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Graph Numbers and Distance Related Parameters of Zero Divisor Graphs

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ABSTRACT

Distance-related parameters have applications in the field of pharmaceutical chemistry, network discovery, robot navigation, and optimizations. Cyclic structures exhibit significant topological features that have become important research areas in the field of computer science and mathematics. Due to the inherent algebraic relationship between graph numbers and distance related parameters, this paper characterizes variants of distance related parameters and graph numbers associated with the zero divisor graphs akin to cyclic structures obtained from classes of completely primary finite rings. In particular, we investigate the local fractional metric dimension and provide certain results concerning graph indices namely the Weiner index and the Zagreb index.

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1 Introduction

The graph distance related parameters has been extensively studied by various authors. Among others, Harary and Melter [10] studied the problem of finding the metric dimension of a graph, [12] showed that the metric dimension of a graph is an NP-complete problem. Distance-based parameters for networks play a vital role in various fields including pharmaceutical chemistry [5], network discovery [4], robot navigation, and optimizations [12]. Besides, many real-life large-scale systems having substantial topological features can be modeled as complex networks such as social networks, information networks, technological networks, and biological networks. This representation has innovative impacts to information processing and co-ordination of these large-scale networks. Management of large-scale networks such as Internet with their tremendous growth and heterogeneity is a challenging mathematical problem which have profound implications for the efficient design of future communication networks. Complex networks are composed of building blocks, and if the building blocks are considered as symmetric networks, then complexity of these networks can be reduced for better analysis and interpretation. The concept of undirected zero-divisor graph of a commutative ring was first studied by Beck in [3] showed that all the elements of a ring R were the vertices of the graph, and he was mainly interested in coloring. This work was further studied by Anderson and Naseer [2]. A different approach of associating a graph $\Gamma(R)$ to R with vertices as $Z^*(R) = Z(R)\backslash\{0\}$ was given in [1]. Two vertices $x, y \in Z^*(R)$ of $\Gamma(R)$ are adjacent if and only if $xy = 0$. They believed that this better illustrates the zero-divisor structure of the ring. The zero-divisor graph of a commutative ring has also been studied in [1, 3, 6, 15] and was extended by Redmond [20] to noncommutative rings. Redmond [15] also extended the zero-divisor of a commutative ring to an ideal-based zero-divisor graph of a commutative ring. For a given ideal I of R, he defined an undirected graph $\Gamma_I(R)$ with vertex set $\{x \in R - I \mid xy \in I$ for some $y \in R - I\}$, where distinct vertices x and y are adjacent if and only if $xy \in I$.

A simple graph $G(V, E)$ consists of a finite nonempty set $V(G)$ of objects called vertices together with a set $E(G)$ of unordered pairs of distinct vertices of G called edges. A graph G is connected if there is a path between every two distinct vertices of G. The distance from a vertex v to u denoted by $d(v, u)$ is the length of the shortest path from v to $u(d(v, v) = 0$ and $d(v, u) = \infty$, if there is no such path) [13]. The diameter of G is $\operatorname{diam}(G) = \sup\{d(v, u) \mid v, u \in V(G)\}$. The neighborhood $N(v)$ of a vertex v denotes the set all vertices of G adjacent to the vertex v and $N[v] = N(v) \cup \{v\}$.

The concept of the metric representation and the metric dimension in terms of the locating number in a zero-divisor graph associated with a commutative ring with unity was introduced in [8] and had been further studied in [7]. Feng and Wang in [8] discussed various properties of the locating set and the locating number which includes the characterization of all finite commutative rings with unity, examination of two equivalence relations on the vertices of $\Gamma(R)$, relationship between the locating set and the cut vertices of $\Gamma(R)$, investigation of the locating number in $\Gamma(R)$ when R is a finite product of the integral domains and so on.

Let G_k be a graph on infinite number of vertices with vertex set $V(G_k) = \{v\} \cup \left\{v_1^{(1)}, v_2^{(1)}, \ldots, v_k^{(1)}\right\} \cup \left\{v_1^{(2)}, v_2^{(2)}, \ldots, v_k^{(2)}\right\} \cup \cdots \cup \left\{v_1^{(i)}, v_2^{(i)}, \ldots, v_k^{(i)}\right\} \cup \cdots$ for $i \geq 1$, and the edges are defined by the rule $vv_t^{(1)}, (1 \leq t \leq k)$, and $v_1^{(j)}v_1^{(j+1)}, v_2^{(j)}v_2^{(j+1)}, \ldots, v_k^{(j)}v_k^{(j+1)}$ for all $j = 1, 2, 3, \ldots$. For $k = 1, G_1$ is an infinite tree with vertex set $V(G_1) = \{v\} \cup \left\{v_1^{(1)}\right\} \cup \left\{v_1^{(2)}\right\} \cup \cdots \cup \left\{v_1^{(i)}\right\} \cup \cdots$ for $i \geq 1$, and the edges are defined by $vv_1^{(1)}$ and $v_1^{(j)}v_1^{(j+1)}$ for all $j = 1, 2, 3, \ldots$. Notice, here that the infinite tree G_1 is rooted at the vertex v.

For $k=2, G_2$ is a graph with vertex set $V\left(G_2\right)=\{v\} \cup\left\{v_1^{(1)}, v_2^{(1)}\right\} \cup\left\{v_1^{(2)}, v_2^{(2)}\right\} \cup \cdots \cup\left\{v_1^{(i)}, v_2^{(i)}\right\} \cup \cdots$ for $i \geq 1$, and the edges are defined by $v v_t^{(1)},(1 \leq t \leq 2)$, and $v_1^{(j)} v_1^{(j+1)}, v_2^{(j)} v_2^{(j+1)}$ for all $j=1,2,3, \ldots$ [11]. Clearly, G_2 is a tree rooted at the vertex v with two infinite branches. The infinite trees G_k are often denoted by $P_{k, 1}$ to indicate trees rooted in v with k infinite branches. It is straight forward to prove that $\operatorname{dim}_M\left(P_{1,1}\right)=1$ and $\operatorname{dim}_M\left(P_{2,1}\right)=2$.

In this paper, we characterize variants of distance related parameters and graph numbers associated with the zero divisor graphs akin to cyclic structures obtained from classes of completely primary finite rings. In particular, we compute the local fractional metric dimension and provide certain results concerning graph indices namely the Weiner index and the Zagreb index. We also explore the connection between graph number and the metric dimension of the zero divisor graph. We investigate these parameters in $R=\Gamma\left(R_1 \oplus R_2 \oplus \ldots \oplus R_n\right)$, where $R_1, R_2, \ldots, R_n$ are n finite commutative rings each having unity 1 and none of $R_i,(1 \leq i \leq n)$.

