y\|^2 - 2\langle x, y \rangle$;
- ii) $\|x + y\|^2 = \|x\|^2 + 2\langle y, x + y \rangle$.

Lemma 2.4. (Blum and Oettli, 1994; Combettes and Hirstoaga, 2005). Let F be a bifunction from $C \times C$ into \mathbb{R} which satisfies Assumption 1.1. For $r > 0$, define $T_r: H \rightarrow C$ as follows:

$$T_r x = \left\{ z \in C : F(z, y) + \frac{1}{r} \langle y - z, z - x \rangle, \forall y \in C \right\}.$$

Then, the following hold:

- (1) T_r is nonempty and single-valued;
- (2) T_r is firmly nonexpansive, i.e., $\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y \rangle, \forall x, y \in H$;
- (3) $F(T_r) = EP(F)$;
- (4) $EP(F)$ is closed and convex.

Lemma 2.5. (Nadler, 1969). Let (X, d) be a metric space and let $A, B \in CB(X)$. Then, for any $u \in A$ and $\varepsilon > 0$, there exists a point $v \in B$ such that $d(u, v) \leq D(A, B) + \varepsilon$. This implies that for every element $u \in A$, there exists an element $v \in B$ such that $d(u, v) \leq 2D(A, B)$.

Lemma 2.6. (Xu, 2002). Let $\{a_n\}$ be a sequence of nonnegative real numbers such that

$$a_{n+1} \leq (1 - \alpha_n)a_n + \alpha_n \delta_n, \text{ for } n \geq n_0,$$

where $\{\alpha_n\} \subset (0, 1)$ and $\{\delta_n\} \subset \mathbb{R}$ satisfying the following conditions:

$$\lim_{n \rightarrow \infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty, \text{ and } \limsup_{n \rightarrow \infty} \delta_n \leq 0.$$

Then, $\lim_{n \rightarrow \infty} a_n = 0$.

Lemma 2.7. (Mainge, 2008). Let $\{b_n\}$ be a sequence of real numbers such that there exists a subsequence $\{n_j\}$ of $\{n\}$ such that $b_{n_j} < b_{n_{j+1}}$, for all $j \in \mathbb{N}$. Then, there exists a nondecreasing sequence $\{n_k\} \subset \mathbb{N}$ such that $n_k \rightarrow \infty$ and the following properties are satisfied by all (sufficiently large) numbers $k \in \mathbb{N}$:

$$b_{n_k} \leq b_{n_{k+1}} \text{ and } b_k \leq b_{n_{k+1}}$$

In fact, $n_k = \max\{i \leq k : b_i \leq b_{i+1}\}$.

MAIN RESULT

In this section, we define an iterative algorithm and prove its strong convergence to a common solution of a finite family of generalized equilibrium problems and a fixed point problem for a multi-valued Lipschitz hemicontractive-type mapping.

Theorem 3.1. Let C be a nonempty, closed and convex subset of a real Hilbert space H . Let $T: C \rightarrow CB(C)$ be a Lipschitz hemicontractive-type multi-valued mapping with Lipschitz con-

stant L . Let $A_m: C \rightarrow H$ be a continuous monotone mapping and let $F_m: C \times C \rightarrow \mathbb{R}$ be a bifunction satisfying Assumption 1.1, for each $m \in \{1, 2, \dots, N\}$. Assume that $\Theta = \bigcap_{m=1}^N EP(F_m, A_m) \cap F(T)$ is nonempty and $Tq = \{q\}$ for all $q \in \Theta$. Let $\{r_{m,n}\} \subset (0, \infty)$ and let $\{a_n\}, \{b_n\}, \{c_n\}, \{e_n\}$ and $\{d_{m,n}\}$ be sequences in $(0, 1)$ such that

- i) $b_n + c_n + e_n = 1$;
- ii) $\sum_{m=1}^N d_{m,n} = 1$;
- iii) $b_n + c_n \leq a_n \leq d < \frac{1}{\sqrt{1+4L^2+1}}$.

Let $\{x_n\}$ be a sequence generated from an arbitrary $x_1, v \in C$ by

$$\begin{cases} F_m(y_{m,n}, z) + \langle A_m y_{m,n}, z - y_{m,n} \rangle \\ + \frac{1}{r_{m,n}} \langle z - y_{m,n}, y_{m,n} - x_n \rangle \geq 0, \forall z \in C, m = 1, 2, \dots, N, \\ w_n = \sum_{m=1}^N d_{m,n} y_{m,n}, \\ z_n = a_n v_n + (1 - a_n) w_n, \\ x_{n+1} = b_n v + c_n u_n + e_n w_n, \end{cases} \dots\dots\dots(3.1)$$

for all $n \geq 1$, where $v_n \in T w_n, u_n \in T z_n$ such that $\|v_n - u_n\| \leq 2D(T w_n, T z_n)$. Then, the sequence $\{x_n\}$ is bounded.

