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Analysis of Adjacency, Laplacian and Distance Matrices of Zero Divisor Graphs of 4-Radical Zero Completely Primary Finite Rings

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ABSTRACT

This study is an extension of our study on matrices of zero divisor graphs of classes of 3-radical zero completely primary finite rings. It focusses on Matrices of a class of finite rings R whose subset of the zero divisors $Z(R)$ satisfies the condition $(Z(R))^4 = (0)$ and $(Z(R))^3 \neq (0)$ for all characteristics of R that is; $p,\ p^2,\ p^3$ and p^4. We have formulated the zero divisor graphs $\Gamma(R)$ of R and associated them with three classes of matrices, namely, the Adjacency matrix $[A]$, the Laplacian matrix $[L]$ and the Distance matrix $[d_{ij}]$. The study has further characterized the properties of the graphs $\Gamma(R)$ and the matrices mentioned.

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Keywords: Completely Primary Finite Rings, Matrices of Zero Divisor Graphs

1 Introduction

The study of various classes of the zero divisor graphs obtained from finite rings has been very active since its inception by Beck [2] in an investigation of graph colourings of commutative finite rings. Perhaps, further and simpler developments in this endeavour can be attributed to Anderson and Livingston and Mulay in [1] and [8] respectively who investigated various properties other than colouring. In particular, Mulay in [8] characterized the cycles and symmetries associated with a class of zero divisor graph obtained from finite rings. The mentioned studies in [1, 2, 8] were all aimed at determining the classification of classes of finite rings using the properties of their graphs. The structures of unit groups, zero divisor graphs and the associated adjacency matrices of Galois rings, square radical and classes of cube radical zero completely primary finite rings are well understood (see for example [5, 6, 7, 9, 10]). In particular, Lao *et al* in [5, 6, 7], considered the automorphism groups of the zero divisor graphs of Galois rings, 2-radical zero and 3-radical radical zero completely primary finite rings, while Ndago *et al* in [9] obtained the properties of the Adjacency and Incidence Matrices from the zero divisor graphs of the 2-radical zero finite rings. Most recently, the authors in [10] extended the study of 3-radical zero finite rings covering the algebraic properties of the Ajacency, Laplacian and Distance Matrices associated with the graphs $\Gamma(R)$ of the 3-radical zero finite completely primary rings. Closely related works can be found in [14, 15] where $R = \mathbb{Z}_p \times \mathbb{Z}_p$ for $p = 2, 3$ and $5,\ \mathbb{Z}_p[i] \times \mathbb{Z}_p[i]$ for $p = 2, 3$ and 5. In each case, an analysis of the determinant, trace, rank and the symmetry of the matrices was done. Further, a research on the adjacency universal spectrum of $\Gamma(R)$ on the ring $\mathbb{Z}_n$ with its compliment was done in [16]. In the study, an investigation on the loopless graph G with matrices $[A]$ and $[D]$ was performed by choosing a universal matrix $U(G)$ whose computation algorithm was $\beta D + \gamma l + \eta j + \alpha A$ with $\alpha(\neq 0), \gamma, \beta, \eta \in R, l$ being the matrix identity and j having entries of 1.

The Adjacency matrix $[A]$, the Laplacian matrix $[L]$ and Distance matrix $[d_{ij}]$ have inherent structural algebraic relationships which give the matrix representation of the zero divisor graphs $\Gamma(R)$ for ease of their algebraic and geometric analyses. Consider the Adjacency matrix $[A]$ and the degree matrix $[D]$ of $\Gamma(R)$, the Laplacian matrix is a square matrix computed through the relation, $[L] = [D] - [A]$. Bilal in [3] investigated the eigenvalues of Laplacian matrix of $\Gamma(R)$ associated with $\mathbb{Z}_n$. The research showed that the Euler's totient function Φ satisfies the relation $\Phi(qp) = \Phi(q)\Phi(p)$ for relatively prime integers p and q in the ring $\mathbb{Z}_p \times \mathbb{Z}_q$. Also, in [13] and [17], signless Laplacian spectrum and Laplacian Eigenvalues of zero divisor graphs of the ring $\mathbb{Z}_n$ were investigated.

Let R be a completely primary finite ring whose subset of zero divisors $Z(R)$ satisfy the condition $(Z(R))^4 = (0)$ and $(Z(R))^3 \neq (0)$. Then it is well known that the characteristic of R is $p,\ p^2,\ p^3$ or p^4. For certain classes of R, the unit groups R^*, the automorphism group $aut(R^*)$, the zero divisor graphs $\Gamma(R)$ as well as the $aut(\Gamma(R))$ are well known. This paper focusses on the adjacency, Laplacian and distance Matrices of the zero divisor graphs of the classes of R.
Throughout the paper, $R, \Gamma(R), deg(v), V(\Gamma(R)), [A]_{ij}, [L]_{ij}$ and $[d_{ij}]$ are used to denote the completely primary finite ring, the zero divisor graph of R, the degree of a vertex in $\Gamma(R)$, a vertex set of the zero divisor graph and adjacency, Laplacian and distance matrices respectively.

2 4-Radical Zero Completely Primary finite Rings of Characteristic p

The following construction can be obtained from [11].

2.1 Construction I

Let $R' = GR(p^r, p)$ be a Galois ring of order p^r and characteristic p. Consider finitely generated R'-modules $U,\ V,$ and W such that $dim_{R'}U = s,\ dim_{R'}V = t$ and $dim_{R'}W = \lambda$ and $s + t + \lambda = h$. Let the R' modules be generated by $\{u_1, u_2 \cdots, u_s\}, \{v_1, v_2, \cdots, v_t\}$ and $\{w_1, w_2, \cdots, w_\lambda\}$ respectively so that $R = R' \oplus U \oplus V \oplus W$ is an additive abelian group. Suppose $s = 1, t = 1$ and $\lambda = h - 2$, then $R = R' \oplus R'u \oplus R'v \oplus \sum_{k=1}^{h-2} R'w_k$ where $pu = 0, pv = 0, pw_k = 0$ such that $1 \leq k \leq h - 2$ for any prime integer p. We define multiplication on R as follows;

$$(a_\circ, a_1, a_2, \cdots, a_h)(b_\circ, b_1, b_2, \cdots, b_h) =$$
$$(a_\circ b_\circ, a_\circ b_1 + a_1 b_\circ, a_\circ b_2 + a_2 b_\circ + a_1 b_1, a_\circ b_3 + a_3 b_\circ + a_1 b_2 + a_2 b_1, \cdots, a_\circ b_h + a_h b_\circ + a_1 b_2 + a_2 b_1).$$

As established in [11], R is turned by this multiplication into a commutative ring with identity $(1, 0, 0, \cdots, 0)$ and further, the set $Z(R)$ of zero divisors of R satisfy the following properties:
$Z(R) = R'u \oplus R'v \oplus \sum_{k=1}^{\lambda} R'w_k,\ (Z(R))^2 = R'v \oplus \sum_{k=1}^{\lambda} R'w_k,\ (Z(R))^3 = \sum_{k=1}^{\lambda} R'w_k, (Z(R))^4 = (0)$.
As a consequence, the next result in the sequel holds for $\Gamma(R)$.

