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Updated Vector Autoregressive Model Incorporating new Information Using the Bayesian Approach

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ABSTRACT

Vector Autoregressive (VAR) models have been applied extensively in modeling time series due to their high precision when used to forecast. In the VAR development, if we have information up to time t, then a VAR(p) model is fitted. However, if new information at time $t+1$, is obtained, then a new VAR(p) model has to be fitted which makes one to go through the process again. Therefore, despite their good performance, a need would arise to incorporate new information that could be obtained after the model has been fitted to update the model instead of fitting a new model each and every time a new information is obtained. This study, therefore, considers incorporating the new information to update the vector autoregressive model of order p using Bayesian approach. First, a VAR model of order 1 is formulated after which this is generalized to the VAR model of order p. We assume that the VAR model is the prior while new information is the likelihood. The performance of updated model is compared with corresponding VAR(p) models and the model is found to perform well based on the small values of the root mean square error (RMSE) in the update and in the prediction for the plots obtained.

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1 Introduction

The Vector Autoregressive (VAR) models were developed by the macroeconometrician Christopher Sims in 1980 where the main aim was to model the joint dynamics and causal relations among a set of macroeconomic variables and they dominate time series econometrics modeling [13]. The joint dynamics includes how each variable in the model is explained by the past history of every variable and how the innovations may be correlated. The VAR models explain not only the serial dependence within each component series $\{X_{ti}\}$ but also interdependence between the different component series $\{X_{ti}\}$ and $\{X_{tj}\}$, $i \neq j$ as seen in [2, 3, 7, 15, 16]. A v-variate vector autoregressive time series model of order p, VAR(p), is given by

$$\mathbf{Y}_t = \mathbf{A_1 Y_{t-1}} + \mathbf{A_2 Y_{t-2}} + \cdots + \mathbf{A_p Y_{t-p}} + \mathbf{u_t} \tag{1.1a}$$

where $\mathbf{Y}_t$ is a $(v \times 1)$ vector of time series variables, p is the number of lags and $\mathbf{u_t}$ is a $(v \times 1)$ vector of white noise process. In expanded form, equation 1.1a can be written as

$$\begin{pmatrix} y_{1,t} \\ y_{2,t} \\ \vdots \\ y_{v,t} \end{pmatrix} = \begin{bmatrix} a_{11,1} & a_{12,1} & \cdots & a_{1v,1} \\ a_{21,1} & a_{22,1} & \cdots & a_{2v,1} \\ \vdots & \vdots & \ddots & \vdots \\ a_{v1,1} & a_{v2,1} & \cdots & a_{vv,1} \end{bmatrix} \begin{pmatrix} y_{1,t-1} \\ y_{2,t-1} \\ \vdots \\ y_{v,t-1} \end{pmatrix} + \begin{bmatrix} a_{11,2} & a_{12,2} & \cdots & a_{1v,2} \\ a_{21,2} & a_{22,2} & \cdots & a_{2v,2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{v1,2} & a_{v2,2} & \cdots & a_{vv,2} \end{bmatrix} \begin{pmatrix} y_{1,t-2} \\ y_{2,t-2} \\ \vdots \\ y_{v,t-2} \end{pmatrix}$$
$$+ \cdots + \begin{bmatrix} a_{11,p} & a_{12,p} & \cdots & a_{1v,p} \\ a_{21,p} & a_{22,p} & \cdots & a_{2v,p} \\ \vdots & \vdots & \ddots & \vdots \\ a_{v1,p} & a_{v2,p} & \cdots & a_{vv,p} \end{bmatrix} \begin{pmatrix} y_{1,t-p} \\ y_{2,t-p} \\ \vdots \\ y_{v,t-p} \end{pmatrix} + \begin{pmatrix} u_{1,t} \\ u_{2,t} \\ \vdots \\ u_{v,t} \end{pmatrix} \tag{1.1b}$$

The VAR model provides forecasts which are superior to those obtained from the univariate time series models, [3]. Traditionally, VAR models are widely much useful in describing the dynamic nature of most economic and financial time series. However, recently the vector autoregressive models have gained much application in a wide range of disciplines such as Medicine, Epidemiology, Economics, Biology and Macroeconomics among others. Indeed, Hamzah $et\ al.$ [3] ascertains that VAR models are the mostly used models for modelling multivariate time series data. This is as seen in the works [1, 4, 5, 6, 10, 12, 17].

Despite the fact that the VAR models have been applied extensively in many areas due to their ability to perform well, there is a concern of what happens in the event that new information is obtained. This is due to the fact that, if we have data up to time t, then we can fit a VAR(p) model. If some new information at time $t + 1$ is obtained, then it requires that again a new model is fitted. In this study, an approach is proposed where we incorporate new information obtained at time $t + 1$ to update the VAR(p) model. To do this we consider the use of the Bayesian approach to cater for new information obtained as time goes on after the model has been developed instead of repeating the process. The fitted VAR model is considered as the prior, new information as the likelihood and the updated VAR model is the posterior. Therefore, the main objective of the study is to provide an approach for updating the VAR(p) model whenever new information is obtained rather that fitting a new model when even a single measurement is obtained. The rest of the paper is structured as follows: first we have model formulation in Section 2 where we begin with VAR(1) model and then generalize to VAR(p) model using the Bayesian approach, then check the performance of the updated model in Section 3 where the root mean square error (RMSE) is computed alongside the plots of the model. The conclusions are then given in Section 4.

2 Model Formulation

In this section we discuss modification of the ordinary VAR(p) model using the Bayesian technique to incorporate new information. The model to be updated is vector autoregressive model of order 1 after which a generalization to the vector autoregressive model of order p is done. However, first, a brief discussion on the existing VAR model is given.