Proof. Let $q \in \Theta$. Then, we have $Tq = \{q\}$ and $F_m(q, z) + \langle A_m q, z - q \rangle \geq 0$, for all $m = 1, 2, \dots, N$ and $z \in C$. Define $G_m: C \times C \rightarrow \mathbb{R}$ by $G_m(x, z) := F_m(x, z) + \langle A_m x, z - x \rangle$ for all $x, z \in C$ and $m \in \{1, 2, \dots, N\}$. Then, in view of Remark 1.2, G_m is a bifunction satisfying Assumption 1.1, for each $m \in \{1, 2, \dots, N\}$ and $q \in EP(F_m, A_m)$ is equivalent to $G_m(q, z) \geq 0$ for all $z \in C$. Hence, using Lemma 2.4, $y_{n,m}$ can be rewritten as $y_{m,n} = T_{r_{m,n}} x_n$ and hence we obtain $q = T_{r_{m,n}} q$. In view of the fact that $T_{r_{m,n}}$ is nonexpansive, by Lemma 2.4, we have that

$$\|y_{m,n} - q\| = \|T_{r_{m,n}} x_n - T_{r_{m,n}} q\| \leq \|x_n - q\| \dots\dots(3.2)$$

Then, from (3.2) and condition (ii), we have the following:

$$\begin{aligned} \|w_n - q\| &= \left\| \sum_{m=1}^N d_{m,n} y_{m,n} - q \right\| \\ &= \left\| \sum_{m=1}^N d_{m,n} y_{m,n} - \sum_{m=1}^N d_{m,n} q \right\| \\ &\leq \sum_{m=1}^N d_{m,n} \|y_{m,n} - q\| \end{aligned}$$

$$\begin{aligned} &\leq \sum_{m=1}^N d_{m,n} \|x_n - q\| \\ &= \|x_n - q\|. \end{aligned} \dots\dots\dots(3.3)$$

Using Lemma 2.2, the fact that T is hemicontractive-type mapping and $v_n \in Tw_n$, we find that

$$\begin{aligned} \|z_n - q\|^2 &= \|a_n(v_n - q) + (1 - a_n)(w_n - q)\|^2 \\ &= a_n \|v_n - q\|^2 + (1 - a_n) \|w_n - q\|^2 \\ &\quad - a_n(1 - a_n) \|w_n - v_n\|^2 \\ &\leq a_n D^2(Tw_n, Tq) + (1 - a_n) \|w_n - q\|^2 \\ &\quad - a_n(1 - a_n) \|w_n - v_n\|^2 \\ &\leq a_n (\|w_n - q\|^2 + \|w_n - v_n\|^2) \\ &\quad + (1 - a_n) \|w_n - q\|^2 \\ &\quad - a_n(1 - a_n) \|w_n - v_n\|^2 \\ &= \|w_n - q\|^2 + a_n^2 \|w_n - v_n\|^2. \end{aligned}$$

From (3.3), it follows that

$$\|z_n - q\|^2 \leq \|x_n - q\|^2 + a_n^2 \|w_n - v_n\|^2. \dots\dots\dots(3.4)$$

In addition, since T is hemicontractive-type mapping and $u_n \in Tz_n$, from (3.1) and (3.4), we get that

$$\begin{aligned} \|u_n - q\|^2 &\leq D^2(Tz_n, Tq) \\ &\leq \|z_n - q\|^2 + \|z_n - u_n\|^2 \\ &\leq \|x_n - q\|^2 + a_n^2 \|w_n - v_n\|^2 + \|z_n - u_n\|^2 \\ &\dots\dots\dots(3.5) \end{aligned}$$

It follows from (3.1) that

$$\begin{aligned} \|w_n - z_n\|^2 &= \|w_n - (a_n v_n + (1 - a_n)w_n)\|^2 \\ &= a_n^2 \|w_n - v_n\|^2. \dots\dots\dots(3.6) \end{aligned}$$

Thus, since $\|v_n - u_n\| \leq 2D(Tw_n, Tz_n)$ and T is a L -Lipschitzian mapping, from (3.6) and Lemma 2.2, we get that

$$\begin{aligned} \|z_n - u_n\|^2 &= \|a_n v_n + (1 - a_n)w_n - u_n\|^2 \\ &= a_n \|v_n - u_n\|^2 + (1 - a_n) \|w_n - u_n\|^2 \\ &\quad - a_n(1 - a_n) \|w_n - v_n\|^2 \\ &\leq (1 - a_n) \|w_n - u_n\|^2 + 4a_n D^2(Tw_n, Tz_n) \\ &\quad - a_n(1 - a_n) \|w_n - v_n\|^2 \\ &\leq (1 - a_n) \|w_n - u_n\|^2 + 4a_n L^2 \|w_n - z_n\|^2 \\ &\quad - a_n(1 - a_n) \|w_n - v_n\|^2 \\ &= (1 - a_n) \|w_n - u_n\|^2 + 4a_n^3 L^2 \|w_n - v_n\|^2 \\ &\quad - a_n(1 - a_n) \|w_n - v_n\|^2 \\ &= (1 - a_n) \|w_n - u_n\|^2 \\ &\quad + a_n(4L^2 a_n^2 + a_n - 1) \|w_n - v_n\|^2. \dots\dots\dots(3.7) \end{aligned}$$

Hence, substituting (3.7) into (3.5), we have that

$$\begin{aligned} \|u_n - q\|^2 &\leq \|x_n - q\|^2 + a_n^2 \|w_n - v_n\|^2 \\ &\quad + (1 - a_n) \|w_n - u_n\|^2 \\ &\quad + a_n(4L^2 a_n^2 + a_n - 1) \|w_n - v_n\|^2 \end{aligned}$$

$$\begin{aligned} &= \|x_n - q\|^2 + (1 - a_n) \|w_n - u_n\|^2 \\ &\quad + a_n(4L^2 a_n^2 + 2a_n - 1) \|w_n - v_n\|^2. \dots\dots\dots(3.8) \end{aligned}$$