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Proposition 2.1. *Let R be a ring of Construction I. Then the zero divisor graph $\Gamma(R)$ satisfy the following properties:*

(i) The cardinality of the vertices, $\mid V(\Gamma(R)) \mid = p^{hr} - 1.$

(ii) Minimum degree, $\delta(\Gamma(R)) = p^r - 1.$

(iii) Maximum degree, $\Delta(\Gamma(R)) = p^{hr} - 2.$

(iv) $\Gamma(R)$ *is incomplete.*

Proof. (i) Since $char(R) = char(R') = p$ and $pu_i = pv_j = pw_k = 0,$
$\mid R'u_i \mid = p^{sr}, \mid R'v_j \mid = p^{tr}, \mid R'w_k \mid = p^{\lambda r} \Longrightarrow \mid Z(R) \mid = p^{sr}.p^{tr}.p^{\lambda r} = p^{(s+t+\lambda)r} = p^{hr}$
but $\mid Z(R)^* \mid = \mid V(\Gamma(R)) \mid = p^{hr} - 1.$
(ii) With the multiplication described, $Ann(Z(R)) = (Z(R))^3$. Suppose the vertex set $V_1 = Ann(Z(R)) \setminus \{0\}$, we thus have that $\mid V_1 \mid = p^r - 1$. Since there are only $p^r - 1$ vertices adjacent to every vertex then the minimum degree of a vertex is $p^r - 1$.
(iii) Since the number of vertices in $\Gamma(R)$ is $p^{hr} - 1$, there exist $x \in V_1$ connected to every vertex in the graph. Therefore, the degree of $x,\ deg(x) = (p^{hr} - 1) - 1 = p^{hr} - 2 \Longrightarrow \Delta(\Gamma(R))$ for the avoidance of self loop.
(iv) Clearly, $\delta(\Gamma(R))$ is not equal to $\Delta(\Gamma(R))$ illustrating that the vertices in $\Gamma(R)$ do not have the same degree of connectedness. That is, not every pair of vertices in $\Gamma(R)$ are connected. Further, due to the fact that $(Z(R))^2 \neq (0)$, the incompleteness of $\Gamma(R)$ follows.

□

2.2 Matrices of Zero Divisor Graphs of a Ring in Construction I

Proposition 2.2. *Let R be a ring of Construction I. The Adjacency and Laplacian matrices satisfy the following properties:*

(i) $[A]_{p^{hr}-1}$ *and* $[L]_{p^{hr}-1}$ *are singular.*

(ii) $rank([A]_{p^{hr}-1}) = p^{hr} - p^{(h-1)r}.$

(iii) $rank([L]_{p^{hr}-1}) = p^{(h-1)r} + 2.$

(iv) $Tr([L]_{p^{hr}-1}) = 2p^{(h+1)r} - 3p^{hr} + p^{2(h-1)r} + 2p^r + 1.$

(v) For $[A]_{p^{hr}-1}$, the number of real and complex eigenvalues

are $p^{(h-1)r}$ and $p^{hr} - p^{(h-1)r} - 1$ respectively.
Indeed, the real eigenvalues $\lambda[A]_{p^{hr}-1} = \begin{cases} 0, & \text{of multiplicity } p^{(h-1)r} - 1; \\ -1, & \end{cases}$
and the complex eigenvalues $\lambda[A]_{p^{hr}-1} = \begin{cases} (p^{(h-1)r} - 2)i, & \text{of multiplicity } p^{hr} - p^{r} - 2; \\ (p^{(h-1)r} - 1)i, & \text{of multiplicity } p^{r} - p^{(h-1)r} + 1. \end{cases}$

(vi) The eigenvalues $\lambda[L]_{p^{hr}-1} = \begin{cases} 0, & \\ p^{hr} - 1, & \\ p^{(h-1)r} - 1, & \\ 1, & \text{of multiplicity } p^{hr} - 4. \end{cases}$

Proof. (i) Given the adjacency matrix $\begin{bmatrix} 0 & 1 & \cdots & \cdots & \cdots & 1 \\ 1 & 0 & 1 & \cdots & \cdots & 1 \\ 1 & 1 & 0 & 1 & \cdots & \vdots \\ 1 & 1 & 1 & 0 & \cdots & 1_{p^{hr}-p^{(h-1)r}} \\ 0 & 0 & \cdots & \cdots & \cdots & 0_{p^{hr}-p^{r}} \\ \vdots & \vdots & 0 & \cdots & 0 & \vdots \\ 0 & \cdots & \cdots & \cdots & \cdots & 0_{p^{hr}-1} \end{bmatrix}$, suppose we take row 1 as the pivot row in obtaining the determinant, let $a_{11}, a_{12}, \cdots, a_{1p^{hr}-1}$ be the elements of the first row of $[A]_{p^{hr}-1}$. Expanding the minor determinants along the first row, we notice that the matrix minors of $a_{1j}, j = 1, 2, \cdots, p^{hr} - 1$ have zero determinants. That is,
$a_{11}(-1)^{1+j} \mid minor\ (a_{11}) \mid = \cdots = a_{1p^{hr}-1}(-1)^{1+(p^{hr}-1)} \mid minor\ (a_{1p^{hr}-1}) \mid = 0.$
Therefore $\sum_{j=1}^{p^{hr}-1}((-1)^{1+j}a_{1j} \mid minor\ (a_{1j}) \mid) = 0$, hence the determinant of $[A]_{p^{hr}-1}$. A similar argument can be extended for the Laplacian matrices $[L]_{p^{hr}-1}$. This proves the singularity for the matrices.
ii) Reducing the adjacency matrix to its echelon form by conducting a row operation on it, we obtain the matrix

$$\begin{bmatrix} 1 & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & 1 & 0 & \cdots & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & \vdots \\ 0 & 0 & 0 & 1 & \cdots & 1_{p^{hr}-p^{(h-1)r}} \\ 0 & 0 & \cdots & \cdots & \cdots & 0_{p^{hr}-p^{r}} \\ \vdots & \vdots & 0 & \cdots & 0 & \vdots \\ 0 & \cdots & \cdots & \cdots & \cdots & 0_{p^{hr}-1} \end{bmatrix}.$$

Clearly, from this reduced echelon form, we obtain $p^{hr}-p^{(h-1)r}$ non zero rows spanning the matrix space. This leads to a rank of $p^{hr}-p^{(h-1)r}$ for the adjacency matrix $[A]_{p^{hr}-1}$.

(iii) Similar to (ii), the Laplacian matrix obtained can be reduced to an echelon form

$$\begin{bmatrix} 1 & 0 & 0 & \cdots & \cdots & \cdots & \cdots & -1 \\ 0 & 1 & 0 & 0 & \cdots & \cdots & \cdots & -1 \\ 0 & 0 & 1 & 0 & 0 & \cdots & \cdots & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & \cdots & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & \cdots & -1 \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 & -1_{p^{(h-1)r}+2} \\ \vdots & 0 & \cdots & & & & 0 & \vdots \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0_{p^{hr}-1} \end{bmatrix}$$

which is of order $(p^{hr}-1)\times(p^{hr}-1)$. This results to $p^{(h-1)r}+2$ linearly independent vectors which span the matrix row space for the Laplacian matrix $[L]_{p^{hr}-1}$, hence its rank.