2 Preliminaries

The following results are useful in the sequel.

Theorem 2.1. *Let R be a commutative ring with unity 1 (not a domain). Then $\operatorname{dim}_M(\Gamma(R))$ is finite if and only if R is finite.*

Proof. Suppose R is finite. Then, it is clear that $\operatorname{dim}_M(\Gamma(R))$ is finite. Now, suppose $\operatorname{dim}_M(\Gamma(R))$ is finite. Let S be the metric basis for $\Gamma(R)$ with $|S|=k$, where k is some non-negative integer. The diameter of $\Gamma(R)$ is not more than 3 . Therefore, $d(x, y) \in\{0,1,2,3\}$ for every $x, y \in Z^*(R)$. For each $x \in Z^*(R)$, the metric representation $D(x \mid S)$ is the k coordinate vector, where each coordinate is in the set $\{0,1,2,3\}$. Thus there are only $(3+1)^k$ possibilities for $D(x \mid S)$. Since $D(x \mid S)$ is unique for each $x \in Z^*(R)$, so $\left|Z^*(R)\right| \leq 4^k$. This implies that $Z^*(R)$ is finite and hence R is finite. □

Theorem 6.1 in [14] gives the metric dimension for the zero-divisor graph $\Gamma\left(R_1 \times R_2 \times \ldots \times R_n\right)$, where $R_1, R_2, \ldots, R_n$ are integral domains, and also gives bounds for the metric dimension of the zero-divisor graph $\Gamma\left(\prod_{i=1}^n \mathbb{Z}_2\right)$. Special emphasis has been given to the graph $\Gamma\left(\prod_{i=1}^n \mathbb{Z}_2\right)$ of a finite Boolean ring, and it is shown that $\operatorname{dim}_M\left(\Gamma\left(\prod_{i=1}^n \mathbb{Z}_2\right)\right) \leq n$, $\operatorname{dim}_M\left(\Gamma\left(\prod_{i=1}^n \mathbb{Z}_2\right)\right)=n-1$, for $n=2,3,4$, and $\operatorname{dim}_M\left(\Gamma\left(\prod_{i=1}^n \mathbb{Z}_2\right)\right)=n$, for $n=5$ [14]. We need to know as how the metric dimension behaves with respect to the product $R_1 \times R_2 \times \cdots \times R_n$, where $R_1, R_2, \ldots, R_n$ are n finite commutative rings with each having unity 1 .

Lemma 2.2. *A finite commutative ring R with unity 1 has exactly one unit if and only if $R \cong \prod_{i=1}^n \mathbb{Z}_2$ for some positive integer n.*

Proof. Clearly, the ring listed has only one unit. Suppose R has exactly one unit. If R is a local ring with maximal ideal M, then $|R|=p^k$ and $|M|=p^m$ for some prime p and integers $0 \leq m<k$. Then, $1=|U(R)|=|R|-|M|$ only when $|R|=2$ and $|M|=1$. If R is not local, then R can be written as the finite product of local rings that is $R \cong R_1 \times R_2 \times \ldots \times R_n$, where $R_1, R_2, \ldots, R_n$ are finite local rings. If any R_i has more than one unit, then R would have more than one unit. □

Theorem 2.3. *Let $R_1, R_2, \ldots, R_n$ be n finite commutative rings (not domains) each having unity 1 with none of $R_i, 1 \leq i \leq n$, being isomorphic to $\prod_{i=1}^{n} \mathbb{Z}_2$ for any positive integer n. Then for a finite commutative ring R with unity 1 and for a finite field $\mathbb{F}_q$ on prime q number of elements,*

(a) $\dim_M(\Gamma(R_1 \times R_2 \times \cdots \times R_n)) \geq \sum_{i=1}^{n} \dim_M(\Gamma(R_i))$,

(b) $\dim_M(\Gamma(R \times \mathbb{F}_q)) = |Z^*(R \times \mathbb{F}_q)| - 2^{n+1} + 2$ *or* $|Z^*(R \times \mathbb{F}_q)| - 2$ *or at least* $|U(R)| + (|Z^*(R)| + 1)q - t - 3$, *where t is any positive integer.*

3 A Survey on the Distance Parameters of Cyclic Structures

Let C_n be a cyclic network with the vertex and edge set given by $V(C_n) = \{a_i \mid 1 \leq i \leq n\}$ and $E(C_n) = \{a_i a_{i+1} \mid 1 \leq i \leq n\}$, respectively, with indices taken $\bmod n$.
The local fractional strong metric dimension of certain complex networks is computed.

Theorem 3.1. *For $n \geq 3$,*

$$\operatorname{lsdim}_f(C_n) = \begin{cases} 1, & \text{if } n \equiv 0(\bmod 2); \\ \frac{n}{n-1}, & \text{if } n \equiv 1(\bmod 2). \end{cases}$$

Proof. To prove the above claim, we consider the following cases: Case $1(n \equiv 0(\bmod 2))$ We take note that, $\gamma(C_n) = |V(C_n)| = n$ and $\beta(C_n) = |\cup L(C_n)| = |V(C_n)| = n$. Hence, we conclude

$$\operatorname{lsdim}_f(C_n) = \sum_{s=1}^{\beta(C_n)} \frac{1}{\gamma(C_n)} = 1.$$

Case $2(n \equiv 1(\bmod 2))$ Here, $\gamma(C_n) = (n-1)$ and $\beta(C_n) = |\cup L(C_n)| = n$. We have

$$\operatorname{lsdim}_f(C_n) = \sum_{s=1}^{\beta(C_n)} \frac{1}{\gamma(C_n)} = \frac{n}{n-1}.$$

□

Theorem 3.2. *For $n \geq 6, \operatorname{lsdim}_f(C_n(1,2)) = n/2(\lceil m + 1/2 \rceil)$.*

Proof. Clearly, $\gamma(C_n(1,2)) = |S\{a_r, a_{r+1}\}| = |S\{a_r, a_{r-1}\}| = 2(\lceil m + 1/2 \rceil)$ where $1 \leq r \leq n$ and $m = \lceil n - 5/4 \rceil$. Moreover, $\beta(C_n(1,2)) = |\cup L(C_n(1,2))| = n$. Therefore, we have

$$\operatorname{lsdim}_f(C_n(1,2)) = \sum_{s=1}^{\beta(C_n(1,2))} \frac{1}{\gamma(C_n(1,2))} = \frac{n}{2(\lceil m + 1/2 \rceil)}.$$