Thus, from (3.3), (3.8), Lemma 2.2 and condition (i), we obtain that

$$\begin{aligned} \|x_{n+1} - q\|^2 &= \|b_n v + c_n u_n + e_n w_n - q\|^2 \\ &\leq b_n \|v - q\|^2 + c_n \|u_n - q\|^2 \\ &\quad + e_n \|w_n - q\|^2 - c_n e_n \|w_n - u_n\|^2 \\ &\leq b_n \|v - q\|^2 + c_n (\|x_n - q\|^2 \\ &\quad + (1 - a_n) \|w_n - u_n\|^2 \\ &\quad + a_n(4L^2 a_n^2 + 2a_n - 1) \|w_n - v_n\|^2) \\ &\quad + e_n \|w_n - q\|^2 - c_n e_n \|w_n - u_n\|^2 \\ &\leq b_n \|v - q\|^2 + (1 - b_n) \|x_n - q\|^2 \\ &\quad + c_n(1 - e_n - a_n) \|w_n - u_n\|^2 \\ &\quad - c_n a_n(1 - 4L^2 a_n^2 - 2a_n) \|w_n - v_n\|^2 \\ &= b_n \|v - q\|^2 + (1 - b_n) \|x_n - q\|^2 \\ &\quad - c_n a_n(1 - 4L^2 a_n^2 - 2a_n) \|w_n - v_n\|^2 \\ &\quad + c_n(b_n + c_n - a_n) \|w_n - u_n\|^2, \dots\dots\dots(3.9) \end{aligned}$$

and from condition (iii), we have that $1 - 4L^2 a_n^2 - 2a_n \geq 1 - 4L^2 d^2 - 2d > 0$ and $b_n + c_n - a_n \leq 0$, for all $n \geq 1$. $\dots\dots\dots(3.10)$

Thus, from (3.9) and (3.10), we find that

$$\begin{aligned} \|x_{n+1} - q\|^2 &\leq b_n \|v - q\|^2 + (1 - b_n) \|x_n - q\|^2 \\ &\leq \max\{\|v - q\|^2, \|x_n - q\|^2\}. \end{aligned}$$

Hence, by induction, the sequence $\{x_n\}$ is bounded. This completes the proof. \square

Theorem 3.2. Let C be a nonempty, closed and convex subset of a real Hilbert space H . Let $T: C \rightarrow CB(C)$ be a Lipschitz hemicontractive-type multi-valued mapping with Lipschitz constant L . Let $A_m: C \rightarrow H$ be a continuous monotone mapping and let $F_m: C \times C \rightarrow \mathbb{R}$ be a bifunction satisfying Assumption 1.1, for each $m \in \{1, 2, \dots, N\}$. Assume that $\theta = \bigcap_{m=1}^N EP(F_m, A_m) \cap F(T)$ is nonempty, closed and convex, $(I - T)$ is demiclosed at zero and $Tq = \{q\}$ for all $q \in \theta$. Let $\{r_{m,n}\} \subset (0, \infty)$ such that $\lim_{n \rightarrow \infty} r_{m,n} = r_m$ for some $0 < r_m < \infty$ and for each $m = 1, 2, \dots, N$, and let $\{a_n\}, \{b_n\}, \{c_n\}, \{e_n\}$ and $\{d_{m,n}\}$ be sequences in $(0, 1)$ such that

- i) $b_n + c_n + e_n = 1$ and $0 < a \leq c_n, e_n \leq b < 1$;
- ii) $\lim_{n \rightarrow \infty} b_n = 0, \sum_{n=1}^{\infty} b_n = \infty$;
- iii) $\sum_{m=1}^N d_{m,n} = 1$ and $0 < c \leq d_{m,n} \leq 1$;
- iv) $b_n + c_n \leq a_n \leq d < \frac{1}{\sqrt{1+4L^2+1}}$.

Let $\{x_n\}$ be a sequence generated from an arbitrary $x_1, v \in C$ by

$$\left\{ \begin{array}{l} F_m(y_{m,n}, z) + \langle A_m y_{m,n}, z - y_{m,n} \rangle \\ + \frac{1}{r_{m,n}} \langle z - y_{m,n}, y_{m,n} - x_n \rangle \geq 0, \forall z \in C, m = 1, 2, \dots, N, \\ w_n = \sum_{m=1}^N d_{m,n} y_{m,n}, \\ z_n = a_n v_n + (1 - a_n) w_n, \\ x_{n+1} = b_n v + c_n u_n + e_n w_n, \end{array} \right. \dots\dots\dots (3.11)$$

for all $n \geq 1$, where $v_n \in T w_n, u_n \in T z_n$ such that $\|v_n - u_n\| \leq 2D(T w_n, T z_n)$. Then, the sequence $\{x_n\}$ converges strongly to $p = P_\theta(v)$.