(iv) Let $\gamma_1,\cdots,\gamma_r \in R'$ with $\overline{\gamma}_1,\cdots,\overline{\gamma}_r \in R'$ form a basis for R' over its prime subfield R'/pR'. From the multiplication defined on R, $Ann(Z(R))=(Z(R))^3=p^3R'$. Let $V_1,\ V_2$ and V_3 be the vertex sets partitioning $V(\Gamma(R))$ such that $V_1=Ann(Z(R)^*)$. This implies that $\mid V_1 \mid = p^r-1$. Therefore, for $x\in V_1$, $deg(x)=p^{hr}-2$.

Consider the vertex set $V_2=\{\gamma_i v+\sum_{k=1}^{h-2} b\gamma_i w_k | b\in R'\}$. Then, $\mid V_2 \mid = p^{(h-1)r}-p^r$ and each vertex $y\in V_2$ is adjacent to a vertex of the form $\gamma_i v+\sum_{k=1}^{h-2}\gamma_i w_k$. Therefore, $deg(y)=p^{(h-1)r}-1$. Let the set $V_3=\{\gamma_i u+a\gamma_i v+\sum_{k=1}^{h-2} c\gamma_i w_k | a,c\in R'\}$. This means that $\mid V_3 \mid = p^{hr}-p^{(h-1)r}$ and $deg(z)\in V_3=p^r-1$ since z is only adjacent to the vertices in the annihilator set V_1.

The trace of the Laplacian matrix is the sum of diagonal entries in the degree matrix $[D]_{p^{hr}-1}$. Thus,

$Tr([L]_{p^{hr}-1}) = (p^{hr}-2)(p^r-1)+(p^{(h-1)r}-1)(p^{(h-1)r}-p^r)+(p^{hr}-p^{(h-1)r})(p^r-1)$. Upon expansion and simplification of this expression, we obtain $Tr([L]_{p^{hr}-1}) = 2p^{(h+1)r} - 3p^{hr}+p^{2(h-1)r}+2p^r+1$.
(v) Solving the equation $\mid \lambda I - A \mid = 0$, we obtain the characteristic polynomial equation $\lambda^{p^{hr}-1}-(p^{hr}-1)\lambda^{p^{hr}-p^r-1}-p^r\lambda^{p^{(h-1)r}}+p^{(h-1)r}\lambda^{p^{(h-1)r}-1}=0$ which can be expressed in factor form as $\lambda^{p^{(h-1)r}-1}(1+\lambda)(\lambda^{p^{(h-1)r}-1}-\lambda^{p^r}-(p^{hr}-p^{(h-1)r})\lambda+p^{(h-1)r})=0$. Finding λ, we solve $\lambda^{p^{(h-1)r}-1}=0 \implies \lambda=0$ of multiplicity $p^{(h-1)r}-1$, $(1+\lambda)=0 \implies \lambda=-1$. The order of the real eigenvalues is obtained by adding the multiplicities $(p^{(h-1)r}-1)+1 = p^{(h-1)r}$.
The equation $(\lambda^{p^{(h-1)r}-1}-\lambda^{p^r}-(p^{hr}-p^{(h-1)r})\lambda+p^{hr})=0$ yields the complex eigenvalues as $(p^{(h-1)r}-2)i$ of multiplicity $p^{hr}-p^r-2$ and $(p^{(h-1)r}-1)i$ of multiplicity $p^r-p^{(h-1)r}+1$. Therefore, the sum of multiplicities of complex eigenvalues are $(p^{hr}-p^r-2)+p^r-p^{(h-1)r}+1 = p^{hr}-p^{(h-1)r}-1$.
(vi) For the Laplacian matrix $[L]_{p^{hr}-1}$, we evaluate $\mid \lambda I - [L]_{p^{hr}-1} \mid = 0$ to obtain the characteristic polynomial equation $-\lambda((-(p^{hr}-1)+\lambda)(-(p^{(h-1)r}-1)+\lambda)(-1+\lambda)^{p^{hr}-4}) = 0$. Finding the values of λ in each factor, we have $-\lambda=0 \implies \lambda=0$. Next, $-(p^{hr}-1)+\lambda=0 \implies \lambda=p^{hr}-1$ and further $-(p^{(h-1)r}-1)+\lambda=0 \implies \lambda=p^{(h-1)r}-1$. Finally, $(-1+\lambda)^{p^{hr}-4}=0 \implies \lambda=1$ of multiplicity $p^{hr}-4$. □

Proposition 2.3. *Let R be a ring of Construction I and $[d_{ij}]$ be the distance matrix then:*

(i) $Tr([d_{ij}])=0$.

(ii) $rank([d_{ij}])=p^{hr}-1$.

(iii) The eigenvalues $\lambda[d_{ij}] = \begin{cases} -1, & \text{of multiplicity } p^r-1; \\ -p^r, & \text{of multiplicity } p^{hr}-2p^r+1; \\ -(p^r-1)i, & \text{of multiplicity } p^r-1, \text{ where } \lambda\in\mathbb{C}. \end{cases}$

(iv) $Det([d_{ij}])=p^{(2hr+1)r}$.

Proof. (i) Since the minimum distance between a vertex and its self $d(v_i,v_i)=0$, it means that every entry d_{ii} of $[d_{ij}]$ is zero and thus $\sum_{i=1}^{p^{hr}-1} d_{ii}=0$. Hence the trace, $Tr([d_{ij}])=0$.

(ii) We carry out an elementary row operation on $[d_{ij}]$ to obtain a row reduced matrix of the form

$$\begin{pmatrix} 1 & 0 & \cdots & \cdots & 0 \\ 0 & 1 & 0 & \cdots & \vdots \\ \vdots & 0 & \ddots & 0 & \\ \vdots & & & \ddots & 0 \\ 0 & \cdots & \cdots & & 1_{p^{hr}-1} \end{pmatrix}.$$

Clearly there are $p^{hr}-1$ linearly independent vectors in the matrix span hence the rank.
(iii) To find the characteristic equation, we solve $\mid \lambda I - [d_{ij}] \mid = 0$ to obtain the equation

$$-(1+\lambda)^{p^r-1}(p^r+\lambda)^{(p^{hr}-2p^r+1)}(\lambda^{p^r-1}-(p^{hr}-1)\lambda^{p^r-2}-(p^{(h+2)r}-1)\lambda-p^{(h+1)r})=0.$$

From the equation, the real eigenvalues are $-(1+\lambda)=0 \implies \lambda=-1$ of multiplicity p^r-1 and $(p^r+\lambda)^{(p^{hr}-2p^r+1)}=0 \implies \lambda=-p^r$ of multiplicity $p^{hr}-2p^r+1$.
Solving the equation $(\lambda^{(p^r-1)}-(-1+p^{hr})\lambda^{(p^r-2)}-(p^{(h+2)r}-1)\lambda-p^{(h+1)r})=0$ yields the complex eigenvalues as $-(p^r-1)i$ of multiplicity p^r-1.
(iv) In obtaining the determinant we evaluate $\sum_{i,j=1}^{p^{hr}-1}(d_{ij}(-1)^{i+j} \mid minor\ (d_{ij}) \mid) = p^{hr}.p^{(h+1)r} = p^{(hr+hr+r)} = p^{(2hr+1)r}$. □