□

Theorem 3.3. *For* $n \geq 6$,

$$\operatorname{lsdim}_f\left(M_{2n}\right)= \begin{cases}1, & \text{if } n \equiv 1(\bmod 2); \\ \frac{n}{n-1}, & \text{if } n \equiv 0(\bmod 2).\end{cases}$$

Proof. The proof of this theorem is subdivided into the following two cases:
Case 1 $(n \equiv 1(\bmod 2))$
We have $\gamma\left(M_{2n}\right)=\left|V\left(M_{2n}\right)\right|=2n$ and $\beta\left(M_{2n}\right)=\left|\cup L\left(M_{2n}\right)\right|=\left|V\left(M_{2n}\right)\right|=2n$. Hence the following can be concluded:

$$\operatorname{lsdim}_f\left(M_{2n}\right)=\sum_{s=1}^{\beta\left(M_{2n}\right)} \frac{1}{\gamma\left(M_{2n}\right)}=1.$$

Case 2 $(n \equiv 0(\bmod 2))$
In this case by considering $\gamma\left(M_{2n}\right)=2(n-1)$ and $\beta\left(M_{2n}\right)=\left|\cup L\left(M_{2n}\right)\right|=2n$. Hence, we have

$$\operatorname{lsdim}_f\left(M_{2n}\right)=\sum_{s=1}^{\beta\left(M_{2n}\right)} \frac{1}{\gamma\left(M_{2n}\right)}=\frac{n}{n-1}.$$

□

Theorem 3.4. *For* $n \geq 6$,

$$\operatorname{lsdim}_f\left(G_m^n\right)= \begin{cases}1, & \text{if } n \equiv 0(\bmod 2); \\ \frac{n}{n-1}, & \text{if } n \equiv 1(\bmod 2).\end{cases}$$

Proof. Proof. The proof can be segregated into the following two cases:
Case 1 $(n \equiv 1(\bmod 2))$
$\gamma\left(G_m^n\right)=m(n-1)$ and $\beta\left(G_m^n\right)=\left|\cup L\left(G_m^n\right)\right|=\left|V\left(G_m^n\right)\right|=mn$.
We have

$$\operatorname{lsdim}_f\left(G_m^n\right)=\sum_{t=1}^{\beta\left(G_m^n\right)} \frac{1}{\gamma\left(G_m^n\right)}=\frac{n}{n-1}.$$

Case 2 $(n \equiv 0(\bmod 2))$
In this , $\gamma\left(G_m^n\right)=mn$ and $\beta\left(G_m^n\right)=\left|\cup L\left(G_m^n\right)\right|=mn$. Hence we conclude that

$$\operatorname{lsdim}_f\left(G_m^n\right)=\sum_{s=1}^{\beta\left(G_m^n\right)} \frac{1}{\gamma\left(G_m^n\right)}=1.$$

□

4 Some Cyclic Structures obtained from $\Gamma(R)$

The following graphs represent cyclic structures associated with graph networks obtained from classes of completely primary finite rings. Consider a finite ring R of idealization given by $R = R_0 \oplus U \oplus V \oplus W \oplus Y$ where U, V, W and Y are R_0-modules generated by various basis elements u_i, v_j, w_k and y_l. If R is closed under product given by

$$\big(r_\circ + \sum_{i=1}^{e} r_i u_i + \sum_{j=1}^{f} s_j v_j + \sum_{k=1}^{g} t_k w_k + \sum_{l=1}^{h} z_l y_l\big)\big(r_\circ^{'} + \sum_{i=1}^{e} r_i^{'} u_i + \sum_{j=1}^{f} s_j^{'} v_j + \sum_{k=1}^{g} t_k^{'} w_k + \sum_{l=1}^{h} z_l^{'} y_l\big)$$

$$= r_\circ r_\circ^{'} + p^a \sum_{i,m=1}^{e} \big(r_i r_m^{'} + pR_0\big) + \sum_{i=1}^{e} \big[r_\circ r_i^{'} + r_i r_\circ^{'} + pR_0\big] u_i + \sum_{j=1}^{f} \big[\big(r_\circ + pR_0\big) s_j^{'} + s_j\big(r_\circ^{'} + pR_0\big) +$$

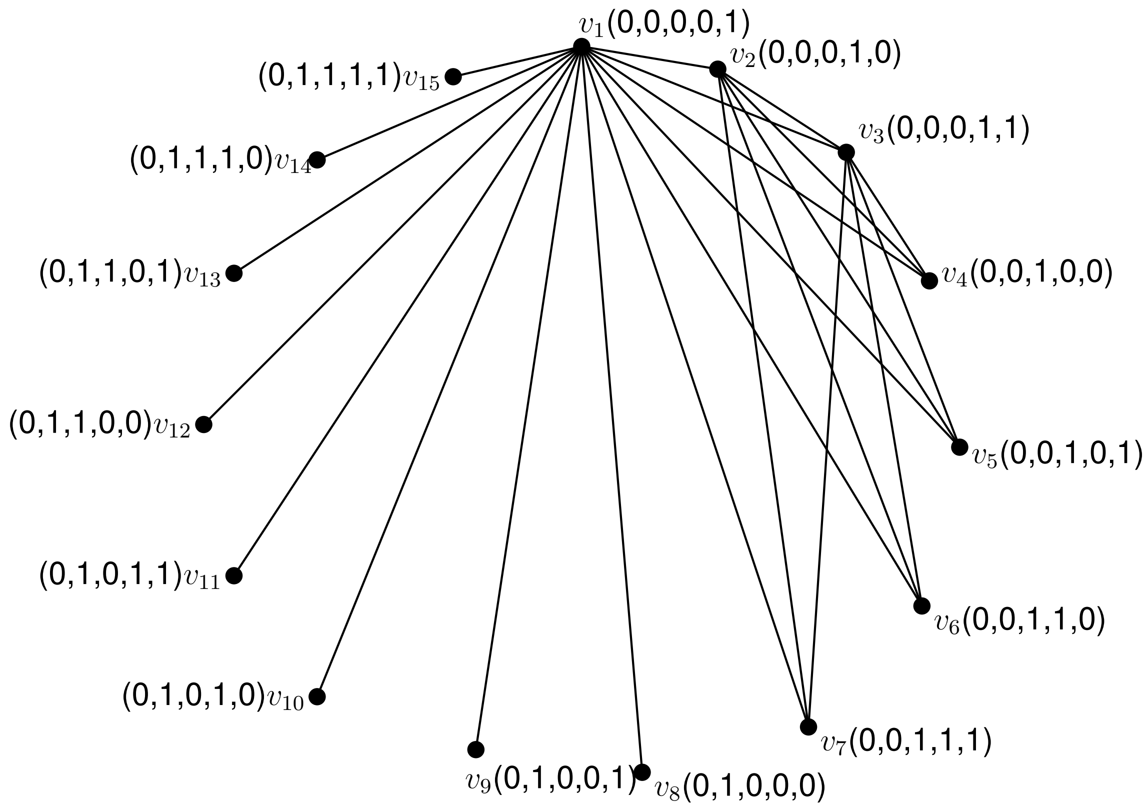
$$\sum_{\nu,\mu=1}^{e} \big(r_\nu r_\mu^{'} + pR_0\big)\big] v_j + \sum_{k=1}^{g} \big[\big(r_\circ + pR_0\big) t_k^{'} + t_k\big(r_\circ^{'} + pR_0\big) + \sum_{i,j} \big(r_i + pR_0\big) s_j^{'} + s_j\big(r_i^{'} + pR_0\big)\big] w_k +$$