Proof. Since θ is nonempty, closed and convex subset of H , then we see that P_θ is well defined. Obviously, from Theorem 3.1 the sequence $\{x_n\}$ and hence $\{y_{m,n}\}, \{w_n\}$ and $\{z_n\}$ are bounded. Now, let $q \in \theta$. Then, using the fact that $T_{r_{m,n}}$ is firmly nonexpansive and $T_{r_{m,n}} q = q$, for all $m = 1, 2, \dots, N$, and Lemma 2.3 (i), we find that

$$\begin{aligned} & \|y_{m,n} - q\|^2 \\ &= \|T_{r_{m,n}} x_n - T_{r_{m,n}} q\|^2 \\ &\leq \langle y_{m,n} - q, x_n - q \rangle \\ &= \frac{1}{2} (\|y_{m,n} - q\|^2 + \|x_n - q\|^2 - \|y_{m,n} - x_n\|^2), \end{aligned}$$

which implies that

$$\|y_{m,n} - q\|^2 \leq \|x_n - q\|^2 - \|y_{m,n} - x_n\|^2.$$

This gives that

$$\begin{aligned} \|w_n - q\|^2 &\leq \sum_{m=1}^N d_{m,n} \|y_{m,n} - q\|^2 \\ &\leq \|x_n - q\|^2 - \sum_{m=1}^N d_{m,n} \|y_{m,n} - x_n\|^2. \end{aligned} \dots\dots\dots (3.12)$$

On the other hand, from (3.11), Lemma 2.2 and Lemma 2.3 (ii), we have

$$\begin{aligned} \|x_{n+1} - q\|^2 &= \|b_n v + c_n u_n + e_n w_n - q\|^2 \\ &\leq \|c_n (u_n - q) + e_n (w_n - q)\|^2 \\ &\quad + 2b_n \langle v - q, x_{n+1} - q \rangle \\ &\leq c_n \|u_n - q\|^2 + e_n \|w_n - q\|^2 \\ &\quad - c_n e_n \|w_n - u_n\|^2 \\ &\quad + 2b_n \langle v - q, x_{n+1} - q \rangle. \dots\dots\dots (3.13) \end{aligned}$$

Thus, substituting (3.8) and (3.12) into (3.13), we obtain that

$$\begin{aligned} & \|x_{n+1} - q\|^2 \\ &\leq c_n (\|x_n - q\|^2 + a_n (4L^2 a_n^2 + 2a_n - 1) \|w_n - v_n\|^2) \\ &\quad + c_n (1 - a_n) \|w_n - u_n\|^2 \\ &\quad + e_n (\|x_n - q\|^2 - \sum_{m=1}^N d_{m,n} \|y_{m,n} - x_n\|^2) \\ &\quad - c_n e_n \|w_n - u_n\|^2 + 2b_n \langle v - q, x_{n+1} - q \rangle \end{aligned}$$

$$\begin{aligned} &= (1 - b_n) \|x_n - q\|^2 \\ &\quad - c_n a_n (1 - 4L^2 a_n^2 - 2a_n) \|w_n - v_n\|^2 \\ &\quad + c_n (b_n + c_n - a_n) \|w_n - u_n\|^2 \\ &\quad - e_n \sum_{m=1}^N d_{m,n} \|y_{m,n} - x_n\|^2 \\ &\quad + 2b_n \langle v - q, x_{n+1} - q \rangle \dots\dots\dots (3.14) \end{aligned}$$

Now, we consider the following two cases:

Case 1. Suppose that there exists $n_0 \in \mathbb{N}$ such that $\{\|x_n - q\|\}_{n \geq n_0}$ is nonincreasing sequence. Then, the boundedness of $\{\|x_n - q\|\}$ implies that $\{\|x_n - q\|\}$ is convergent. From (3.10) and (3.14), it follows that

$$\begin{aligned} & c_n a_n (1 - 4L^2 a_n^2 - 2a_n) \|w_n - v_n\|^2 \\ &\leq (1 - b_n) \|x_n - q\|^2 - \|x_{n+1} - q\|^2 \\ &\quad + 2b_n \langle v - q, x_{n+1} - q \rangle \end{aligned}$$

Thus, from (3.10), the assumptions of $\{c_n\}$ and $\{a_n\}$, and the fact that $b_n \rightarrow 0$ as $n \rightarrow \infty$ and $\{x_n\}$ is bounded, we find that

$$\|w_n - v_n\| \rightarrow 0 \text{ as } n \rightarrow \infty. \dots\dots\dots (3.15)$$

This implies that

$$\lim_{n \rightarrow \infty} d(w_n, T w_n) = 0. \dots\dots\dots (3.16)$$

From (3.10) and (3.14), we see that

$$\begin{aligned} e_n d_{m,n} \|y_{m,n} - x_n\|^2 &\leq (1 - b_n) \|x_n - q\|^2 \\ &\quad - \|x_{n+1} - q\|^2 \\ &\quad + 2b_n \langle v - q, x_{n+1} - q \rangle. \end{aligned}$$

This together with conditions (i), (ii) and (iii) imply that

$$\lim_{n \rightarrow \infty} \|y_{m,n} - x_n\| = 0. \dots\dots\dots (3.17)$$

In addition, from the Lipschitz condition of T , (3.6) and (3.15), we get that

$$\begin{aligned} \|w_n - u_n\| &\leq \|w_n - v_n\| + \|v_n - u_n\| \\ &\leq \|w_n - v_n\| + 2L \|w_n - z_n\| \\ &= \|w_n - v_n\| + 2L a_n \|w_n - v_n\|. \end{aligned}$$