3 4-Radical Zero Finite Completely Primary Rings of Characteristic p^2

3.1 Construction II

Let $R' = GR(p^{2r}, p^2)$ be a Galois ring of order p^{2r} and characteristic p^2. Consider R' modules U, V and W which are generated finitely by $\{u_1, \cdots, u_s\}, \{v_1, v_2, \cdots, v_t\}$ and $\{w_1, w_2, \cdots, w_\lambda\}$ respectively so that $R = R' \oplus U \oplus V \oplus W$ is additive abelian group and $s+t+\lambda = h$. Assume $s = h-1, t = 1$ and $\lambda = 0$ so that $R = R' \oplus \sum_{i=1}^{h-1} R'u_i \oplus R'v$ where $pu_i \neq 0, p^2u_i = 0$ and $pv = 0$ with $1 \leq i \leq s$. The following defines multiplication on R.

$$(a_\circ, a_1, a_2, , \cdots, a_{h-1}, \bar{a}_h)(b_\circ, b_1, b_2, , \cdots, b_{h-1}, \bar{b}_h) =$$
$$(a_\circ b_\circ + p\sum_{i,j=1}^{h-1} a_i b_j, a_\circ b_1 + a_1 b_\circ, \cdots, a_\circ b_{h-1} + a_{h-1} b_\circ, a_\circ \bar{b}_h + \bar{a}_h b_\circ)$$

where $\bar{a}_h, \bar{b}_h \in R'/pR'$. The multiplication so defined turns R into a commutative finite ring of identity $(1, 0, 0, \cdots, \bar{0})$ as verified in [4].
$Z(R)$ satisfies the following properties;
$Z(R) = pR' \oplus \sum_{i=1}^{s} R'u_i \oplus R'v,\ (Z(R))^2 = pR' \oplus p\sum_{i=1}^{s} R'u_i \oplus R'v,\ (Z(R))^3 = p\sum_{i=1}^{s} R'u_i,$
$(Z(R))^4 = (0).$
The following result describes some properties of $\Gamma(R)$ of the ring constructed in this section.

Proposition 3.1. *Let R be a ring of Construction II. Then:*

(i) The cardinality, $\mid V(\Gamma(R)) \mid = p^{2hr} - 1.$

(ii) The maximumn degree, $\Delta(\Gamma(R)) = p^{2hr} - 2.$

(iii) $\Gamma(R)$ *is an incomplete graph.*

(iv) The minimum degree, $\delta(\Gamma(R)) = p^{hr} - 1.$

Proof. (i) Given that the structure of zero divisors is given by $Z(R) = pR' \oplus \sum_{i=1}^{s} R'u_i \oplus R'v$ and due to the fact that $pu_i \neq 0,\ p^2u_i = 0$ and $pv = 0$ with $1 \leq i \leq s,\ \mid pR' \mid = p^r,\ \mid R'u_i \mid = p^{2r}$ and $\mid R'v \mid = p^r$. Therefore, $\mid Z(R) \mid = p^r(p^{2r(h-1)})p^r = p^{2hr}$. Since $\mid Z(R)^* \mid = \mid Z(R) \setminus \{0\} \mid,\ \mid Z(R)^* \mid = p^{2hr} - 1 = \mid V(\Gamma(R)) \mid$.
(ii) Let $\gamma_1, \cdots, \gamma_r \in R'$ with $\gamma_1 = 1$ such that $\overline{\gamma}_1, \cdots, \overline{\gamma}_r \in R'$ is a basis for R' over its prime subfield R'/pR'. Let $V_1 = Ann(Z(R)) \setminus \{0\}$. From the multiplication described, $Ann(Z(R)) = \{pc_1\gamma_i u_1 + \cdots + pc_{h-1}\gamma_i u_{h-1} + b\gamma_i v | c_1, \cdots, c_{h-1}, b \in R'\}$. Vertices in V_1 are adjacent to every vertex in $\Gamma(R)$. Therefore, every $y \in V_1$ is of degree $p^{2hr} - 2$ for an avoidance of self loop. Hence the maximum degree $\Delta(\Gamma(R)) = p^{2hr} - 2$.
(iii) This is clear due to the fact that $(Z(R))^2 \neq (0)$.
(iv) Let V_1 be the set described in (ii), $deg(y) \in V_1 = p^{2hr} - 2$ and $\mid V_1 \mid = p^{hr} - 1.$ Any vertex of minimum degree is not adjacent to any other vertex in $V(\Gamma(R))$ a part from the vertices in the set V_1. Since there are $p^{hr} - 1$ vertices in set V_1, it implies that $\delta(\Gamma(R)) = p^{hr} - 1$.

□

The results below describe the properties of the matrices associated with $\Gamma(R)$ of the ring constructed in this Section.

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3.2 Matrices of the Zero Divisor Graph of the Ring in Construction II

Proposition 3.2. *Let R be a ring of Construction II. Suppose $[A]_{p^{2hr}-1}$ and $[L]_{p^{2hr}-1}$ are the Adjacency and Laplacian matrices respectively;*

(i) Both matrices are singular.

(ii) $rank([A]_{p^{2hr}-1}) = p^{2hr} - p^{hr}$.

(iii) $rank([L]_{p^{2hr}-1})$ *is* $p^{hr} + 2$.

(iv) The number of real and complex eigenvalues λ for $[A]_{p^{2hr}-1}$
$$= \begin{cases} p^{hr}, & \lambda \in \mathbb{R}; \\ p^{2hr} - p^{hr} - 1, & \lambda \in \mathbb{C}. \end{cases}$$

(v) The eigenvalues $\lambda[L]_{p^{2hr}-1} = \begin{cases} 0, \\ p^{2hr} - p^{hr}, \\ p^{hr} + p^{r}, \\ 1, & \text{of multiplicity } p^{hr}. \end{cases}$

(vi) $Tr([L]_{p^{2hr}-1}) = p^{2hr} + p^{hr} + p^{r}$.

