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Short communication

NOTE ON FINITE p - GROUPS

Haile Michael Bereda

University Paul Sabatier, Mathematiqus 118, Route de Narbonne
31062 Toulouse Cedex 4, France

ABSTRACT: In this note, we consider finite non-abelian p-groups ($p \geq 3$) in which the derived group is cyclic. As far as we know, these groups have not yet been classified. This will be done in a forthcoming paper.

The notation and terminology employed will be as follows. If G is a p-group, G' stands for the derived group of G. A subgroup of G is of type (p,p) if it is elementary abelian of order p^2 G is said to be regular if for every pair of elements a, b in G,

$$(ab)^p = a^p b^p c^p,$$

Where c is an element of the derived group of the subgroup generated by a and b. $[a, b] = a^{-1}b^{-1}ab$ as usual, for a, b in G. For positive integers m and n, $m|n$ means m divides n. Given a real number r, i(r) is the integer parts of r. If M and N are isomorphic groups, we write $M \approx N$. M_n ($n \geq 1$) is the n^{th} term of the descending central series of M. Z is the set of rational integers.

We use the following elementary but basic fact. If G is a finite nilpotent group, every normal subgroup of G different from the identity subgroup, {1}, intersects the centre Z(G) of G non trivially.

We need the following lemmas.

Lemma 1

Let a, b be pair of elements of a group G.

a) If [a,b] commutes with a, then for all $n \in \mathbf{Z}$,

$$[a^n,b]=[a,b]^n.$$

b) If [a,b] commutes with a and b, then for all positive integers n,

$$(ab)^n = a^n b^n \, [b,a]^{\binom{n}{2}}$$

where $\binom{n}{2}$ is the binomial coefficient $\frac{n(n-1)}{2}$.

Proof

(a) We proceed by induction on $n>0$ since $[a^0,b]=[1,b]=1=[a,b]^0$. For $n=1$, there is nothing to prove. Suppose $n>1$ and the assertion true for n-1. Then,

$$[a^n,b]=[aa^{n-1},b]=a^{-n+1}\,[a,b]a^{n-1}\,[a^{n-1},b]=[a,b]\,[a^{n-1},b]=[a,b]\,[a,b]^{n-1}=[a,b]^n.$$

Furthermore,

$1=[a^na^{-n},b]=a^n[a^n,b]a^{-n}[a^{-n},b]=a^n[a,b]^na^{-n}[a^{-n},b]=[a,b]^n[a^{-n},b]$, hence $[a^{-n},b]=[a,b]^{-n}$.

(b) For n = 1, the assertion is trivial since $\genfrac{}{}{0pt}{}{1}{2}$ 0 by convention. Suppose n > 1 and the assertion true for n-1. Then,

$$\begin{aligned}
(ab)^n &= (ab)^{n-1}ab = a^{n-1}b^{n-1}[b,a]^{\binom{n-1}{2}}ab = a^{n-1}b^{n-1}ab[b,a]^{\binom{n-1}{2}}\\
&= a^{n-1}ab^{n-1}b^{-(n-1)}a^{-1}b^{n-1}ab[b,a]^{\binom{n-1}{2}} = a^nb^{n-1}[b^{n-1},a]b[b,a]^{\binom{n-1}{2}}\\
&= a^nb^{n-1}[b,a]^{n-1}b[b,a]^{\binom{n-1}{2}} = a^nb^n[b,a]^{\binom{n}{2}}
\end{aligned}$$

Lemma 2

Let M and N be normal subgroups of a p-group G such that $N \subset M$ and $|M/N| = p^m$. Then, for all integers k satisfying $0 \leq k \leq m$, there exists a normal subgroup R of G such that $N \subset R \subset M$ and $|R/N| = p^k$.

Proof

Consider the normal series $\{1\} \subset N \subset M \subset G$. This can be refined into a series of normal subgroups of G (Huppert, 1967, I,11.7) in such a way that the factor group of any two consecutive members of this series of normal subgroups is of order p. Hence, R can be chosen among the members of this series such that $|R/N| = p^k$.

Lemma 3

Let p be an odd prime and N a normal non-cyclic subgroup of a p-group G. Then, N contains a normal subgroup A of G of type (p,p).

Proof

We proceed by induction on $|G|$, i.e., we suppose the lemma true for all p-groups of order less than $|G|$ and prove that it remains true for G. If $|G| = p^2$, then $G = N = A$.

Suppose $|G| > p^2$. By virtue of Lemma 2, N contains a normal subgroup L of G such that $|L| = p$. Consider G/L.

If N/L is cyclic, then N is abelian since $L \subset Z(G)$. Since N is not cyclic, we have $m(N) = 2$, where m(N) denotes the minimal number of generators of N. $A = \langle x \in N \mid x^p = 1 \rangle$ is a characteristics subgroup of N of type (p,p). Since N is normal in G, A is normal in G as a characteristics subgroup of a normal subgroups of G.