Hence

$$\lim_{n \rightarrow \infty} \|w_n - u_n\| = 0. \dots\dots\dots (3.18)$$

Again from (3.11) and triangle inequality, we get that

$$\begin{aligned} \|x_{n+1} - x_n\| &\leq \|x_{n+1} - w_n\| + \|w_n - x_n\| \\ &= \|b_n (v - w_n) + c_n (u_n - w_n)\| \\ &\quad + \|\sum_{m=1}^N d_{m,n} y_{m,n} - x_n\| \\ &\leq b_n \|v - w_n\| + c_n \|u_n - w_n\| \\ &\quad + \sum_{m=1}^N d_{m,n} \|y_{m,n} - x_n\|. \end{aligned}$$

Therefore, from (3.17), (3.18) and the fact that $b_n \rightarrow 0$ as $n \rightarrow \infty$, we obtain

$$\lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = 0. \dots\dots\dots (3.19)$$

It follows from (3.10) and (3.14) that $\|x_{n+1} - q\|^2 \leq (1 - b_n)\|x_n - q\|^2 + 2b_n \langle v - q, x_{n+1} - q \rangle$(3.20)

Now, let $p = P_\theta(v)$. Then, we show that $\limsup_{n \rightarrow \infty} \langle v - p, x_{n+1} - p \rangle \leq 0$. Since $\{x_{n+1}\}$ is a bounded sequence in a real Hilbert space H , which is a reflexive Banach space, then there exists a subsequence $\{x_{n_i+1}\}$ of $\{x_{n+1}\}$ such that $x_{n_i+1} \rightarrow y$ as $i \rightarrow \infty$ and

$$\limsup_{n \rightarrow \infty} \langle v - p, x_{n_i+1} - p \rangle = \lim_{i \rightarrow \infty} \langle v - p, x_{n_i+1} - p \rangle.$$

Since, C is weakly closed, we have $y \in C$ and from (3.19), it follows that $x_{n_i} \rightarrow y$ as $i \rightarrow \infty$.

Using (3.17), we see that $\|w_{n_i} - x_{n_i}\| \rightarrow 0$ as $i \rightarrow \infty$ and so $w_{n_i} \rightarrow y$ as $i \rightarrow \infty$. Thus, demiclosedness of $(I - T)$ at zero and (3.16) imply that $y \in F(T)$.

On the other hand, from the fact that $(I - T_{r_m, n_i})$ is demiclosed at zero and (3.17), we obtain that $y = T_{r_m, n_i}y$ and thus $y \in EP(F_m, A_m)$, for all $m \in \{1, 2, \dots, N\}$. That is,

$$y \in \bigcap_{m=1}^N EP(F_m, A_m).$$

Therefore, $y \in \theta$. Hence, since $p = P_\theta(v)$ and $x_{n_i} \rightarrow y$ as $i \rightarrow \infty$, from the property of a metric projection, equation (2.1), we have that

$$\limsup_{n \rightarrow \infty} \langle v - p, x_{n_i+1} - p \rangle = \lim_{i \rightarrow \infty} \langle v - p, x_{n_i+1} - p \rangle = \langle v - p, y - p \rangle \leq 0. \dots\dots\dots(3.21)$$

Thus, since $p \in \theta$, from (3.20), (3.21), condition (ii) and Lemma 2.6, we conclude that $\|x_n - p\| \rightarrow 0$ as $n \rightarrow \infty$. That is, $x_n \rightarrow p = P_\theta(v)$.

Case 2. Suppose that there exists a subsequence $\{n_j\}$ of $\{n\}$ such that $\|x_{n_j} - q\| < \|x_{n_j} - q\|$, for all $j \in \mathbb{N}$. Then, by Lemma 2.7, there exists a nondecreasing sequence $\{n_k\} \subset \mathbb{N}$ such that $n_k \rightarrow \infty$, and

$$\|x_{n_k} - q\| \leq \|x_{n_{k+1}} - q\| \text{ and } \|x_k - q\| \leq \|x_{n_{k+1}} - q\|, \dots\dots\dots(3.22)$$

for all $k \in \mathbb{N}$. Thus, replacing n by n_k and using (3.22), (3.14), (3.10) and the fact that $b_n \rightarrow 0$ as $n \rightarrow \infty$, we find that $\|w_{n_k} - v_{n_k}\| \rightarrow 0$ and $\|y_{m, n_k} - x_{n_k}\| \rightarrow 0$ as $k \rightarrow \infty$.

Hence, following an argument similar to that in Case 1, we obtain that

$$\limsup_{i \rightarrow \infty} \langle v - p, x_{n_{k+1}} - p \rangle \leq 0 \dots\dots\dots(3.23)$$

Now, since $p \in \theta$, from (3.20), we get that $\|x_{n_{k+1}} - p\|^2 \leq (1 - b_{n_k})\|x_{n_k} - p\|^2 + 2b_{n_k} \langle v - p, x_{n_{k+1}} - p \rangle$ (3.24)

and hence, since $p \in \theta$, (3.22) and (3.24) imply that

$$b_{n_k} \|x_{n_k} - p\|^2 \leq \|x_{n_k} - p\|^2 - \|x_{n_{k+1}} - p\|^2 + 2b_{n_k} \langle v - p, x_{n_{k+1}} - p \rangle \leq 2b_{n_k} \langle v - p, x_{n_{k+1}} - p \rangle.$$

Then, from the fact that $b_{n_k} > 0$, we have that $\|x_{n_k} - p\|^2 \leq 2 \langle v - p, x_{n_{k+1}} - p \rangle$. It follows from (3.23) that $\|x_{n_k} - p\| \rightarrow 0$ as $k \rightarrow \infty$.