Proof. The proofs for (i), (ii) and (iii) can easily be followed from proposition 2.2.
(iv). Solving the equation $\mid \lambda I - [A]_{p^{2hr}-1} \mid = 0$ results to a characteristic equation of the form $\lambda^{p^{2hr}-1} - (p^{2hr} - 1)\lambda^{p^{2hr}-p^{hr}-1} - p^{hr}\lambda^{p^{hr}} + p^{hr} = 0$ which factorizes as $\lambda^{p^{hr}-1}(1 + \lambda)(\lambda^{p^{hr}-1} - \lambda^{p^{r}} - (p^{2hr} - p^{hr})\lambda + p^{hr}) = 0$. Finding the values of λ from the equation, we obtain $\lambda^{p^{hr}-1} = 0 \Longrightarrow \lambda = 0$ of multiplicity $p^{hr} - 1$ and $\lambda + 1 = 0 \Longrightarrow \lambda = -1$, as the real eigenvalues. Therefore, by evaluating the sum of the multiplicities of real eigenvalues, we obtain the number of real eigenvalues to be $p^{hr} - 1 + 1 = p^{hr}$.
The equation from the remaining factor, $(\lambda^{p^{hr}-1} - \lambda^{p^{r}} - (p^{2hr} - p^{hr})\lambda + p^{hr}) = 0$ yields $(p^{2hr} - 1) - p^{hr} = p^{2hr} - p^{hr} - 1$ complex eigenvalues due to the fact that the adjacency matrix $[A]_{p^{2hr}-1}$ is a square matrix with $p^{2hr} - 1$ rows and columns.
(v). For the Laplacian matrix $[L]_{p^{2hr}-1}$, the equation $\mid \lambda I - [L]_{p^{2hr}-1} \mid = 0$ results to the characteristic polynomial equation of the form $-\lambda(-(p^{2hr} - p^{hr}) + \lambda)(-(p^{hr} + p^{r}) + \lambda)(-1 + \lambda)^{p^{hr}} = 0$. Upon solving the equation, $-\lambda = 0 \Longrightarrow \lambda = 0,\ -(p^{2hr} - p^{hr}) + \lambda = 0 \Longrightarrow \lambda = p^{2hr} - p^{hr}$ and $-(p^{hr} + p^{r}) + \lambda = 0 \Longrightarrow \lambda = p^{hr} + p^{r}$. Finally, $(-1 + \lambda)^{p^{hr}} = 0 \Longrightarrow \lambda = 1$

of multiplicity p^{hr}. Hence the eigenvalues for $[L]_{p^{2hr}-1}$.
(vi). Since trace can be computed as the sum of eigenvalues, $Tr([L]_{p^{2hr}-1}) = \sum_{i=1}^{p^{2hr}-1} \lambda_i \implies Tr([L]_{p^{2hr}-1}) = 0 + p^{2hr} - p^{hr} + p^{hr} + p^r + 1(p^{hr}) = p^{2hr} + p^{hr} + p^r$ as required. □

Proposition 3.3. *Let R be a ring of Construction II and $[d_{ij}]$, the distance matrix then;*

(i) $Tr([d_{ij}]) = 0$.

(ii) $rank([d_{ij}]) = p^{2hr} - 1$.

(iii) The eigenvalues $\lambda = \begin{cases} -1, & \text{of multiplicity } p^{(h+2)r} - 2; \\ -p^r, & \text{of multiplicity } p^{(h+2)r} - 1; \\ \frac{1}{2}(\sigma \pm \sqrt{\sigma^2 - 4\tau}) & . \end{cases}$

(iv) $Det([d_{ij}]) = p^{(2h+2)r}$.

Proof. (i) Follows from the fact that $d(v_i, v_i) = 0$, thus entries d_{ii} of the main diagonal are all $0'$s hence the trace.

(ii) Given the general distance matrix $[d_{ij}]_{p^{2hr}-1} = \begin{pmatrix} 0 & 1 & 1 & \cdots & \cdots & 1 \\ 1 & 0 & 1 & 1 & \cdots & 1 \\ \vdots & & \vdots & \ddots & & \vdots \\ 0 & 0 & \cdots & 0 & \cdots & 0_{p^{hr}-1} \end{pmatrix}$,
consider the set $V = \{v_1, \cdots, v_{p^{2hr}-1}\}$ consisting of vectors which are linearly independent from a row reduced echelon form of matrix $[d_{ij}]_{p^{2hr}-1}$ such that $v_1 = \begin{pmatrix} 1 \\ 0 \\ . \\ . \\ . \\ 0 \end{pmatrix}, v_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ . \\ . \\ 0 \end{pmatrix}, \cdots, v_{p^{2hr}-1} = \begin{pmatrix} 0 \\ . \\ . \\ . \\ 0 \\ 1 \end{pmatrix}$. Clearly, the set V is of dimension $p^{2hr} - 1$ equivalent to the dimension of the matrix thus the matrix space is spanned by vectors in V. Therefore

the $rank([d_{ij}]) = p^{2hr} - 1$.
(iii) We solve the equation $\mid [d_{ij}] - \lambda I \mid= 0$ to obtain the characteristic polynomial $-(1+\lambda)^{p^{(h+2)r}-2}(p^r+\lambda)^{p^{(h+2)r}-1}(\lambda^2-(p^{(h-1)r}(p^{(h+2)r}-p^{hr}-1)\lambda+(2p^{(h+2)r}+2p^{hr}-4)(p^{hr}+3))$. Finding λ in each factor, we solve $(p^r+\lambda)^{p^{(h+2)r}-1} = 0 \implies \lambda = -p^r$ of multiplicity $p^{(h+2)r}-1$. Further, $-(1+\lambda)^{p^{(h+2)r}-2} = 0 \implies \lambda = -1$ with a multiplicity of $p^{(h+2)r}-2$. For the quadratic part, we solve $\lambda^2-(p^{(h-1)r}(p^{(h+2)r}-p^{hr}-1))\lambda+(2p^{(h+2)r}+2p^{hr}-4)(p^{hr}+3) = 0$. If we let $(p^{(h-1)r}(p^{(h+2)r}-p^{hr}-1)\lambda = \sigma$ and $(2p^{(h+2)r}+2p^{hr}-4)(p^{hr}+3) = \tau$, we obtain $\frac{1}{2}(\sigma \pm \sqrt{\sigma^2-4\tau})$.
(iv) This follows form the proof of the determinant of distance matrix in proposition 2.3. □

4 The 4-Radical Zero Finite Completely Primary Rings of Characteristic p^3

4.1 Construction III

Let $R' = GR(p^{3r}, p^3)$ be a Galois ring of characteristic p^3 and of order p^{3r}. Consider finitely generated R' modules U, V and W with dimensions s, t and λ respectively whose generating sets are $\{u_1, \cdots, u_s\}, \{v_1, \cdots, v_t\}$ and $\{w_1, \cdots, w_\lambda\}$ where $s+t+\lambda = h$ so that $R = R' \oplus U \oplus V \oplus W$ is an additive abelian group. Consider $s = h-1, t = 1$ and $\lambda = 0$ so that $R = R' \oplus \sum_{i=1}^{h-1} R'u_i \oplus R'v$ where $p^2u_i \neq 0,\ p^3u_i = 0$ where $1 \leq i \leq s$ and $pv = 0$. The following multiplication is defined on R:
$(a_\circ, \overline{a}_1, \overline{a}_2, \cdots, \overline{a}_{h-1}, \tilde{a}_h)(b_\circ, \overline{b}_1, \overline{b}_2, \cdots, \overline{b}_{h-1}, \tilde{b}_h) = (a_\circ b_\circ, a_\circ\overline{b}_1+\overline{a}_1 b_\circ, \cdots, a_\circ\overline{b}_{h-1}+\overline{a}_{h-1}b_\circ, a_\circ\tilde{b}_h+\tilde{a}_h b_\circ + \sum_{i,j=1}^{h-1}\overline{a_i b_j})$ where $\overline{a}_i, \overline{b}_j \in R'/p^2R'$ and $\tilde{a}_h, \tilde{b}_h \in R'/pR'$. From [12], it is verifiable that R is turned into a commutative ring with identity $(1, \overline{0}, \cdots, \overline{0}, \tilde{0})$ by the multiplication The set of zero divisors $Z(R)$ satisfy the properties below;
$Z(R) = pR' \oplus \sum_{i=1}^{s} R'u_i \oplus R'v,\ \ (Z(R))^2 = p^2R' \oplus p\sum_{i=1}^{s} R'u_i \oplus R'v,\ \ (Z(R))^3 = pR'v,\ (Z(R))^4 = (0)$.