Suppose now N/L non-cyclic. Since the order of G/L is less than $|G|$, by the inductive hypothesis there exists a normal subgroup M of G such that $L \subset M \subset N$ and M/L is of type (p,p). we have $|M| = p^3$. If M is of exponent p, by virtue of Lemma 2, there exists a normal subgroup A of G of type (p,p). Suppose then

M is of exponent greater than p. If M is abelian, we are done. Suppose M non-abelian. It is well known that $|M/M'| \geq p^2$ (Huppert, 1967, III,7.1), hence $|M'| = p$ since $p^3 = |M| = (M{:}M')\,|M'|$. We have also $M' \subset Z(M)$, because M′ is characteristic in M. By Lemma 1(b), for all $x,y \in M$,

$$(xy)^p = x^p y^p [y,x]^{\binom{p}{2}},$$

and since p is odd, we have $[y,x]^{\binom{p}{2}} = 1$. Hence,

$$(x\,y)^p = x^p\,y^p.$$

Consequently $a \mapsto a^p$ is an endomorphism, say f, of M. Since M is an exponent greater than p, we have $f(M) \neq \{1\}$. From M/M' is of type (p,p) it follows that $f(M) \subset M'$, hence $|f(M)| = p$. Ker $(f) = A$ is then of order p^2 and in fact of type (p,p). Since $A = \langle X \in M \mid X^p = 1 \rangle$ is characteristic in the normal subgroup M of G, A is normal in G.

Theorem 1

Let p be an odd prime and G a non-abelian p-group of order p^n in which G′ is cyclic. Then $P^{i\left(\frac{n}{2}\right)+1}$ divides $|G/G'|$.

Proof

We carry out the proof by contradiction. Let G be a counterexample of minimal order. That is, the conclusion of the theorem holds for all p-groups of order less than $|G|$ but it does not hold for G. Then $P^{i\left(\frac{n}{2}\right)}$ divides $|G/G'|$, and since G′, is cyclic, every subgroup of G′ is normal in G. Let A be the subgroup of G′ of order p. Then A is normal in G and $A \subset Z(G)$. Let $s: G \to G/A$ be the natural homomorphism. $s(G') = (s(G))'$ is a cyclic group and $|s(G)| < |G|$. Hence, $P^{i\left(\frac{n-1}{2}\right)+1}$ divides $|s(G)/s(G')|$. But $s(G)/s(G') \approx G/G'$, so

$p^{i\left(\frac{n-1}{2}\right)+1} \| G/G' |$.

If n=2m+1, then $i(\frac{n}{2}) = i(m+\frac{1}{2}) = m, i(\frac{n-1}{2}) = m$, hence $p^{i\left(\frac{n}{2}\right)+1} \| G/G' |$ and we get a contradiction.

If n=2m, then $i(\frac{n}{2}) = m, i(\frac{n-1}{2}) = i(m-\frac{1}{2}) = m-1$ and $p^m \| G/G' |, p^{m+1}$ does not divide $|G/G'|$, hence $|G/G'| = p^m = p^{\frac{n}{2}}$ and $|G'| = p^m$.

Set A= $<a>$. We have a$\in$Z(G). Let M be a normal subgroup of G of type (p,p). M exists by virtue of Lemma 3. We have M$\neq$G′ and G′$\not\subset$M. Consequently, G/M is not abelian, $|G/M| = P^{2m-2}$ and, by the choice of G, $p^m \| (G/M)' |$. But (G/M)′ = G′M/M$\approx$G′/G′$\cap$M and G′/G′$\cap$M is cyclic as a factor group of a cyclic group. Hence, by comparing orders, we get M$\cap$G′={1}. It follows that M$\subset$Z(G). Let x$\in$M-{1}.Then N = $<$x, a$>$ $>$ is of type (p,p) and normal in G since N$\subset$Z(G). This shows that we again get a contradiction because a$\in$G′ and N$\cap$G′ $\neq${1}. The proof is complete.

The bound as stated in Theorem 1 is the best possible. Indeed, there are non-abelian p-groups (p$\geq$3) of order p^n such that $|G/G'|$ is equal to $p^{i\left(\frac{n}{2}\right)+1}$.

If n=2m+1, then take $G = < x,y : y^{p^m} = x^{p^{m+1}} = 1, y^{-1}xy = x^{1+p} >$. We obtain

$$|G/G'| = P^{m+1} = p^{i\left(\frac{n}{2}\right)+1}.$$

When n=2m, consider $G = < x,y : y^{p^{m-1}} = x^{p^{m+1}} = 1, y^{-1}xy = x^{1+p^2} >$. In this case, we get $|G'| = | < x^{p^2} > | = P^{m-1}$ and $|G/G'| = P^{m+1} = P^{i\left(\frac{n}{2}\right)+1}$.

The case of 2-groups is completely different from that of p-groups where p is odd as shown by the following result.

Theorem 2

For any integers m and n satisfying $2 \leq n \leq m$, there exists a group G such that $|G| = 2^m$, $|G/G'| = 2^n$, G′ is cyclic.

Proof

Let M be any abelian group of order 2^{n-2} and let N denote the dihedral group of order 2^{m-n+2} . Let G be the direct product of M and N. Then $|G| = 2^m$, $G' = N'$ is cyclic of order 2^{m-n} and $|G/G'| = 2^n$.

Theorem 3

The group in Theorem 1 is regular.

Proof

Let $H = <a,b>$ be a subgroup of G, where a and b do not commute; this is possible, since G is non-abelian. $H' \subset G'$ and G′ is cyclic imply that H′ is cyclic. Let $H' = <c>$. Then $\{1\} \subset H_3 \subset H'$ with $H_3 \neq H'$ and consequently $H_3 \subset <c^p>$. We can now apply Lemma 1(b) to H/H_3: there exists $d \in H_3 \subset <c^p>$ such that

$$(ab)^p = a^p b^p \, [b,a]^{\binom{p}{2}} d.$$

Since $p \geq 3$, $\binom{p}{2}$ is a multiple of p. Hence $[b,a]^{\binom{p}{2}} d = c^{mp}$. The proof of Theorem 3 is complete.

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