2.1 Existing VAR(p) Model

A general VAR(p) model is as given in equation 1.1a. It should be noted that the VAR models can further be classified into two types namely: the reduced form, equation 1.1a and the structural VAR model as given in [7]. In the reduced form VAR model, each variable is a function of its own past and the past values of the other variables. On the other hand, the structural vector autoregressive models add the restrictions that allow identification of causal relationships beyond those that can be identified with the reduced form [7, 14]. In addition, structural form is used when the error terms are uncorrelated and that the variables can have a contemporaneous impact on other variables [14].
The identification or fitting of an ordinary VAR model involves model specification, estimation of model parameters and model checking to test whether the model is adequate. The order, p, of VAR is chosen which minimizes the Schwartz and Hannan-Quinn criteria as outlined by [8]. The Schwartz criterion is given by

$$SC(p) = \ln|\widehat{\Sigma}_u(p)| + \frac{\ln T}{T}pv^2$$

On the other hand, the Hannan-Quinn criterion is given by

$$HQ(p) = \ln|\widehat{\Sigma}_u(p)| + \frac{2\ln\ln T}{T}pv^2$$

where, for both criteria, $\widehat{\Sigma}_u$ is the estimated white noise covariance matrix, T is the sample size and v is the number of time series components. The criteria compare the residuals of the models and estimate the relative information loss of representing the original data using each of the model. In addition, the criteria weigh the quality of fit (covariance of residuals) against the complexity (number of free parameters) and therefore the model with least criterion value is considered [14]. The parameters of a fitted VAR model can be estimated by ordinary least squares estimation method under the assumptions that error term has mean of zero, the variables are stationary and no outliers. The developed model is then subjected to diagnostic checking for its adequacy and this involves checking whether the residuals are white noise, normally distributed and uncorrelated. Afterwards, the model is used to forecast which is the most important function of the VAR models. Apart from forecasting, the VAR models can be used to give the dynamics that are predicted by the models in addition to estimating the model's parameters which involves Granger-causality statistics, impulse response function and forecast error decomposition as given in [8]. Granger-causality involves testing whether one variable is statistically significant when predicting another variable while impulse response function traces the dynamic path of variables in the system to shocks to other variables in the system. On the other hand, forecast error decomposition separates the forecast error variance into proportions attributed to each variable in the model which enables understanding of how much of an impact one variable has on another variable in the VAR model [8].

2.2 Updated Vector Autoregressive VAR(p) Model

In this section, the updated Vector Autoregressive model is discussed. First, the updated VAR(1) model is discussed after which the proposed updated VAR(p) model is given. A v-variate VAR model of order 1 is given by

$$Y_t = A_1 Y_{t-1} + u_t, \qquad u_t \sim \mathcal{N}(0, Q). \tag{2.1}$$

Now, let the relation between Y_t, which is assumed to be the state at time t, and X_t, measurements at time t, be given by

$$X_t = P_t Y_t + \eta_t, \qquad \eta_t \sim \mathcal{N}(0, R), \tag{2.2}$$

where P is a matrix that may depend on time t and η_t is the measurement error which is white noise. Equation 2.1 is a transition equation giving transition from state t to state $t+1$ while 2.2 is known as measurement equation. Equations 2.1 and 2.2 now form a system of models referred to as state-space models given by

$$Y_t = A_1 Y_{t-1} + u_t, \qquad u_t \sim \mathcal{N}(0, Q) \tag{2.3a}$$

$$X_t = P_t Y_t + \eta_t, \qquad \eta_t \sim \mathcal{N}(0, R) \tag{2.3b}$$

where: Y_t is an $v \times 1$ state vector, X_t is a $n \times 1$ vector of measurement or observable variables, P_t is a $n \times v$ measurement matrix, A_1 is a $v \times v$ state transition matrix which may be time dependent, u_t is a $v \times 1$ vector of transition equation errors and η_t is a $n \times 1$ vector of measurement errors.

The goal is to get the estimates of the states Y_t given the observations X_t for the representation given by 2.3a and 2.3b. To achieve this, we do it in two steps, namely; the prediction and the update step. In the prediction step, we assume that the previous belief $p(Y_{t-1}|X_{t-1})$ is known and we wish to get $p(Y_t|X_{t-1})$.

$$p(Y_t|X_{t-1}) = \int p(Y_t, Y_{t-1}|X_{t-1}) dY_{t-1}$$

From conditional probability we have that

$$p(Y_t|X_{t-1}) = \int p(Y_t|Y_{t-1}, X_{t-1}) p(Y_{t-1}|X_{t-1}) dY_{t-1}$$

But Y_t is independent of X_{t-1} and therefore

$$p(Y_t|X_{t-1}) = \int p(Y_t|Y_{t-1}) p(Y_{t-1}|X_{t-1}) dY_{t-1} \tag{2.4}$$

The probability density functions $p(Y_{t-1}|X_{t-1})$ and $p(Y_t|Y_{t-1})$ are Gaussian, where

$$\begin{aligned} p(Y_{t-1}|X_{t-1}) &= \mathcal{N}\left(E[Y_{t-1}|X_{t-1}], Var[Y_{t-1}|X_{t-1}]\right) \\ &= \mathcal{N}\left(\hat{Y}_{t-1|t-1}, S_{t-1|t-1}\right) \end{aligned} \tag{2.5}$$

and

$$\begin{aligned} p(Y_t|Y_{t-1}) &= \mathcal{N}\left(E[Y_t|Y_{t-1}], Var[Y_t|Y_{t-1}]\right) \\ &= \mathcal{N}\left(A_{1,t-1} Y_{t-1}, Q\right) \end{aligned} \tag{2.6}$$

Substituting equations 2.5 and 2.6 in the prediction posterior, equation 2.4, we have

$$p(Y_t|X_{t-1}) = \int \mathcal{N}\left(A_{1,t-1} Y_{t-1}, Q\right) \mathcal{N}\left(\hat{Y}_{t-1|t-1}, S_{t-1|t-1}\right) dY_{t-1}$$