The results in the sequel describe some properties of $\Gamma(R)$ of the ring constructed in this Section.

Proposition 4.1. *Let R be a ring of Construction III. Then:*

(i) The cardinality, $\mid V(\Gamma(R)) \mid= p^{3hr} - 1$.

(ii) The maximum degree, $\Delta(\Gamma(R)) = p^{3hr} - 2.$

(iii) The minimum degree, $\delta(\Gamma(R)) = p^{hr} - 1.$

(iv) The graph $\Gamma(R)$ *is incomplete.*

Proof. (i) Given that $Z(R) = pR' \oplus \sum_{i=1}^{s} R'u_i \oplus R'v$ and that $p^2u_i \neq 0,\ p^3u_i = 0$ and $pv = 0$, it is easy to see that $\mid pR' \mid = p^{2r},\ \mid R'u_i \mid = p^{3r}$ and $\mid R'v \mid = p^r$. Therefore, $\mid Z(R) \mid = p^{2r}(p^{3r(h-1)})p^r = p^{3hr}$. Since $\mid Z(R) \setminus \{0\} \mid = \mid (Z(R))^* \mid = p^{3r} - 1, \Longrightarrow \mid (Z(R))^* \mid = \mid V(\Gamma(R)) \mid = p^{3r} - 1.$

The Proofs for (ii) and (iii) are described in the next Proposition. For (iv), the fact that $(Z(R))^2 \neq (0)$ explains the incompleteness of $\Gamma(R)$. □

Proposition 4.2. *Let* R *be a ring of Construction III. Suppose* V_1, V_2, V_3, V_4 *and* V_5 *are the partitions of* $V(\Gamma(R))$. *Then the degrees of vertices* $v \in V(\Gamma(R))$

$$= \begin{cases} p^{3hr} - 2, & v \in V_1 \text{ and } \mid V_1 \mid = p^{hr} - 1; \\ p^{2hr} - 2, & v \in V_2 \text{ and } \mid V_2 \mid = p^{2hr} - p^{hr}; \\ deg(v) \in (X \cup Y) = V_3, & v \in V_3 \text{ and } \mid V_3 \mid = p^{(h+1)r} - p^{(h-1)r}\ ; \\ deg(v) \in (W \cup Z) = V_4, & v \in V_4 \text{ and } \mid V_4 \mid = 2p^{(h+2)r}; \\ p^{(h+1)r} - p^{hr} + p^{(h-1)r} - 1, & v \in V_5 \text{ and } \mid V_5 \mid = p^{3hr} - 2p^{(h+2)r} + p^{(h+1)r}. \end{cases}$$

Proof. We describe the connectedness of $\Gamma(R)$ for the ring in this section as follows: Let $\gamma_1, \cdots, \gamma_r \in R'$ with $\gamma_1 = 1$ such that $\overline{\gamma}_1, \cdots, \overline{\gamma}_r \in R'$ is the basis of R' over its prime subfield R'/pR'. From the defined multiplication, $Ann(Z(R)) = \{p^2\gamma_i u_1 + \cdots + p^2\gamma_i u_{h-1} + b\gamma_i v \mid b \in R'\}$. Let $V_1 = Ann(Z(R))^*$, therefore the order of $V_1,\ \mid V_1 \mid = p^{hr} - 1$. Every $v \in V_1$ is adjacent to each vertex in $\Gamma(R)$ and therefore the degree, $deg(v) \in V_1 = p^{3hr} - 2$. Similarly, consider set $V_2 = \{p^2r_\circ + p^2\gamma_i u_1 + \cdots + p^2\gamma_i u_{h-1} + b\gamma_i v \mid p^2r_\circ \neq 0, b \in R'\}$. Each vertex $v \in V_2$ is connected to other vertices in $\Gamma(R)$ apart from the vertices of the form $pr_\circ + \gamma_i u_1 + \cdots + \gamma_i u_{h-1} + b\gamma_i v, b \in R'$ where $r_\circ$ is not a multiple of p. Thus, $\mid V_2 \mid = p^{2hr} - p^{hr}$ and $deg(v) \in V_2 = p^{2hr} - 2$.
Next, suppose $X = \{p^2r_\circ + p\gamma_i u_1 + \cdots + p\gamma_i u_{h-1} + b\gamma_i v\} \setminus V_1 \cup V_2$. It means that the order of X, $\mid X \mid = p^{(h+1)r} - p^{hr}$. Each vertex in set X is connected to a vertex in either set $V_1,\ V_2,\ X$ or Y where $Y = \{p\gamma_i u_1 + \cdots + p\gamma_i u_{h-1} + b\gamma_i v \mid b \in R'\} \setminus V_1$. This implies that $\mid Y \mid = p^{hr} - p^{(h-1)r}$ hence, $deg(v) \in X = p^{(h-1)r} - 1 + p^{hr} - p^{(h-1)r} + p^{hr} - p^{(h-1)r} + p^{(h+1)r} - p^{hr} - 1 = p^{(h+1)r} + p^{hr} - 2p^{(h-1)r} - 2$ and each $v \in Y$ is adjacent to either a vertex in V_1, V_2, X or Y. Thus $deg(v) \in Y = p^{(h+1)r} + p^{hr} - 2p^{(h-1)r} - 2.$