$$\sum_{l=1}^{h} \big[\big(r_\circ + pR_0\big) z_l^{'} + z_l\big(r_\circ^{'} + pR_0\big) + \sum_{i,k} \big(r_i + pR_0\big) t_k^{'} + t_k\big(r_i^{'} + pR_0\big) + \sum_{\kappa,\tau=1}^{f} \big(s_\kappa s_\tau^{'} + pR_0\big] y_l$$

where $a = 1, 2, 3$ or 4 depending on whether $charR = p^2, p^3, p^4$ or p^5. This multiplication turns R into a commutative ring with identity $(1, 0, 0, 0, 0)$ [9].
Given a zero divisor graph $\Gamma(R)$ with vertex set $V(\Gamma(R))$. If $x\ and\ y$ are any two vertices of the graph, then x and y lie in the edge of the graph $E(\Gamma(R))$ if and only if $xy = 0$. Using this adjacency property, we have the following cyclic representation of the geometries of $\Gamma(R)$ for various characteristics.

Example 4.1. *If R is the ring of the construction above, where $R = R_0 \oplus U \oplus V \oplus W \oplus Y$ then for $charR = p = 2$*
$R = \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$ then the zero divisors will be
$Z(R) = \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$ In this case $p = 2$, $r = 1$, $e = 1$, $f = 1$, $g = 1$ and $h = 1$.
So the $\Gamma(R)$ is $4-$ partite with $dim(\Gamma(R)) = 2$, $gr(\Gamma(R)) = 3$ and $b(\Gamma(R)) = \frac{1}{4}$.

When $charR = p = 2$ *the graph is as follows;*

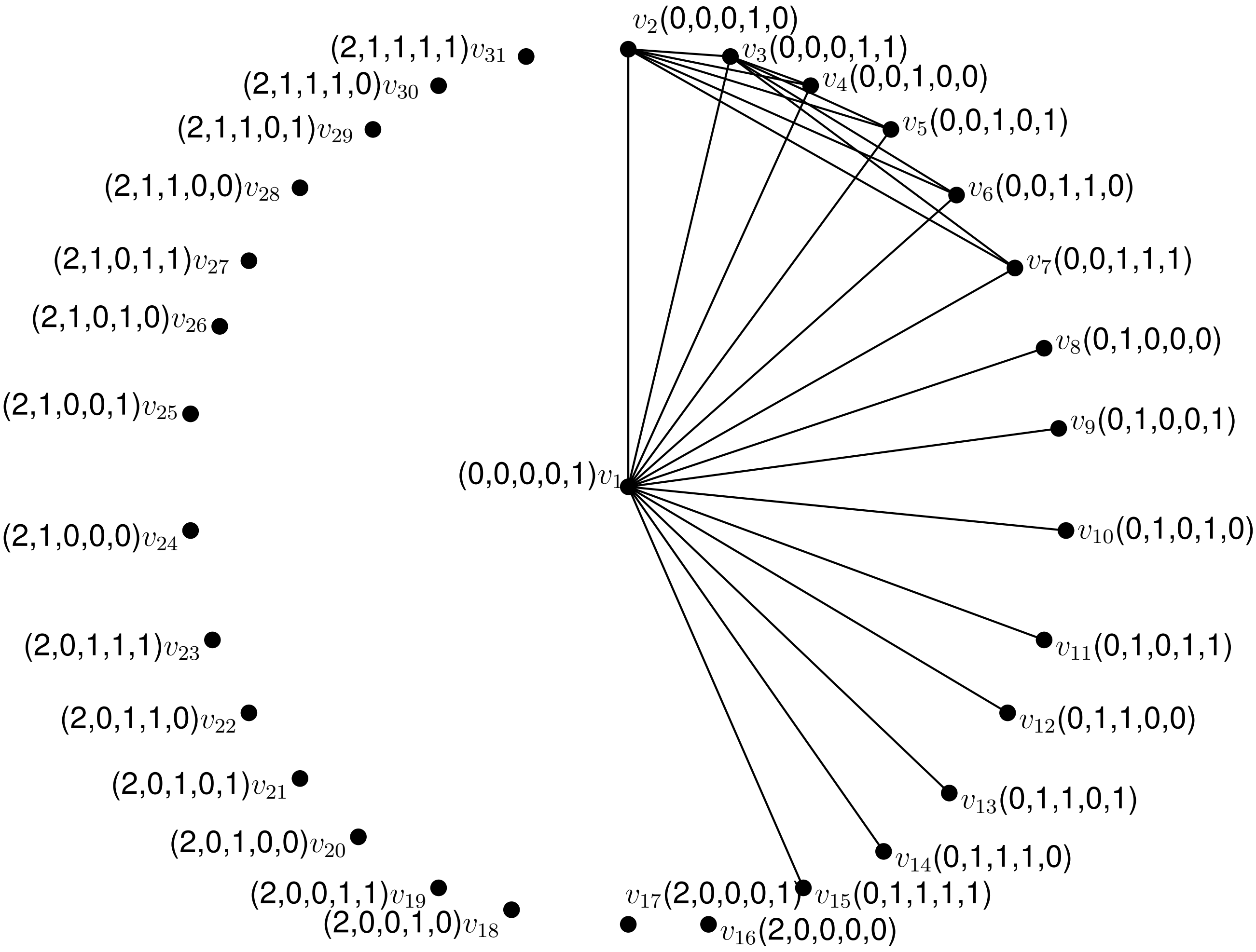


Example 4.2. *Let* $Z(R) = 2Z(R) = \mathbb{Z}_4 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus (Z)_2$. *In this case* $p = 2$, $r = 1$,$e = 1$ $f = 1$,$g = 1$, $g = 1$ *and* $h = 1$. *Then the set of vertices* $V(\Gamma(R))$ *is given by*

Let $Z(R) = 2Z(R) = \mathbb{Z}_4 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus (Z)_2$. *In this case* $p = 2$, $r = 1$,$e = 1$ $f = 1$,$g = 1$, $g = 1$ *and* $h = 1$. *Then*

$\Gamma(R)$ is $8-$ partite with $diam(\Gamma(R)) = 2$, $gr(\Gamma(R)) = 3$ and $b(\Gamma(R)) = \frac{7}{24}$.