But $\int \mathcal{N}\left(\hat{Y}_{t-1|t-1}, S_{t-1|t-1}\right) dY_{t-1} = 1$, then

$$\begin{aligned} p(Y_t|X_{t-1}) &= \mathcal{N}\left(A_{1,t-1}\hat{Y}_{t-1}, S_{t|t-1}\right) \\ &= \mathcal{N}\left(\hat{Y}_{t|t-1}, S_{t|t-1}\right) \end{aligned} \tag{2.7}$$

where the predicted mean in equation 2.7 is given by

$$\begin{aligned} \hat{Y}_{t|t-1} &= E[Y_t|X_{t-1}] \\ &= E[A_{1,t-1}Y_{t-1} + u_t|X_{t-1}] \\ &= E[A_{1,t-1}Y_{t-1}|X_{t-1}] + E[u_t|X_{t-1}] \end{aligned} \tag{2.8}$$

But since u_t are independent and identically distributed and not dependent on X_{t-1}, then equation 2.8 becomes

$$\begin{aligned} \hat{Y}_{t|t-1} &= A_{1,t-1}E[Y_{t-1}|X_{t-1}] + E[u_t] \\ &= A_{1,t-1}\hat{Y}_{t-1|t-1} \end{aligned} \tag{2.9}$$

since $E(u_t) = 0$. On the other hand, the predicted covariance $S_{t|t-1}$ is given by

$$\begin{aligned} S_{t|t-1} &= Var[Y_t|X_{t-1}] \\ &= Var[A_{1,t-1}Y_{t-1} + u_t|X_{t-1}] \\ &= Var[A_{1,t-1}Y_{t-1}|X_{t-1}] + Var[u_t|X_{t-1}] \end{aligned} \tag{2.10}$$

But since u_t is independent of X_{t-1}, then equation 2.10 becomes

$$\begin{aligned} S_{t|t-1} &= A_{1,t-1}Var[Y_{t-1}|X_{t-1}]A_{1,t-1}^T + Var[u_t] \\ &= A_{1,t-1}S_{t-1|t-1}A_{1,t-1}^T + Q \end{aligned} \tag{2.11}$$

where $Var(u_t) = Q$. In the update step, the new measurement X_t is used to obtain the posterior $p(Y_t|X_t)$. From Bayes' theorem,

$$\begin{aligned} p(Y_t|X_t) &= \frac{p(X_t|Y_t)p(Y_t)}{p(X_t)} \\ &= \frac{p(X_t, X_{t-1}|Y_t)p(Y_t)}{p(X_t, X_{t-1})} \\ &= \frac{p(X_t|X_{t-1}, Y_t)p(X_{t-1}|Y_t)p(Y_t)}{p(X_t|X_{t-1})p(X_{t-1})} \end{aligned} \tag{2.12}$$

But

$$p(X_{t-1}|Y_t) = \frac{p(X_{t-1}, Y_t)}{p(Y_t)} = \frac{p(Y_t, X_{t-1})}{p(Y_t)} = \frac{p(Y_t|X_{t-1})p(X_{t-1})}{p(Y_t)} \tag{2.13}$$

and therefore substituting 2.13 in 2.12 we have

$$\begin{aligned} p(Y_t|X_t) &= \frac{p(X_t|X_{t-1}, Y_t)p(Y_t|X_{t-1})p(X_{t-1})p(Y_t)}{p(X_t|X_{t-1})p(X_{t-1})p(Y_t)} \\ &= \frac{p(X_t|X_{t-1}, Y_t)p(Y_t|X_{t-1})}{p(X_t|X_{t-1})} \\ &= \frac{p(X_t|Y_t)p(Y_t|X_{t-1})}{p(X_t|X_{t-1})} \end{aligned} \tag{2.14}$$

Furthermore,

$$\begin{aligned} p(X_t|X_{t-1}) &= \int p(X_t, Y_t|X_{t-1})dY_t = \int p(X_t|Y_t, X_{t-1})p(Y_t|X_{t-1})dY_t \\ &= \int p(X_t|Y_t)p(Y_t|X_{t-1})dY_t \end{aligned} \tag{2.15}$$

Substituting 2.15 in 2.14 we have

$$p(Y_t|X_t) = \frac{p(X_t|Y_t)p(Y_t|X_{t-1})}{\int p(X_t|Y_t)p(Y_t|X_{t-1})dY_t} \tag{2.16}$$

From the measurement equation we have that $p(X_t|Y_t) = \mathcal{N}[P_tY_t, R]$ and since $p(Y_t|X_{t-1}) = \mathcal{N}[\hat{Y}_{t|t-1}, S_{t|t-1}]$, then 2.16 becomes

$$\begin{aligned} p(Y_t|X_t) &= \frac{p(X_t|Y_t)p(Y_t|X_{t-1})}{\int p(X_t|Y_t)p(Y_t|X_{t-1})dY_t} \\ &= \frac{\mathcal{N}[P_tY_t, R]\mathcal{N}[\hat{Y}_{t|t-1}, S_{t|t-1}]}{\int \mathcal{N}[P_tY_t, R]\mathcal{N}[\hat{Y}_{t|t-1}, S_{t|t-1}]dY_t} \end{aligned} \tag{2.17}$$

In the numerator to 2.17, we have that

$$\begin{aligned} \mathcal{N}[P_tY_t, R]\mathcal{N}[\hat{Y}_{t|t-1}, S_{t|t-1}] &= \frac{1}{\sqrt{det(2\pi R)}} e^{-\frac{1}{2}(X_t - P_tY_t)^T R^{-1}(X_t - P_tY_t)} \times \\ & \quad \frac{1}{\sqrt{det(2\pi S_{t|t-1})}} e^{-\frac{1}{2}(Y_t - \hat{Y}_{t|t-1})^T S_{t|t-1}^{-1}(Y_t - \hat{Y}_{t|t-1})} \\ &= \frac{1}{2\pi\sqrt{det(R)det(S_{t|t-1})}} e^{-\frac{1}{2}[M]} \end{aligned} \tag{2.18}$$

where $M = (X_t - P_tY_t)^T R^{-1}(X_t - P_tY_t) + (Y_t - \hat{Y}_{t|t-1})^T S_{t|t-1}^{-1}(Y_t - \hat{Y}_{t|t-1})$. But from **[?]**, M can be written as

$$\begin{aligned} M &= (X_t - P_tY_t)^T R^{-1}(X_t - P_tY_t) + (Y_t - \hat{Y}_{t|t-1})^T S_{t|t-1}^{-1}(Y_t - \hat{Y}_{t|t-1}) \\ &= (X_t - P_t\hat{Y}_{t|t-1})^T (R + P_tS_{t|t-1}P_t^T)^{-1}(X_t - P_t\hat{Y}_{t|t-1}) \\ & \quad +(Y_t - \hat{Y}_{t|t})^T (S_{t|t-1} + P_t^T R^{-1}P_t)(Y_t - \hat{Y}_{t|t}) \end{aligned} \tag{2.19}$$