Further, let $V_3 = X \cup Y$. and consider set $W = \{pr_\circ + p\gamma_i u_1 + \cdots + p\gamma_i u_{h-1} + b\gamma_i v \mid b \in R'\} \setminus V_1 \cup V_2 \cup V_3$. Therefore, the order of $W, \mid W \mid = p^{(h+2)r} - (p^{(h-1)r} + p^{(h+1)r} - p^{hr} + p^{hr} - p^{(h-1)r}) = p^{(h+2)r} - p^{(h+1)r}$. Each $v \in W$ is either adjacent to a vertex in V_1 or V_2 therefore, $deg(v) \in W = p^{(h-1)r} - 1 + p^{hr} - p^{(h-1)r} = p^{hr} - 1$.
Similarly, let $Z = \{p^2 r_\circ + \gamma_i u_1 + \cdots + \gamma_i u_{h-1} + b\gamma_i v \mid b \in R'\}$. It means that the order of $Z, \mid Z \mid = p^r(p^{hr} - p^{(h-1)r})p^r = p^{(h+2)r} - p^{(h+1)r}$. Each vertex, $v \in Z$ is either connected to a vertex in V_1 or Y. So, $deg(v) \in Z = p^{(h-1)r} - 1 + p^{hr} - p^{(h-1)r} = p^{hr} - 1$. We finally consider the set $V_4 = W \cup Z$. and let set $V_5 = \{pr_\circ + \gamma_i u_1 + \cdots + \gamma_i u_{h-1} + b\gamma_i v \mid b \in R'\} \setminus Z$. Then, $\mid V_5 \mid = p^{(h-1)r}(p^{(h-1)r})p^r - (p^{(h+2)r} - p^{(h+1)r} = p^{(h-1)r}(p^{(h+1)r} - p^{hr}) - (p^{(h+2)r} - p^{(h+1)r}) = p^{(h+3)r} - 2p^{(h+2)r} + p^{(h+1)r}$. Therefore the degree of every vertex in V_5 is $p^{(h-1)r} - 1 + (p^{(h+1)r} - p^{hr}) = p^{(h+1)r} - p^{hr} + p^{(h-1)r} - 1$.

□

4.2 Matrices of the Zero Divisor Graph of a Ring in Construction III

The following results describe some properties of the Adjacency, Laplacian and distance matrices associated with $\Gamma(R)$ of the ring described in this section.

Proposition 4.3. *Let R be a ring of Construction III. The adjacency and Laplacian matrices have the following properties;*

(i) $[A]_{p^{3hr}-1}$ *and* $[L]_{p^{3hr}-1}$ *are both singular and symmetric.*

(ii) $rank([A]_{p^{3hr}-1}) = p^{3hr} - p^{2hr} + p^r + 1$.

(iii) $rank([L]_{p^{3hr}-1}) = p^{3hr} - p^{(h-1)r}$.

(iv) The number of real and complex eigenvalues $\lambda = \begin{cases} p^{2hr} - p^{hr} + 1, & \lambda \in \mathbb{R}; \\ p^{3hr} - p^{2hr} - p^{hr}, & \lambda \in \mathbb{C}. \end{cases}$

for both the adjacency and Laplacian matrices.

Proof. The steps for the proof of (i),(ii) and (iii) are similar to the ones in proposition 2.2. We provide the proof for (iv) as follows.
Upon solving the equation $\mid \lambda I - [A]_{p^{3hr}-1} \mid = 0$, we obtain the real eigenvalues by evaluating $-\lambda^{(p^{2hr}-p^{hr}-p^r)}(1+\lambda)^{p^r+1} = 0$. This implies that $\lambda = 0$ of multiplicity $p^{2hr} -$

$p^{hr} - p^r$ and $\lambda = -1$ of multiplicity $p^r + 1$. Therefore real eigenvalues are $p^{2hr} - p^{hr} - p^r + p^r + 1 = p^{2hr} - p^{hr} + 1$ in number. The number of complex eigenvalues in $[A]_{p^{3hr}-1}$ is $(p^{3hr} - 1) - (p^{2hr} - p^{hr} + 1) = p^{3hr} - p^{2hr} - p^{hr}$.
For the Laplacian matrix, simplifying $| \lambda I - [L]_{p^{3hr}-1} |= 0$ results to the characteristic equation of the form $-(-1 + \lambda)^{(p^{2hr}-p^{hr}-p^r)}\lambda^{p^r+1} = 0$. Solving the equation yields real eigenvalues $\lambda = 0$ of multiplicity $p^r + 1$ and $(-1 + \lambda)^{(p^{2hr}-p^r-1)} = 0$ implying that $\lambda = 1$ of multiplicity $p^{2hr} - p^{hr} - p^r$. Therefore, the number of real eigenvalues are $p^{2hr} - p^{hr} - p^r + p^r + 1 = p^{2hr} - p^{hr} + 1$. From this and given that the matrix is of order $p^{3hr} - 1$, the complex eigenvalues are $p^{3hr} - p^{2hr} - p^{hr}$ in number. □

Proposition 4.4. *Let R be a ring of Construction III and $[d_{ij}]$, the distance matrix. Then;*

(i) $Tr([d_{ij}]) = 0$.

(ii) $rank([d_{ij}]) = p^{2hr} - 2$.

(iii) The eigenvalues $\lambda = \begin{cases} -1, & \text{of multiplicity } p^{2hr}; \\ -p^{2r}, & \text{of multiplicity } p^{2hr} - 1; \\ p^{2r} + 1 & . \end{cases}$

(iv) $Det([d_{ij}]) = p^{hr}$.

Proof. The steps for the proof are similar to those in propositions 3.3. □

5 4-Radical Zero Finite Completely Primary Rings of Characteristic p^4

5.1 Construction IV

Let $R' = GR(p^{4r}, p^4)$ be a Galois ring of order p^{4r} and characteristic p^4. Consider finitely generated R'-modules U, V and W generated by $\{u_1, u_2, \cdots, u_s\}, \{v_1, v_2, \cdots, v_t\}$ and $\{w_1, w_2, \cdots, w_\lambda\}$ respectively. Let $dim_{R'}U = s, dim_{R'}V = t$ and $dim_{R'}W = \lambda$, so that $R = R' \oplus U \oplus V \oplus W$ is an additive abelian group and $s + t + \lambda = h$. Assume $s = h, t = 0$ and $\lambda = 0$ so that

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$R = R' \oplus \sum_{i=1}^{s} R'u_i$ with $pu_i = 0,\ 0 \leq i \leq s$. The multiplication on R is defined by;

$$(a_\circ, \overline{a}_1, \cdots, \overline{a}_h)(b_\circ, \overline{b}_1, \cdots, \overline{b}_h) =$$
$$(a_\circ b_\circ, a_\circ \overline{b}_1 + \overline{a}_1 b_\circ, \cdots, a_\circ \overline{b}_h + \overline{a}_h b_\circ)$$

where $\overline{a}_i,\ \overline{b}_j \in R'/pR'$ and $1 \leq i, j \leq s$. R is turned by this multiplication into a commutative ring with identity $(1, \overline{0}, \cdots, \overline{0})$. The set $Z(R)$ satisfy the following properties; $Z(R) = pR' \oplus \sum_{i=1}^{s} R'u_i,\ (Z(R))^2 = p^2R',\ (Z(R))^3 = p^3R',\ (Z(R))^4 = (0)$.

The following result describes the zero divisor graph $\Gamma(R)$ of the ring constructed in this Section.

Proposition 5.1. *Let R be a ring of Construction IV. Let V_1, V_2, V_3 and V_4 be the order of partitions of vertices in $V(\Gamma(R))$. Then:*

(i) The cardinality, $\mid V(\Gamma(R)) \mid = p^{(h+3)r} - 1$.