This together with (3.24) and the fact that $b_{n_k} \rightarrow 0$ imply that $\|x_{n_{k+1}} - p\| \rightarrow 0$ as $k \rightarrow \infty$. Since $p \in \theta$, we get that $\|x_k - p\| \leq \|x_{n_{k+1}} - p\|$ for all $k \in \mathbb{N}$. Thus, we obtain that $x_k \rightarrow p$ as $k \rightarrow \infty$. Therefore, we conclude that the sequence $\{x_n\}$ generated by (3.11) converges strongly to the point $p = P_\theta(v)$. This completes the proof. \square

As a direct consequence of our main result, we obtain the following results.

Corollary 3.3. Let C be a nonempty, closed and convex subset of a real Hilbert space H . Let $T: C \rightarrow CB(C)$ be a Lipschitz pseudocontractive multi-valued mapping with Lipschitz constant L . Let $A_m: C \rightarrow H$ be a continuous monotone mapping and let $F_m: C \times C \rightarrow \mathbb{R}$ be a bifunction satisfying Assumption 1.1, for each $m \in \{1, 2, \dots, N\}$. Assume that $\theta = \bigcap_{m=1}^N EP(F_m, A_m) \cap F(T)$ is nonempty, closed and convex, $(I - T)$ is demiclosed at zero and $Tq = \{q\}$ for all $q \in F(T)$. Let $\{r_{m, n}\} \subset (0, \infty)$ such that $\lim_{n \rightarrow \infty} r_{m, n} = r_m$ for some $0 < r_m < \infty$ and for each $m = 1, 2, \dots, N$, and let $\{a_n\}, \{b_n\}, \{c_n\}, \{e_n\}$ and $\{d_{n, m}\}$ be sequences in $(0, 1)$ such that

- i) $b_n + c_n + e_n = 1$ and $0 < a \leq c_n, e_n \leq b < 1$;
- ii) $\lim_{n \rightarrow \infty} b_n = 0, \sum_{n=1}^\infty b_n = \infty$;
- iii) $\sum_{m=1}^N d_{m, n} = 1$ and $0 < c \leq d_{m, n} \leq 1$;
- iv) $b_n + c_n \leq a_n \leq d < \frac{1}{\sqrt{1+4L^2+1}}$.

Let $\{x_n\}$ be a sequence generated from an arbitrary $x_1, v \in C$ by

$$\begin{cases} F_m(y_{m,n}, z) + \langle A_m y_{m,n}, z - y_{m,n} \rangle \\ + \frac{1}{r_{m,n}} \langle z - y_{m,n}, y_{m,n} - x_n \rangle \geq 0, \forall z \in C, m = 1, 2, \dots, N, \\ w_n = \sum_{m=1}^N d_{m,n} y_{m,n}, \\ z_n = a_n v_n + (1 - a_n) w_n, \\ x_{n+1} = b_n v + c_n u_n + e_n w_n, \end{cases}$$

for all $n \geq 1$, where $v_n \in Tw_n, u_n \in Tz_n$ such that $\|v_n - u_n\| \leq 2D(Tw_n, Tz_n)$. Then, the sequence $\{x_n\}$ converges strongly to $p = P_\theta(v)$.

Proof. Since a Lipschitz pseudocontractive multi-valued mapping T with $F(T) \neq \emptyset$ and $Tq = \{q\}$, $\forall q \in F(T)$ is Lipschitz hemicontractive-type mapping, we obtain the desired result from Theorem 3.2. \square

If, in Theorem 3.2, we assume that $A_m \equiv 0$, for all $m \in \{1, 2, \dots, N\}$, then we obtain the following corollary on a finite family of equilibrium problems and fixed point problem of multi-valued Lipschitz hemicontractive-type mapping.

Corollary 3.4. Let C be a nonempty, closed and convex subset of a real Hilbert space H . Let $T: C \rightarrow CB(C)$ be a Lipschitz hemicontractive-type multi-valued mapping with Lipschitz constant L . Let $F_m: C \times C \rightarrow \mathbb{R}$ be a bifunction satisfying Assumption 1.1, for each $m \in \{1, 2, \dots, N\}$. Assume that $\theta = \bigcap_{m=1}^N EP(F_m) \cap F(T)$ is nonempty, closed and convex, $(I - T)$ is demiclosed at zero and $Tq = \{q\}$ for all $q \in \theta$. Let $\{r_{m,n}\} \subset (0, \infty)$ such that $\lim_{n \rightarrow \infty} r_{m,n} = r_m$ for some $0 < r_m < \infty$ and for each $m = 1, 2, \dots, N$, and let $\{a_n\}, \{b_n\}, \{c_n\}, \{e_n\}$ and $\{d_{m,n}\}$ be sequences in $(0, 1)$ such that

- i) $b_n + c_n + e_n = 1$ and $0 < a \leq c_n, e_n \leq b < 1$;
- ii) $\lim_{n \rightarrow \infty} b_n = 0, \sum_{n=1}^{\infty} b_n = \infty$;
- iii) $\sum_{m=1}^N d_{m,n} = 1$ and $0 < c \leq d_{m,n} \leq 1$;
- iv) $b_n + c_n \leq a_n \leq d < \frac{1}{\sqrt{1+4L^2+1}}$.