From which

$$det(R) \times det(S_{t|t-1}) = det(R + P_tS_{t|t-1}P_t^T) \times det(S_{t|t-1} + P_t^T R^{-1}P_t) \tag{2.20}$$

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Substituting equations 2.19 and 2.20 in equation 2.18 we have

$$\begin{aligned}\mathcal{N}[P_tY_t, R]\mathcal{N}[\hat{Y}_{t|t-1}, S_{t|t-1}] &= \frac{1}{\sqrt{det(2\pi(R+P_tS_{t|t-1}P_t^T))}} \times \\ & e^{-\frac{1}{2}(X_t-P_t\hat{Y}_{t|t-1})^T(R+P_tS_{t|t-1}P_t^T)^{-1}(X_t-P_t\hat{Y}_{t|t-1})} \times \\ & \frac{1}{\sqrt{det(2\pi(S_{t|t-1}+P_t^TR^{-1}P_t)^{-1})}} \times \\ & e^{-\frac{1}{2}(Y_t-\hat{Y}_{t|t})^T(S_{t|t-1}+P_t^TR^{-1}P_t)(Y_t-\hat{Y}_{t|t})} \\ &= \mathcal{N}[P_t\hat{Y}_{t|t-1}, R+P_tS_{t|t-1}P_t^T] \times \\ & \mathcal{N}[\hat{Y}_{t|t}, (S_{t|t-1}+P_t^TR^{-1}P_t)^{-1}] \end{aligned} \tag{2.21}$$

The denominator in 2.17 can be expressed as

$$\begin{aligned}\int \mathcal{N}[P_tY_t, R]\mathcal{N}[\hat{Y}_{t|t-1}, S_{t|t-1}]dY_t &= \int \mathcal{N}[P_t\hat{Y}_{t|t-1}, R+P_tS_{t|t-1}P_t^T] \times \\ & \mathcal{N}[\hat{Y}_{t|t}, (S_{t|t-1}+P_t^TR^{-1}P_t)^{-1}]dY_t \\ &= \mathcal{N}[P_t\hat{Y}_{t|t-1}, R+P_tS_{t|t-1}P_t^T] \times \\ & \int \mathcal{N}[\hat{Y}_{t|t}, (S_{t|t-1}+P_t^TR^{-1}P_t)^{-1}]dY_t \\ &= \mathcal{N}[P_t\hat{Y}_{t|t-1}, R+P_tS_{t|t-1}P_t^T] \end{aligned} \tag{2.22}$$

where $\int \mathcal{N}[\hat{Y}_{t|t}, (S_{t|t-1}+P_t^TR^{-1}P_t)^{-1}]dY_t = 1$. Therefore, the updated posterior is given by

$$\begin{aligned}p(Y_t|X_t) &= \frac{\mathcal{N}[P_t\hat{Y}_{t|t-1}, R+P_tS_{t|t-1}P_t^T]\mathcal{N}[\hat{Y}_{t|t}, (S_{t|t-1}+P_t^TR^{-1}P_t)^{-1}]}{\mathcal{N}[P_t\hat{Y}_{t|t-1}, R+P_tS_{t|t-1}P_t^T]} \\ &= \mathcal{N}[\hat{Y}_{t|t}, (S_{t|t-1}+P_t^TR^{-1}P_t)^{-1}] \end{aligned} \tag{2.23}$$

Defining the inverse-covariance of the update as

$$\hat{S}_{t|t}^{-1} = S_{t|t-1}^{-1} + P_t^TR^{-1}P_t \tag{2.24}$$

then we have that

$$p(Y_t|X_t) = \mathcal{N}[\hat{Y}_{t|t}, \hat{S}_{t|t}] \tag{2.25}$$

By definition, see [9],

$$\hat{S}_{t|t}^{-1}\hat{Y}_{t|t} = S_{t|t-1}^{-1}\hat{Y}_{t|t-1} + P_t^TR^{-1}X_t \tag{2.26}$$

Thus to obtain $\hat{S}_{t|t}$, we apply the Woodbury matrix identity given as

$$(E+FGH)^{-1} = E^{-1} - E^{-1}F(G^{-1}+HE^{-1}F)^{-1}HE^{-1} \tag{2.27}$$

see [9]. Hence, applying 2.27 to 2.24 we have that

$$\begin{aligned}[\hat{S}_{t|t}^{-1}]^{-1} = \hat{S}_{t|t} &= (S_{t|t-1}^{-1} + P_t^TR^{-1}P_t)^{-1} \\ &= S_{t|t-1} - S_{t|t-1}P_t^T(R+P_tS_{t|t-1}P_t^T)^{-1}P_tS_{t|t-1} \\ &= (I - S_{t|t-1}P_t^T(R+P_tS_{t|t-1}P_t^T)^{-1}P_t)S_{t|t-1} \\ &= (I-K_tP_t)S_{t|t-1} \end{aligned} \tag{2.28}$$

where $K_t = \frac{S_{t|t-1}P_t^T}{R+P_tS_{t|t-1}P_t^T}$. To obtain the updated state, suppose that 2.26 is multiplied by $\hat{S}_{t|t}$ so that we have

$$\hat{S}_{t|t}\hat{S}_{t|t}^{-1}\hat{Y}_{t|t} = (I - K_tP_t)S_{t|t-1}[S_{t|t-1}^{-1}\hat{Y}_{t|t-1} + P_t^TR^{-1}X_t] \tag{2.29}$$