(ii) $deg(v) = \begin{cases} p^{(h+3)r} - 2, & v \in V_1 \text{ and } \mid V_1 \mid = p^{(h+2)r} - 1; \\ p^{hr} + p^{(h-1)r} + p^r, & v \in V_2 \text{ and } \mid V_2 \mid = p^{hr}; \\ p^{(h+1)r} - p^r, & v \in V_3 \text{ and } \mid V_3 \mid = p^{hr} + p^{(h-1)r}; \\ p^{hr} - p^{(h-1)r} + 1, & v \in V_4 \text{ and } \mid V_4 \mid = p^{(h+1)r} - p^{hr}. \end{cases}$

Proof. (i) Given $Z(R) = pR' \oplus \sum_{i=1}^{s} R'u_i$ and that $pu_i = 0$, then, $\mid Z(R) \mid = \mid V(\Gamma(R)) \mid$. Further, $\mid pR' \mid = p^{3r}$ and $\mid R'u \mid = p^{hr}$. Therefore, $\mid Z(R) \mid = p^{3r}(p^{hr}) = p^{(h+3)r}$ and $\mid Z(R) \setminus \{0\} \mid = p^{(h+3)r} - 1 =$
$\mid V(\Gamma(R)) \mid$.
(ii) Let $\gamma_1, \gamma_2, \cdots, \gamma_r \in R'$ with $\gamma_1 = 1$ such that $\overline{\gamma}_1, \overline{\gamma}_2, \cdots \overline{\gamma}_r \in R'$ forms a basis for R' over its prime subfield R'/pR'. From the multiplication given, $Ann(Z(R)) = \{p^3r_\circ + b\gamma_i u_1 + \cdots + b\gamma_i u_h \mid b \in R'\}$. Let $V_1 = Ann(Z(R)) \setminus \{0\}$. This implies that $\mid V_1 \mid = p^{(h-1)r} - 1$. Each vertex $v \in V_1$ is connected to every other vertex in $V(\Gamma(R))$. Therefore, $deg(v)$ in the set V_1 is $p^{(h+3)r} - 1 - 1 = p^{(h+3)r} - 2$.
Let $V_2 = \{p^3r_\circ + b\gamma_i u_1 + \cdots + b\gamma_i u_h \mid b \in R'\}$. Clearly $\mid V_2 \mid = p^{hr}$ and every $v \in V_2$ is adjacent to a vertex of the form $pr_\circ + b\gamma_i u_1 + \cdots + b\gamma_i u_h$ therefore, $deg(v)$ in the set V_2 is $p^{hr} + p^{(h-1)r} + p^r$.
Further, let $V_3 = \{p^2r_\circ + b\gamma_i u_1 + \cdots + b\gamma_i u_h \mid b \in R'\}$ then $\mid V_3 \mid = p^{hr} + p^{(h-1)r}$. Each $v \in V_3$ is adjacent to the vertex of the form $p^2r_\circ + b\gamma_i u_1 + \cdots + b\gamma_i u_h$ therefore, $deg(v)$ in V_3 is $p^{(h+1)r} - p^r$.
Finally, let $V_4 = \{pr_\circ + \gamma_i u_1 + \cdots + \gamma_i u_h\} \setminus V_1 \cup V_3$. Therefore, $\mid V_4 \mid = p^{(h+1)r} - (p^{(h-1)r} +$

$p^{hr} - p^{(h-1)r}) = p^{(h+1)r} - p^{hr}$. Each $v \in V_4$ is either adjacent to a vertex in V_1 or V_2. So, $deg(v)$ in the set V_4 is $p^{hr} - (p^{(h-1)r} - 1) = p^{hr} - p^{(h-1)r} + 1$. □

5.2 Matrices of the Zero Divisor Graph of a Ring in Construction IV

Proposition 5.2. *Let R be a ring of Construction IV. The adjacency and Laplacian matrices satisfy the following properties;*

(i) $[A]_{p^{(h+3)r}-1}$ and $[L]_{p^{(h+3)r}-1}$ are both singular.

(ii) $rank([A]_{p^{(h+3)r}-1}) = p^{hr} + p^{(h-2)r} + 1$.

(iii) $rank([L]_{p^{(h+3)r}-1}) = p^{(h+1)r} + p^{hr} + 2$.

(iv) The number of real and complex eigenvalues $\lambda = \begin{cases} p^{(h+1)r} + 2p^{(h-1)r}, & \lambda \in \mathbb{R}; \\ p^{(h+2)r} - p^{(h+1)r} - 2p^{(h-1)r} - 1, & \lambda \in \mathbb{C}. \end{cases}$

for both $[A]_{p^{(h+3)r}-1}$ and $[L]_{p^{(h+3)r}-1}$.

Proof. We provide a proof for (iv). The proof for (i),(ii) and (iii) are clear. Upon obtaining the characteristic polynomial for the adjacency matrix, we find the real eigenvalues from the equation
$-\lambda^{(p^{(h+1)r}+p^{(h-1)r}-1)}(1+\lambda)^{(p^{(h-1)r}+1)} = 0$. The solution to this results to $\lambda = 0$ of multiplicity $p^{(h+1)r}+p^{(h-1)r}-1$ and $(1+\lambda)^{(p^{(h-1)r}+1)} = 0$ implying that $\lambda = -1$ of multiplicity $p^{(h-1)r}+1$. Therefore, the number of real eigenvalues from the characteristic polynomial equation of the adjacency matrix is $p^{(h+1)r} + p^{(h-1)r} + p^{(h-1)r} - 1 + 1 = p^{(h+1)r} + 2p^{(h-1)r}$.
Given that the adjacency matrix $[A]_{p^{(h+3)r}-1}$ is a square matrix with $p^{(h+2)r} - 1$ rows and columns and its characteristic polynomial has both real and complex parts, we have that the number of complex eigenvalues are $(p^{(h+2)r} - 1) - p^{(h+1)r} + 2p^{(h-1)r} = p^{(h+2)r} - p^{(h+1)r} - 2p^{(h-1)r} - 1$.

For the Laplacian matrix, the characteristic polynomial equation is of the form
$-(\lambda^{p^{(h+2)r}-1}+\lambda p^{(h+1)r}-p^{(h-1)r})(-1+\lambda)^{(p^{(h+1)r}+p^{(h-1)r}-1)}\lambda^{(p^{(h-1)r}+1)} = 0$. From the equation, we obtain the real eigenvalues by solving the equation $(-1+\lambda)^{(p^{(h+1)r}+p^{(h-1)r}-1)}\lambda^{(p^{(h-1)r}+1)} = 0$. This implies that $\lambda = 0$ of multiplicity $p^{(h-1)r} + 1$ and $\lambda = 1$ of multiplicity $p^{(h+1)r} + p^{(h-1)r} - 1$. Similarly, we can find the values of λ in the remaining factor by solving the equation $-(\lambda^{p^{(h+2)r}-1} + \lambda p^{(h+1)r} - p^{(h-1)r}) = 0$ to obtain the complex eigenvalues. □

6 Conclusion

In this paper, we have established that the zero divisor graphs of classes of 4-Radical Zero Completely Primary Finite Rings can be expressed in terms of matrices. Therefore, this provides an illustration for better analysis of the graphs from the perspective of matrix algebraic properties. The focus of this research was on the Adjacency, Laplacian and Distance matrices associated with the zero divisor graphs of the classes of rings in constructions I to IV. A further research on other types of matrices can be explored.

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