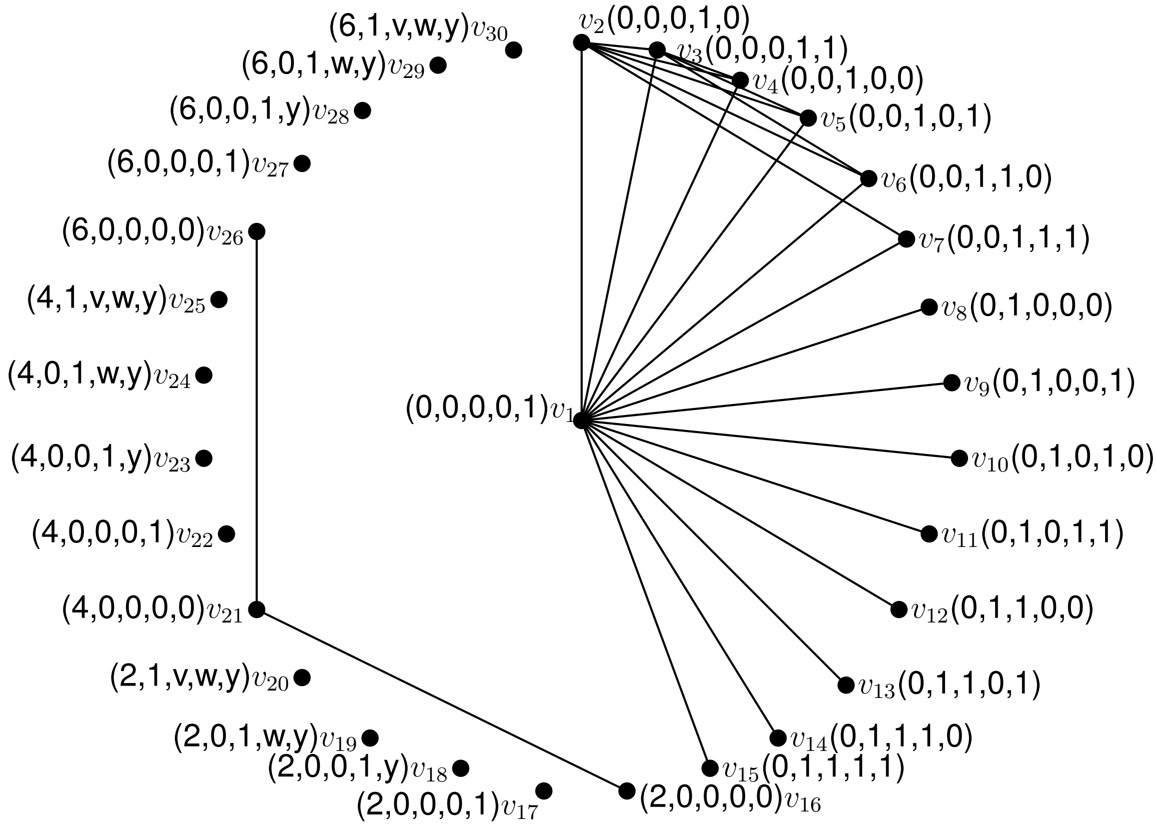
When $charR = P^2$ the graph is as follows;



Example 4.3. *Let* $Z(R) = 2Z(R) = 2\mathbb{Z}_8 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus (Z)_2$. *In this case* $p = 2$, $r = 1$,$e = 1$ $f = 1$,$g = 1$, $g = 1$ *and* $h = 1$. *Then*

$\Gamma(R)$ is $16-$ partite with $diam(\Gamma(R)) = 2$, $gr(\Gamma(R)) = 3$ and $b(\Gamma(R)) = \frac{5}{16}$.