Let $\{x_n\}$ be a sequence generated from an arbitrary $x_1, v \in C$ by

$$\begin{cases} F_m(y_{m,n}, z) + \frac{1}{r_{m,n}} \langle z - y_{m,n}, y_{m,n} - x_n \rangle \geq 0, \\ \quad \forall z \in C, m = 1, 2, \dots, N, \\ w_n = \sum_{m=1}^N d_{m,n} y_{m,n}, \\ z_n = a_n v_n + (1 - a_n) w_n, \\ x_{n+1} = b_n v + c_n u_n + e_n w_n, \end{cases}$$

for all $n \geq 1$, where $v_n \in Tw_n, u_n \in Tz_n$ such that

$\|v_n - u_n\| \leq 2D(Tw_n, Tz_n)$. Then, the sequence $\{x_n\}$ converges strongly to $p = P_\theta(v)$.

If, in Theorem 3.2, we assume that $F_m \equiv 0$, for all $m \in \{1, 2, \dots, N\}$, then we have the following result on the problem of finding a common point of the common solution set of a finite family of variational inequality problems and fixed point set of Lipschitz hemicontractive-type mapping.

Corollary 3.5. Let C be a nonempty, closed and convex subset of a real Hilbert space H . Let $T: C \rightarrow CB(C)$ be a Lipschitz hemicontractive-type multi-valued mapping with Lipschitz constant L . Let $A_m: C \rightarrow H$ be a continuous monotone mapping, for each $m \in \{1, 2, \dots, N\}$. Assume that $\theta = \bigcap_{m=1}^N VI(C, A_m) \cap F(T)$ is nonempty, closed and convex, $(I - T)$ is demiclosed at zero and $Tq = \{q\}$ for all $q \in \theta$. Let $\{r_{m,n}\} \subset (0, \infty)$ such that $\lim_{n \rightarrow \infty} r_{m,n} = r_m$ for some $0 < r_m < \infty$ and for each $m = 1, 2, \dots, N$, and let $\{a_n\}, \{b_n\}, \{c_n\}, \{e_n\}$ and $\{d_{m,n}\}$ be sequences in $(0, 1)$ such that $b_n + c_n + e_n = 1$ and $0 < a \leq c_n, e_n \leq b < 1$;

- i) $\lim_{n \rightarrow \infty} b_n = 0, \sum_{n=1}^{\infty} b_n = \infty$;
- ii) $\sum_{m=1}^N d_{m,n} = 1$ and $0 < c \leq d_{m,n} \leq 1$;
- iii) $b_n + c_n \leq a_n \leq d < \frac{1}{\sqrt{1+4L^2+1}}$.

Let $\{x_n\}$ be a sequence generated from an arbitrary $x_1, v \in C$ by

$$\begin{cases} \langle A_m y_{m,n}, z - y_{m,n} \rangle + \frac{1}{r_{m,n}} \langle z - y_{m,n}, y_{m,n} - x_n \rangle \geq 0, \\ \quad \forall z \in C, m = 1, 2, \dots, N, \\ w_n = \sum_{m=1}^N d_{m,n} y_{m,n}, \\ z_n = a_n v_n + (1 - a_n) w_n, \\ x_{n+1} = b_n v + c_n u_n + e_n w_n, \end{cases}$$

for all $n \geq 1$, where $v_n \in Tw_n, u_n \in Tz_n$ such that $\|v_n - u_n\| \leq 2D(Tw_n, Tz_n)$. Then, the sequence $\{x_n\}$ converges strongly to $p = P_\theta(v)$.

If, in Theorem 3.2, we assume that T is a single-valued hemicontractive mapping from C into itself, then we obtain the following result.

Corollary 3.6. Let C be a nonempty, closed and convex subset of a real Hilbert space H . Let $T: C \rightarrow C$ be a Lipschitz hemicontractive mapping with Lipschitz constant L . Let $A_m: C \rightarrow H$ be a continuous monotone mapping and let $F_m: C \times C \rightarrow \mathbb{R}$ be a bifunction satisfying Assumption 1.1 for each $m \in \{1, 2, \dots, N\}$. Assume that $\theta = \bigcap_{m=1}^N EP(F_m, A_m) \cap F(T)$ is nonempty, closed and convex, $(I - T)$ is demiclosed at zero. Let $\{r_{m,n}\} \subset (0, \infty)$ such that $\lim_{n \rightarrow \infty} r_{m,n} = r_m$ for some $0 < r_m < \infty$ and for each $m = 1, 2, \dots, N$, and let $\{a_n\}, \{b_n\}, \{c_n\}, \{e_n\}$ and $\{d_{m,n}\}$ be sequences in $(0, 1)$ such that

- i) $b_n + c_n + e_n = 1$ and $0 < a \leq c_n, e_n \leq b < 1$;
- ii) $\lim_{n \rightarrow \infty} b_n = 0$, $\sum_{n=1}^{\infty} b_n = \infty$;
- iii) $\sum_{m=1}^N d_{m,n} = 1$ and $0 < c \leq d_{m,n} \leq 1$;
- iv) $b_n + c_n \leq a_n \leq d < \frac{1}{\sqrt{1+4L^2+1}}$.