Thus

$$\begin{aligned}
\hat{Y}_{t|t} &= (I - K_tP_t)\hat{Y}_{t|t-1} + (I - K_tP_t)S_{t|t-1}P_t^TR^{-1}X_t \\
&= \hat{Y}_{t|t-1} - K_tP_t\hat{Y}_{t|t-1} + S_{t|t-1}P_t^TR^{-1}X_t - K_tP_tS_{t|t-1}P_t^TR^{-1}X_t \\
&= \hat{Y}_{t|t-1} + (S_{t|t-1}P_t^T(R + P_tS_{t|t-1}P_t^T)^{-1}(R + P_tS_{t|t-1}P_t^T)R^{-1} \\
&\quad -K_tP_tS_{t|t-1}P_t^TR^{-1})X_t - K_tP_t\hat{Y}_{t|t-1} \\
&= \hat{Y}_{t|t-1} + (K_t(I + P_tS_{t|t-1}P_t^TR^{-1}) - K_tP_tS_{t|t-1}P_t^TR^{-1})X_t - K_tP_t\hat{Y}_{t|t-1} \\
&= \hat{Y}_{t|t-1} + (K_t + K_tP_tS_{t|t-1}P_t^TR^{-1} - K_tP_tS_{t|t-1}P_t^TR^{-1})X_t - K_tP_t\hat{Y}_{t|t-1} \\
&= \hat{Y}_{t|t-1} + K_t(X_t - P_t\hat{Y}_{t|t-1}) \\
&= A_{1,t-1}\hat{Y}_{t|t-1} + K_t(X_t - P_t\hat{Y}_{t|t-1})
\end{aligned} \tag{2.30}$$

Therefore, the equations for the updated VAR(1) model are given as

$$\hat{Y}_{t|t-1} = A_{1,t-1}\hat{Y}_{t|t-1} \tag{2.31a}$$

$$\hat{S}_{t|t-1} = A_{1,t-1}S_{t-1}A_{1,t-1}^T + Q \tag{2.31b}$$

$$K_t = \frac{\hat{S}_{t|t-1}P_t^T}{P_t\hat{S}_{t|t-1}P_t^T + R} \tag{2.31c}$$

$$\hat{Y}_{t|t} = A_{1,t-1}\hat{Y}_{t|t-1} + K_t\left(X_t - P_t\hat{Y}_{t|t-1}\right) \tag{2.31d}$$

$$\hat{S}_{t|t} = S_{t|t-1} - K_tP_tS_{t|t-1} \tag{2.31e}$$

where $K_t = \frac{\hat{S}_{t|t-1}P_t^T}{P_t\hat{S}_{t|t-1}P_t^T+R}$ is known as the gain while the term $\left(X_t - P_t\hat{Y}_{t|t-1}\right)$ is referred to as the innovation, or the residual in the measurement, which is equivalent to the measurement noise. Having obtained the algorithm for the updated VAR(1) model, then we propose that the updated vector autoregressive model of order p, VAR(p) model, is

$$\begin{aligned}
\hat{Y}_{t|t} &= A_{1,t-1}\hat{Y}_{t|t-1} + K_t\left(X_t - P_t\hat{Y}_{t|t-1}\right) + A_{2,t-2}\hat{Y}_{t-1|t-2} \\
&\quad + K_{t-1}\left(X_{t-1} - P_{t-1}\hat{Y}_{t-1|t-2}\right) + \cdots + A_{p,t-p}\hat{Y}_{t-p+1|t-p} \\
&\quad + K_{t-p+1}\left(X_{t-p+1} - P_{t-p+1}\hat{Y}_{t-p+1|t-p}\right)
\end{aligned} \tag{2.32a}$$

and the corresponding covariance is

$$\begin{aligned}
\hat{S}_{t|t} &= \hat{S}_{t|t-1} - K_tP_t\hat{S}_{t|t-1} - K_{t-1}P_{t-1}\hat{S}_{t-1|t-2} - \cdots - \\
&\quad K_{t-p+1}P_{t-p+1}\hat{S}_{t-p+1|t-p}
\end{aligned} \tag{2.32b}$$

This can be used to update the existing VAR(p) model given the new information which is considered as the likelihood. Therefore, the algorithm for the updated generalized vector autoregressive model of order p is

Algorithm 1 Algorithm for generalized updated VAR(p) model

1: Predict the state: $\hat{Y}_{t|t-1} = A_{1,t-1}\hat{Y}_{t|t-1} + \cdots + A_{p,t-p}\hat{Y}_{t|t-p}$ and error covariance: $\hat{S}_{t|t-1} = A_{1,t-1}S_{t-1}A^T_{1,t-1} + \cdots + A_{p,t-p}S_{t-p}A^T_{p,t-p} + Q$

2: Compute the gains: $K_{t-p+1} = \frac{\hat{S}_{t-p+1|t-p}P^T_{t-p+1}}{P_{t-p+1}\hat{S}_{t-p+1|t-p}P^T_{t-p+1}+R}$

3: Update the state: $\hat{Y}_{t|t} = A_{1,t-1}\hat{Y}_{t|t-1} + K_t\left(X_t - P_t\hat{Y}_{t|t-1}\right) + A_{2,t-2}\hat{Y}_{t-1|t-2} + K_{t-1}\left(X_{t-1} - P_{t-1}\hat{Y}_{t-1|t-2}\right) + \cdots + A_{p,t-p}\hat{Y}_{t-p+1|t-p} + K_{t-p+1}\left(X_{t-p+1} - P_{t-p+1}\hat{Y}_{t-p+1|t-p}\right)$

4: Update the error covariance: $\hat{S}_{t|t} = \hat{S}_{t|t-1} - K_tP_t\hat{S}_{t|t-1} - K_{t-1}P_{t-1}\hat{S}_{t-1|t-2} - \cdots - K_{t-p+1}P_{t-p+1}\hat{S}_{t-p+1|t-p}$

3 Testing Performance of the Updated Model

In this section, performance of the updated model is given. First, consider the Bivariate VAR(2) model given by

$$y_t = \nu + \begin{bmatrix} 0.5 & 0.1 \\ 0.4 & 0.5 \end{bmatrix} y_{t-1} + \begin{bmatrix} 0 & 0 \\ 0.25 & 0 \end{bmatrix} y_{t-2} + u_t \tag{3.1}$$

where it is assumed that $\Sigma_u = \begin{bmatrix} 0.09 & 0 \\ 0 & 0.04 \end{bmatrix}$ and ν is assumed to be a null matrix, see [8]. We use equation 3.1 to test the performance of the updated model under **Algorithm 1**.