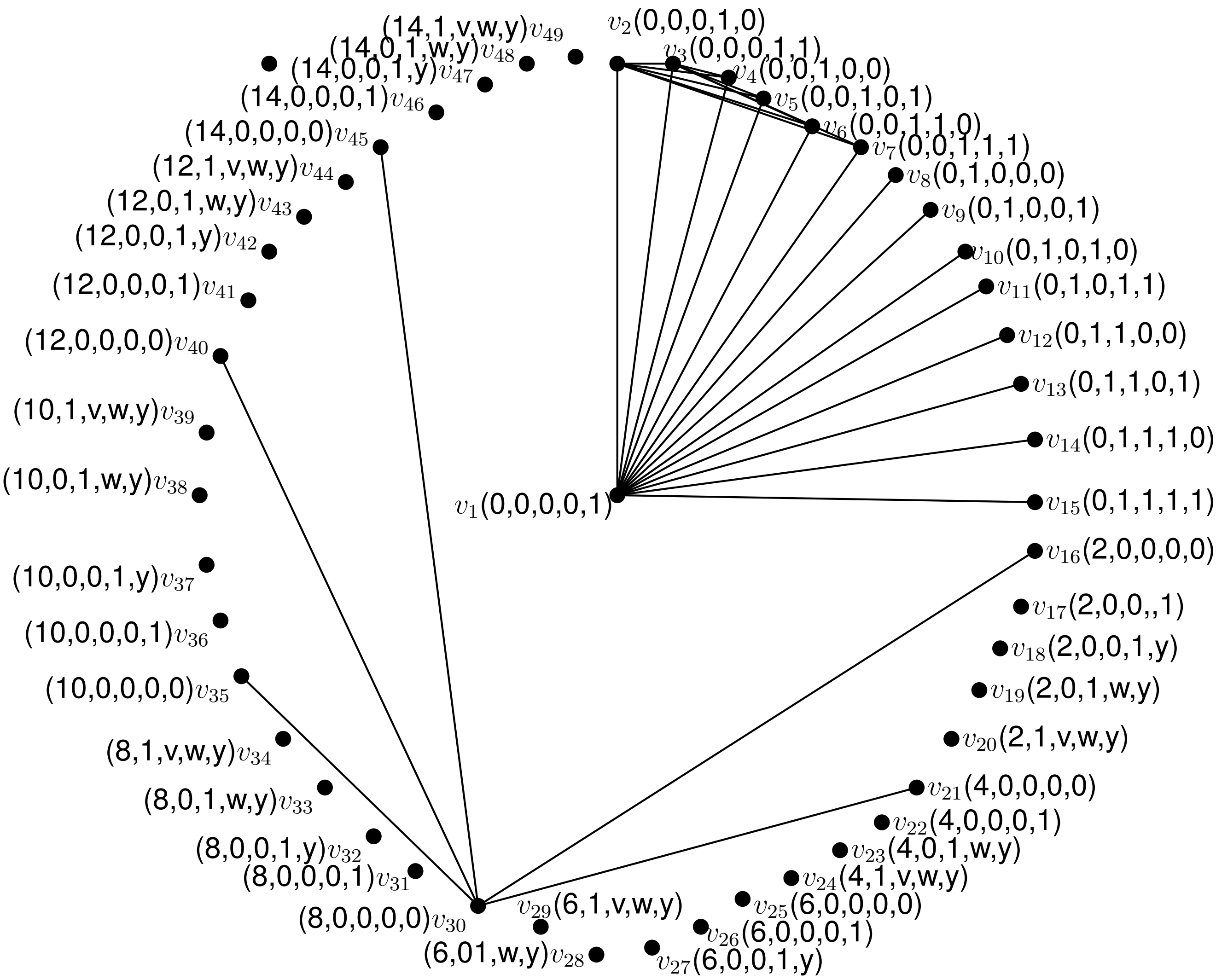
When $charR = p^3$ the graph will appear as below;



Example 4.4. *Let $Z(R) = 2\mathbb{Z}_{16} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus (Z)_2$. In this case $p = 2$, $r = 1$,$e = 1$ $f = 1$,$g = 1$, $g = 1$ and $h = 1$. Then*

$\Gamma(R)$ is $32-$ partite with $diam(\Gamma(R)) = 2$, $gr(\Gamma(R)) = 3$ and $b(\Gamma(R)) = \frac{31}{96}$.

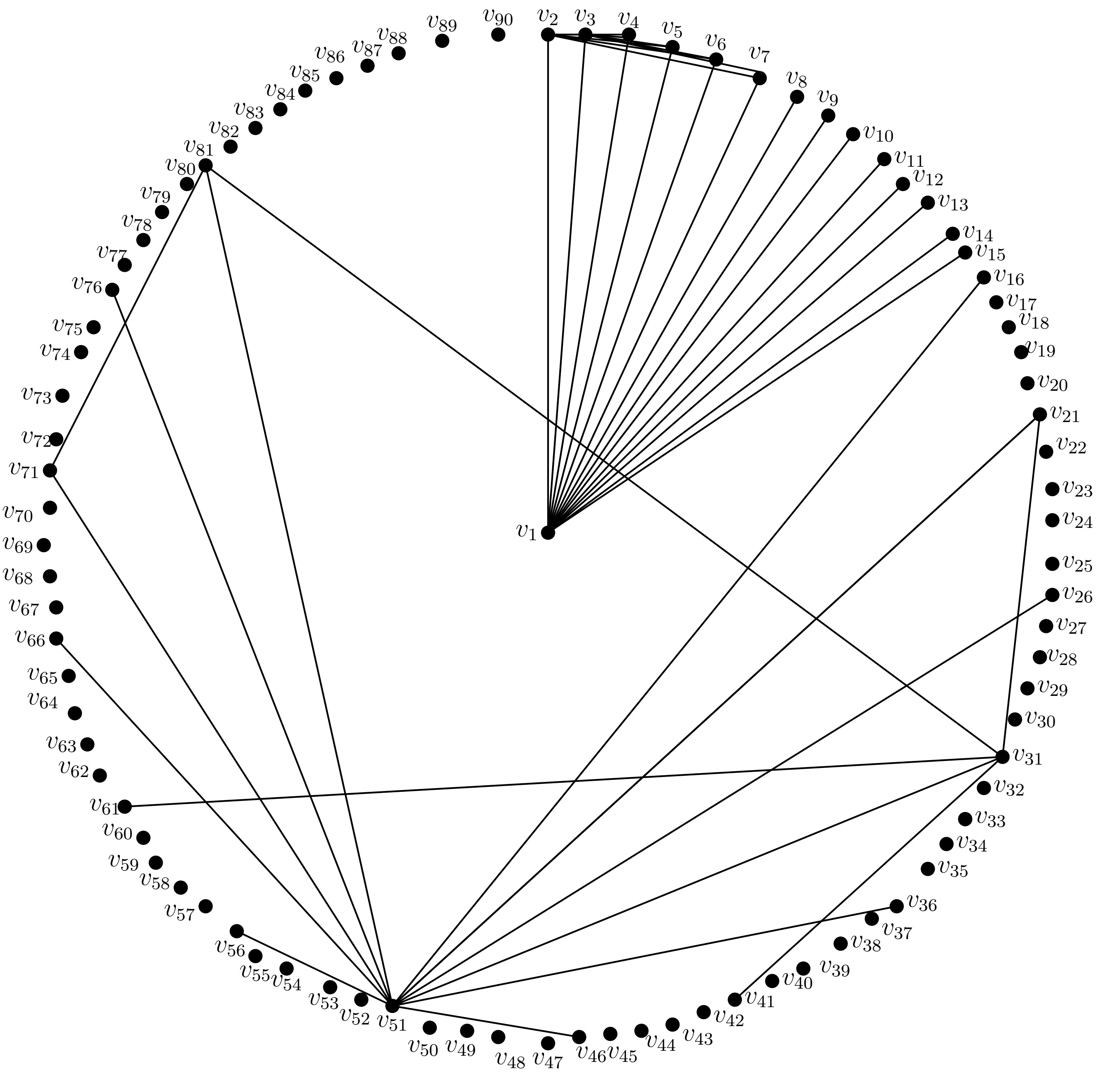
When $charR = p^4$ the graph is as follows;



Example 4.5. *Let* $Z(R) = 2\mathbb{Z}_{16} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus (Z)_2$. *In this case* $p = 2$, $r = 1$,$e = 1$ $f = 1$,$g = 1$, $g = 1$ *and* $h = 1$. *Then*

$\Gamma(R)$ is $64-$ partite with $diam(\Gamma(R)) = 2$, $gr(\Gamma(R)) = 3$ and $b(\Gamma(R)) = \frac{21}{64}$.