Let $\{x_n\}$ be a sequence generated from an arbitrary $x_1, v \in C$ by

$$\begin{cases} F_m(y_{n,m}, z) + \langle A_m y_{n,m}, z - y_{n,m} \rangle \\ + \frac{1}{r_{m,n}} \langle z - y_{n,m}, y_{n,m} - x_n \rangle \geq 0, \forall z \in C, m = 1, 2, \dots, N, \\ w_n = \sum_{m=1}^N d_{m,n} y_{n,m}, \\ z_n = a_n T w_n + (1 - a_n) w_n, \\ x_{n+1} = b_n v + c_n T z_n + e_n w_n, \end{cases}$$

for all $n \geq 1$. Then, the sequence $\{x_n\}$ converges strongly to $p = P_{\theta}(v)$.

If, in Theorem 3.2, we assume that $N = 1$, then we get the following corollary.

Corollary 3.7. (Habtu Zegeye *et al.*, 2017). Let C be a nonempty, closed and convex subset of a real Hilbert space H . Let $T: C \rightarrow CB(C)$ be a Lipschitz hemicontractive-type multi-valued mapping with Lipschitz constant L . Let $A: C \rightarrow H$ be a continuous monotone mapping and let $F: C \times C \rightarrow \mathbb{R}$ be a bifunction satisfying Assumption 1.1. Assume that $\theta = EP(F, A) \cap F(T)$ is nonempty, closed and convex, $(I - T)$ is demiclosed at zero and $Tq = \{q\}$ for all $q \in \theta$. Let $\{r_n\} \subset (0, \infty)$ such that $\lim_{n \rightarrow \infty} r_n = r$ for some $0 < r < \infty$, and let $\{a_n\}, \{b_n\}, \{c_n\}$, and $\{e_n\}$ be sequences in $(0, 1)$ such that

- i) $b_n + c_n + e_n = 1$ and $0 < a \leq c_n, e_n \leq b < 1$;
- ii) $\lim_{n \rightarrow \infty} b_n = 0$, $\sum_{n=1}^{\infty} b_n = \infty$;
- iii) $b_n + c_n \leq a_n \leq d < \frac{1}{\sqrt{1+4L^2+1}}$.

Let $\{x_n\}$ be a sequence generated from an arbitrary $x_1, v \in C$ by

$$\begin{cases} F(y_n, z) + \langle A y_n, z - y_n \rangle \\ + \frac{1}{r_n} \langle z - y_n, y_n - x_n \rangle \geq 0, \forall z \in C, \\ z_n = a_n v_n + (1 - a_n) y_n, \\ x_{n+1} = b_n v + c_n u_n + e_n y_n, \end{cases}$$

for all $n \geq 1$, where $v_n \in T y_n, u_n \in T z_n$ such that $\|v_n - u_n\| \leq 2D(T y_n, T z_n)$. Then, the sequence $\{x_n\}$ converges strongly to $p = P_{\theta}(v)$.

If, in Theorem 3.2, we assume that $T = I$, where I is the identity mapping on C , then we obtain the following corollary on a finite family of generalized equilibrium problems.

Corollary 3.8. Let C be a nonempty, closed and convex subset of a real Hilbert space H . Let $A_m: C \rightarrow H$ be a continuous monotone mapping and let $F_m: C \times C \rightarrow \mathbb{R}$ be a bifunction satisfying

Assumption 1.1, for each $m \in \{1, 2, \dots, N\}$. Assume that $\theta = \bigcap_{m=1}^N EP(F_m, A_m) \cap F(T)$ is nonempty. Let $\{r_{m,n}\} \subset (0, \infty)$ such that $\lim_{n \rightarrow \infty} r_{m,n} = r_m$ for some $0 < r_m < \infty$ and for each $m = 1, 2, \dots, N$, and let $\{b_n\}$ and $\{d_{m,n}\}$ be sequences in $(0, 1)$ such that

- i) $\lim_{n \rightarrow \infty} b_n = 0$, $\sum_{n=1}^{\infty} b_n = \infty$;
- ii) $\sum_{m=1}^N d_{m,n} = 1$ and $0 < c \leq d_{m,n} \leq 1$

Let $\{x_n\}$ be a sequence generated from an arbitrary $x_1, v \in C$ by

$$\begin{cases} F_m(y_{m,n}, z) + \langle A_m y_{m,n}, z - y_{m,n} \rangle \\ + \frac{1}{r_{m,n}} \langle z - y_{m,n}, y_{m,n} - x_n \rangle \geq 0, \forall z \in C, m = 1, 2, \dots, N, \\ w_n = \sum_{m=1}^N d_{m,n} y_{m,n}, \\ x_{n+1} = b_n v + (1 - b_n) w_n, \end{cases}$$

for all $n \geq 1$. Then, the sequence $\{x_n\}$ converges strongly to $p = P_{\theta}(v)$.

Remark 3.9. Theorem 3.2 extends the results of Tesfalem Hadush *et al.* (2016), Habtu Zegeye *et al.* (2017), Razani and Yazdi (2012), Hao (2011), Wang *et al.* (2007), Ceng *et al.* (2010), Huang and Ma (2014) in the sense that our iterative algorithm provides strong convergence to a common element of the set of common solutions of a finite family of generalized equilibrium problems and the set of fixed points of a Lipschitz hemicontractive-type multi-valued mapping. We have used the demiclosedness principle in the proof of our Theorem 3.2 which makes a little simpler than using Assumption 1.1.

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