Setting $A_1 = \begin{bmatrix} 0.5 & 0.1 \\ 0.4 & 0.5 \end{bmatrix}$, $A_2 = \begin{bmatrix} 0 & 0 \\ 0.25 & 0 \end{bmatrix}$, $P_t = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $Q = \Sigma_u = \begin{bmatrix} 0.09 & 0 \\ 0 & 0.04 \end{bmatrix}$, $R = \begin{bmatrix} 0.09 & 0 \\ 0 & 0.04 \end{bmatrix}$ and $S_0 = \begin{bmatrix} 0.09 & 0 \\ 0 & 0.04 \end{bmatrix}$ we obtain the output in Figs. 1-2 which gives the output for the first variable and the second variable respectively. The first subplot in Figs. 1-2 represents the output for VAR(2), modified VAR(2) estimate and modified VAR(2) prediction, denoted by the blue line, red line and the yellow line, respectively while the second subplot represents the RMSE in the estimate and prediction, denoted by the blue and the red lines, respectively.

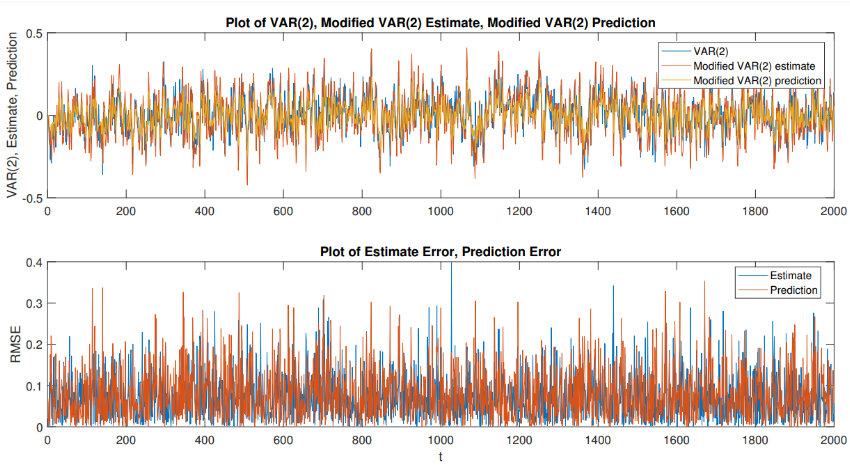

Figure 1: Bivariate VAR(2) - Variable 1

The first subplot gives comparison of the VAR(2), modified VAR(2) estimate and modified VAR(2) prediction, denoted by the blue line, red line and the yellow line, respectively while the second subplot shows the errors between VAR(2) and the modified VAR(2) and between VAR(2) and the modified VAR(2) prediction for variable 1.

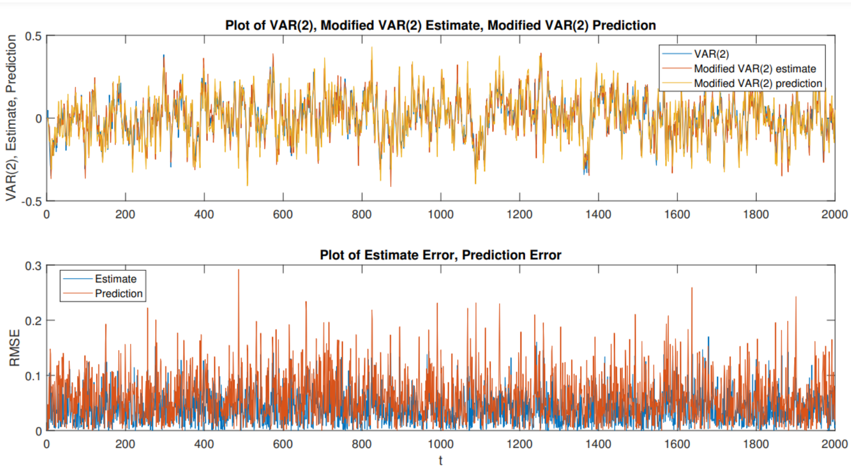

Figure 2: Bivariate VAR(2) - Variable 1

The first subplot gives comparison of the VAR(2), modified VAR(2) estimate and modified VAR(2) prediction, denoted by the blue line, red line and the yellow line, respectively while the second subplot shows the errors between VAR(2) and the modified VAR(2) and between VAR(2) and the modified VAR(2) prediction for variable 2.

From Figs. 1-2, it is observed that the errors between VAR(2) and the modified VAR(2) estimate are less, between 0 and 0.4 for variable 1 and between 0 and 0.3 for variable 2. Furthermore, the errors between VAR(2) and the modified VAR(2) prediction are as well low, between 0 and 0.4 for variable 1 and between 0 and 0.3 for variable 2. This indicates that the updated model performs well due to the small values of RMSE obtained in the estimate and in the prediction.