When $charR = p^5$ the graph is as follows;



Remark 4.1. *The graphs provided in the above examples represent complex structures akin to cyclic structures and their metric dimensions and local fractional metric dimensions obey the bounds provided in the previous section of this paper.*

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5 Graph Numbers

5.1 The Wiener Index of $\Gamma(R)$

The Wiener index denoted as W and also known as the path number or the Wiener number is a graph index defined on a graph by n nodes and defined as

$$W = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}(d)_{ij}$$

where $(d)_{ij}$ is the graph distance matrix. The Wiener index $W(\Gamma(R))$ of the graph G with vertex count $\mid V(\Gamma(R)) \mid$ has a relationship with the average disorder number of the zero divisor graph $A(\Gamma(R)) = \frac{2W(\Gamma(R))}{|V(\Gamma(R))|}$ and the average distance $\mu(\Gamma(R))$ between the vertices of $\Gamma(R)$ which is given by

$$\mu(\Gamma(R)) = \frac{W(\Gamma(R))}{\binom{\mid V(\Gamma(R)) \mid}{2}}.$$

In Topological Graph Theory, computations for Wiener indices for cyclic carbon-chained organic compounds and its applications is fundamental. The index is useful in determination of the boiling points and polarity number of alkanes and their branched isomers. Further, the most and common natural field in the application of the Wiener index is the quantitative structure relationships especially in the estimation of emission spectra of the ultra violet radiations of α and β-unsaturated ketone. We therefore present the following results on the Wiener index of $\Gamma(R)$ and other results describing average disorder number and the average distance indices of $\Gamma(R)$ due to their close interdependence with the Wiener index.

Proposition 5.1. *Let $\Gamma(R)$ be the zero divisor graph of the classes of 5-index zero finite rings. Then for any prime integer $p, r \in \mathbb{Z}^+$ and s fixed, the Wiener index, $W(\Gamma(R))$*

$$= \begin{cases} \frac{1}{2}(2p^{(\frac{2(s^2+3s)}{2}-1)r} + p^{2(\frac{(s^2+3s)}{2}-1)r} - p^{(\frac{(s^2+3s)}{2})r} - 7p^{(\frac{(s^2+3s)}{2}-1)r} + 2), & \text{if } char(R) = p; \\ \frac{1}{2}(2p^{(\frac{2(s^2+3s)}{2}+2)r} + p^{2(\frac{(s^2+3s)}{2})r} - p^{(\frac{(s^2+3s+2)}{2})r} - 7p^{(\frac{(s^2+3s)}{2})r} + 2), & char(R) = p^2; \\ \frac{1}{2}(2p^{(\frac{(2s^2+8s+4)}{2})r} + p^{2(\frac{(s^2+3s+2)}{2})r} - p^{(\frac{(s^2+5s+2)}{2})r} - 7p^{(\frac{(s^2+3s+2)}{2})r} + 2), & char(R) = p^3; \\ \frac{1}{2}(2p^{(\frac{(2s^2+10s+6)}{2})r} + p^{2(\frac{(2s^2+10s+4)}{2})r} - p^{2(\frac{(s^2+5s+4)}{2})r} - 7p^{(\frac{(s^2+5s+2)}{2})r} + 2), & char(R) = p^4,\ p^5. \end{cases}$$

Proposition 5.2. *Let $\Gamma(R)$ be the zero divisor graph of the classes of 5-index zero finite rings and $W(\Gamma(R))$ be its Wiener index. Then for any prime integer p, positive integer r and s fixed, the average* *distance of* $\Gamma(R)$

$$\mu(\Gamma(R)) = \begin{cases} \frac{\frac{1}{2}(2p^{(\frac{(2(s^2+3s)}{2}-1)r}+p^{2(\frac{(s^2+3s)}{2}-1)r}-p^{(\frac{(s^2+3s)}{2})r}-7p^{(\frac{(s^2+3s)}{2}-1)r}+2)}{(p^{(\frac{(s^2+3s)}{2})r}-1)(p^{(\frac{(s^2+3s)}{2})r}-2)}, & \text{for } char(R)=p; \\ \frac{\frac{1}{2}(2p^{(\frac{2(s^2+3s)}{2}+2)r}+p^{(\frac{2(s^2+3s)}{2})r}-p^{(\frac{(s^2+3s+2)}{2})r}-7p^{(\frac{(s^2+3s)}{2})r}+2)}{(p^{(\frac{(s^2+3s+2)}{2})r}-1)(p^{(\frac{(s^2+3s+2)}{2})r}-2)}, & char(R)=p^2; \\ \frac{\frac{1}{2}(2p^{(\frac{(2s^2+8s+4)}{2})r}+p^{2(\frac{(s^2+3s+2)}{2})r}-p^{(\frac{(s^2+5s+2)}{2})r}-7p^{(\frac{(s^2+3s+2)}{2})r}+2)}{(p^{(\frac{(s^2+5s+2)}{2})r}-1)(p^{(\frac{(s^2+5s+2)}{2})r}-2)}, & char(R)=p^3; \\ \frac{\frac{1}{2}(2p^{(\frac{(2s^2+10s+6)}{2})r}+p^{2(\frac{(2s^2+10s+4)}{2})r}-p^{2(\frac{(s^2+5s+4)}{2})r}-7p^{(\frac{(s^2+5s+2)}{2})r}+2)}{(p^{(\frac{(s^2+5s+4)}{2})r}-1)(p^{(\frac{(s^2+5s+4)}{2})r}-2)}, & char(R)=p^4,p^5. \end{cases}$$