Next, suppose we consider the tri-variate VAR(1) model given in [**?**] where

$$y_t = \nu + \begin{bmatrix} 0.5 & 0 & 0 \\ 0.1 & 0.1 & 0.3 \\ 0 & 0.2 & 0.3 \end{bmatrix} y_{t-1} + u_t \tag{3.2}$$

where we assume ν is a null matrix, $\Sigma_u = Q = \begin{bmatrix} 2.25 & 0 & 0 \\ 0 & 1 & 0.5 \\ 0 & 0.5 & 0.74 \end{bmatrix}$, $R = \begin{bmatrix} 2.25 & 0 & 0 \\ 0 & 1 & 0.5 \\ 0 & 0.5 & 0.74 \end{bmatrix}$ and $S_0 = \begin{bmatrix} 2.25 & 0 & 0 \\ 0 & 1 & 0.5 \\ 0 & 0.5 & 0.74 \end{bmatrix}$. We test the performance of the updated model under the model given by equation 3.2 whose output is given in Figs. 3-5 for the first, second and third variables respectively. The first subplot in Figs. 3-5 represents the output for VAR(1), modified VAR(1) estimate and modified VAR(1) prediction denoted by the blue line, red line and the yellow line, respectively. The second subplot in Figs. 3-5 represents the RMSE in the estimate and prediction denoted by the blue and the red lines, respectively.

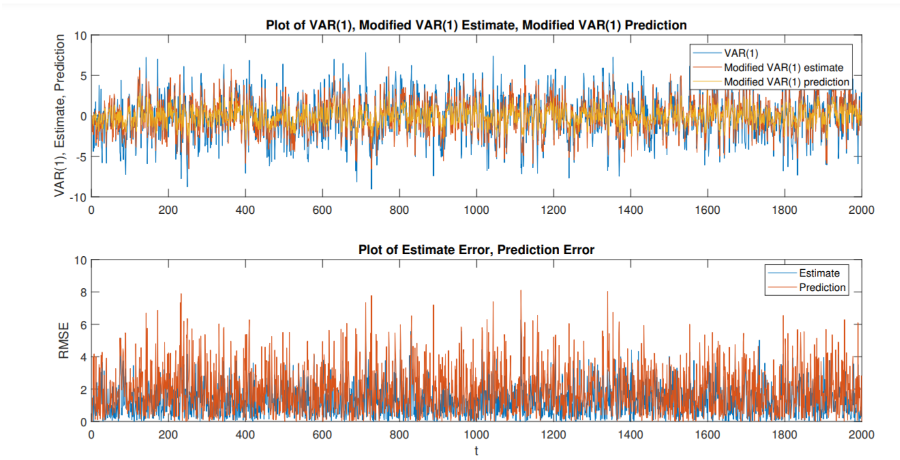

Figure 3: Trivariate VAR(1) - Variable 1

The first subplot gives comparison of the VAR(1), modified VAR(1) estimate and modified VAR(1) prediction, denoted by the blue line, red line and the yellow line, respectively while the second subplot shows the errors between VAR(1) and the modified VAR(1) and between VAR(1) and the modified VAR(1) prediction for variable 1.

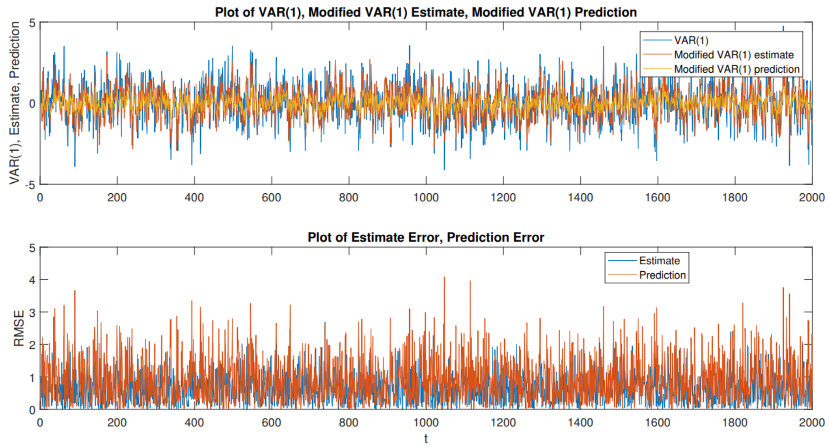

Figure 4: Trivariate VAR(1) - Variable 2

The first subplot gives comparison of the VAR(1), modified VAR(1) estimate and modified VAR(1) prediction, denoted by the blue line, red line and the yellow line, respectively while the second subplot shows the errors between VAR(1) and the modified VAR(1) and between VAR(1) and the modified VAR(1) prediction for variable 2. The first subplot gives comparison

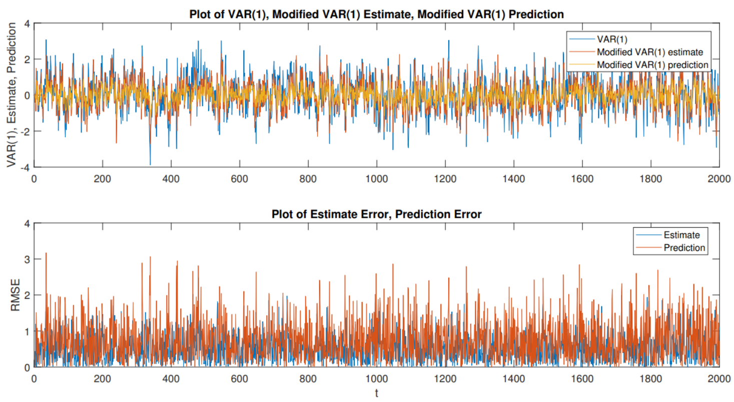

Figure 5: Trivariate VAR(1) - Variable 3

of the VAR(1), modified VAR(1) estimate and modified VAR(1) prediction, denoted by the blue line, red line and the yellow line, respectively while the second subplot shows the errors between VAR(1) and the modified VAR(1) and between VAR(1) and the modified VAR(1) prediction for variable 3.