Proposition 5.3. *Let* $\Gamma(R)$ *be the zero divisor graph of the classes of 5-index zero finite rings. Then for any prime integer* p, *positive integer* r *and* s *fixed, the average disorder number of the zero divisor graph*

$$A(\Gamma(R)) = \begin{cases} \frac{2p^{(\frac{(2(s^2+3s)}{2}-1)r}+p^{2(\frac{(s^2+3s)}{2}-1)r}-p^{(\frac{(s^2+3s)}{2})r}-7p^{(\frac{(s^2+3s)}{2}-1)r}+2}{(p^{(\frac{((s^2+3s)}{2})r}-1)}, & \text{for } char(R)=p; \\ \frac{3p^{(\frac{2(s^2+3s)}{2}+2)r}+p^{(\frac{2(s^2+3s)}{2})r}-p^{(\frac{(s^2+3s+2)}{2})r}-7p^{(\frac{(s^2+3s)}{2})r}+2}{(p^{(\frac{((s^2+3s+2)}{2})r}-1)}, & char(R)=p^2; \\ \frac{3p^{(\frac{(2s^2+8s+4)}{2})r}+p^{2(\frac{(s^2+3s+2)}{2})r}-p^{(\frac{(s^2+5s+2)}{2})r}-7p^{(\frac{(s^2+3s+2)}{2})r}+2}{(p^{(\frac{((s^2+5s+2)}{2})r}-1)}, & char(R)=p^3; \\ \frac{3p^{(\frac{(2s^2+10s+6)}{2})r}+p^{2(\frac{(2s^2+10s+4)}{2})r}-p^{2(\frac{(s^2+5s+4)}{2})r}-7p^{(\frac{(s^2+5s+2)}{2})r}+2}{(p^{(\frac{((s^2+5s+4)}{2})r}-1)}, & char(R)=p^4,p^5. \end{cases}$$

5.2 The Zagreb Indices of $\Gamma(R)$

Let $G = \Gamma(R)$ be a simple graph such that $G = (V, E)$ whose vertex set $V(G)$ consist of elements $\{v_1, \cdots, v_n\}$ such that $\mid V(G) \mid= n$ and the set of edges $E(G)$ of order m. Given that the minimum degree of G is denoted by $\delta(G)$ and $\Delta(G)$ the maximum degree. Let $d_i = deg(v_i)_{\Gamma(R)}, i = 1, 2, \cdots, n$ be the vertex degrees of $v_i \in \Gamma(R)$ so that $d_i \geq d_2 \geq \cdots \geq d_n$. The first Zagreb index is the sum of the squares of degrees of the vertices and the second Zagreb index is the sum of the products of the degrees of the pairs of adjacent vertices. We denote the first and second Zagreb indices of $\Gamma(R)$ by $Z_1(\Gamma(R))$ and $Z_2(\Gamma(R))$ respectively. Therefore,

$$Z_1(\Gamma(R)) = \sum_{v_i \in V(\Gamma(R))} d_i^2,$$

$$Z_2(\Gamma(R)) = \sum_{v_i - v_j \in E(\Gamma(R))} d_i d_j.$$

5.2.1 The First Zagreb Index, $Z_1(\Gamma(R))$

Proposition 5.4. *Let $\Gamma(R)$ be the zero divisor graph of the classes of 5-index zero finite rings and $\Gamma(R)$ be a zero divisor graph of order $p^{(e+f+g+h)r}-1$ with m edges. If $\Delta(\Gamma(R))$ and $\delta(\Gamma(R))$ are the maximum and minimum degrees of $\Gamma(R)$ then for any prime integer $p, r \in \mathbb{Z}^+$,*

(i) $Z_1(\Gamma(R)) \geq \frac{((\Delta(\Gamma(R)))^2+(2m-\Delta(\Gamma(R))))^2}{(p^{(e+f+g+h)r}-2)} + \frac{2(p^{(e+f+g+h)r}-3)}{(p^{(e+f+g+h)r}-2)^2}.(\Delta_2(\Gamma(R))-\delta(\Gamma(R)))^2$, *where $\Delta_2(\Gamma(R))$ is the second maximum degree of $\Gamma(R)$.*

(ii) $Z_1(\Gamma(R)) \leq 4m^2+2((\Delta(\Gamma(R)))^2-4m((\Delta(\Gamma(R)))-((p^{(h+(k-1)r)}-2)((p^{(h+(k-1)r)}-3))[\frac{T(\Gamma(R))}{(p^{(e+f+g+h)r}-2)\Delta(\Gamma(R))}(I(\Gamma(R)))-\frac{1}{\Delta(\Gamma(R))}]^{\frac{2}{p^{(e+f+g+h)r}-3}}$.

Proposition 5.5. *Let R be the classes of rings with h as the dimension of the modules in R'. Let $\Gamma(R)$ be a zero divisor graph such that $\mid \Gamma(R) \mid = p^{(h+(k-1))r}-1$ with m edges. If $\Delta(\Gamma(R))$ is the maximum degree of each $v_i \in \Gamma(R)$ then for any prime integer $p, r \in \mathbb{Z}^+$,*

$$Z_1(\Gamma(R)) \leq (p^{(e+f+g+h)r}-1)m - \Delta(\Gamma(R))((p^{(h+(k-1))r}-1)-\Delta(\Gamma(R))) + \frac{2(m-\Delta(\Gamma(R)))}{p^{(e+f+g+h)r}-3}$$

5.2.2 The second Zagreb Index, $Z_2(\Gamma(R))$

Proposition 5.6. *Let $\Gamma(R)$ be the zero divisor graph of the classes of rings such that $\mid V(\Gamma(R)) \mid = p^{(h+(k-1))r}-1$ and $\Delta(\Gamma(R))), \delta(\Gamma(R))$ its maximum and minimum degrees respectively. If m is the number of edges of $\Delta(\Gamma(R))$ then for any prime integer p, positive integers r, k,*

(i) $Z_2(\Gamma(R)) \geq 2m^2 - m(p^{(h+(k-1))r}-2)\Delta(\Gamma(R)) + \frac{1}{2}(\Delta(\Gamma(R))-2)[(\Delta(\Gamma(R)))^2 + \frac{(2m-\Delta(\Gamma(R)))^2}{p^{(h+(k-1))r}-2} + \frac{2(p^{(e+f+g+h)r}-1)}{(p^{(e+f+g+h)r}-2)^2}(\Delta(\Gamma(R))-\delta(\Gamma(R)))^2]$.

(ii) $Z_2(\Gamma(R)) \geq 2m^2 - m(p^{(e+f+g+h)r)\delta(\Gamma(R))+\frac{1}{2}(\delta(\Gamma(R))-1)[m(p^{(e+f+g+h)r})-\Delta(\Gamma(R))(p^{(e+f+g+h)r}} - \Delta(\Gamma(R)) + \frac{2(m-\Delta(\Gamma(R)))^2}{p^{(e+f+g+h)r}-3}]$.

6 Conclusion

In conclusion, the study succeeds in presenting some findings related to the metric dimensions, Weiner index, and Zagreb index of $\Gamma(R)$ of interest.

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