From Figs. 3-5 it can be seen that the updated model performs well due to the small RMSE values (ranging between 0 and 10) obtained in the update and in the prediction in each of the variables for the model. Furthermore, we check the performance of the updated model by considering the model in five dimensions. In five dimension, then A_1 and P_t are 5×5 matrices. Suppose now that the state space model is given by

$$\begin{pmatrix} y_{1,t} \\ y_{2,t} \\ y_{3,t} \\ y_{4,t} \\ y_{5,t} \end{pmatrix} = \begin{bmatrix} 0.99 & 0 & 0 & 0 & 0 \\ 0 & 0.99 & 0 & 0 & 0 \\ 0 & 0 & 0.99 & 0 & 0 \\ 0 & 0 & 0 & 0.99 & 0 \\ 0 & 0 & 0 & 0 & 0.99 \end{bmatrix} \begin{pmatrix} y_{1,t-1} \\ y_{2,t-1} \\ y_{3,t-1} \\ y_{4,t-1} \\ y_{5,t-1} \end{pmatrix} + \begin{pmatrix} u_{1,t} \\ u_{2,t} \\ u_{3,t} \\ u_{4,t} \\ u_{5,t} \end{pmatrix}$$

$$\begin{pmatrix} x_{1,t} \\ x_{2,t} \\ x_{3,t} \\ x_{4,t} \\ x_{5,t} \end{pmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} y_{1,t} \\ y_{2,t} \\ y_{3,t} \\ y_{4,t} \\ y_{5,t} \end{pmatrix} + \begin{pmatrix} \eta_{1,t} \\ \eta_{2,t} \\ \eta_{3,t} \\ \eta_{4,t} \\ \eta_{5,t} \end{pmatrix}$$

Upon setting $Q = \begin{bmatrix} 0.001 & 0 & 0 & 0 & 0 \\ 0 & 0.001 & 0 & 0 & 0 \\ 0 & 0 & 0.001 & 0 & 0 \\ 0 & 0 & 0 & 0.001 & 0 \\ 0 & 0 & 0 & 0 & 0.001 \end{bmatrix}$,

$$R = \begin{bmatrix} 0.001 & 0 & 0 & 0 & 0 \\ 0 & 0.001 & 0 & 0 & 0 \\ 0 & 0 & 0.001 & 0 & 0 \\ 0 & 0 & 0 & 0.001 & 0 \\ 0 & 0 & 0 & 0 & 0.001 \end{bmatrix}$$

and

$$S_0 = \begin{bmatrix} 0.001 & 0 & 0 & 0 & 0 \\ 0 & 0.001 & 0 & 0 & 0 \\ 0 & 0 & 0.001 & 0 & 0 \\ 0 & 0 & 0 & 0.001 & 0 \\ 0 & 0 & 0 & 0 & 0.001 \end{bmatrix}$$

we have the plots in Figs. 6-10 which represent the first, second, third, fourth and fifth variables respectively. The first subplot in Figs. 6-10 represents the output for VAR(1), modified VAR(1) estimate and modified VAR(1) prediction denoted by the blue line, red line and the yellow line, respectively while the second subplots represents the RMSE in the estimate and prediction denoted by the blue and the red lines, respectively.

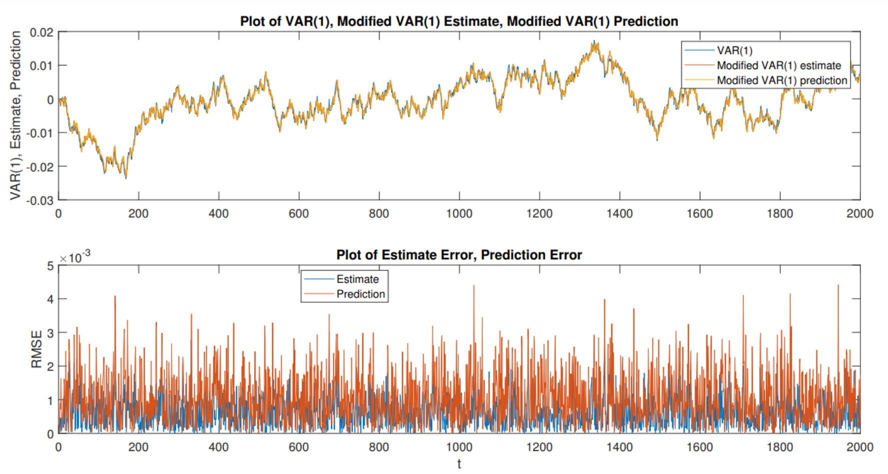

Figure 6: Pentavariate VAR(1) - Variable 1

The first subplot gives comparison of the VAR(1), modified VAR(1) estimate and modified VAR(1) prediction, denoted by the blue line, red line and the yellow line, respectively while the second subplot shows the errors between VAR(1) and the modified VAR(1) and between VAR(1) and the modified VAR(1) prediction for variable 1.

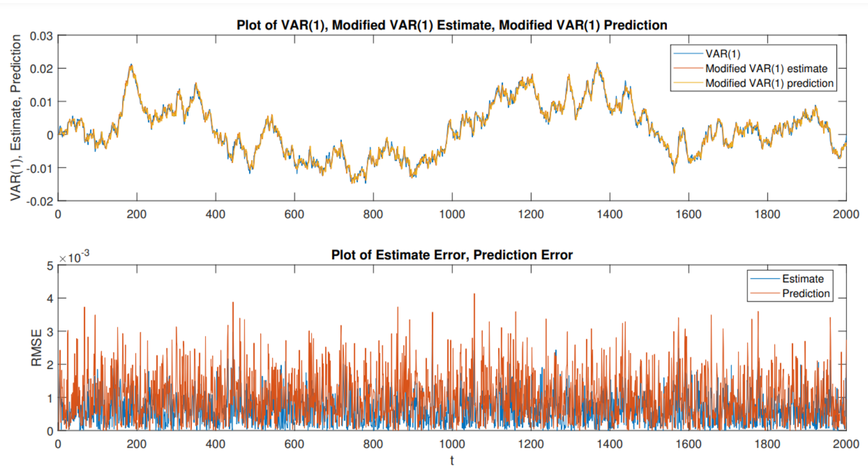

Figure 7: Pentavariate VAR(1) - Variable 2

The first subplot gives comparison of the VAR(1), modified VAR(1) estimate and modified VAR(1) prediction, denoted by the blue line, red line and the yellow line, respectively while the second subplot shows the errors between VAR(1) and the modified VAR(1) and between VAR(1) and the modified VAR(1) prediction for variable 2.

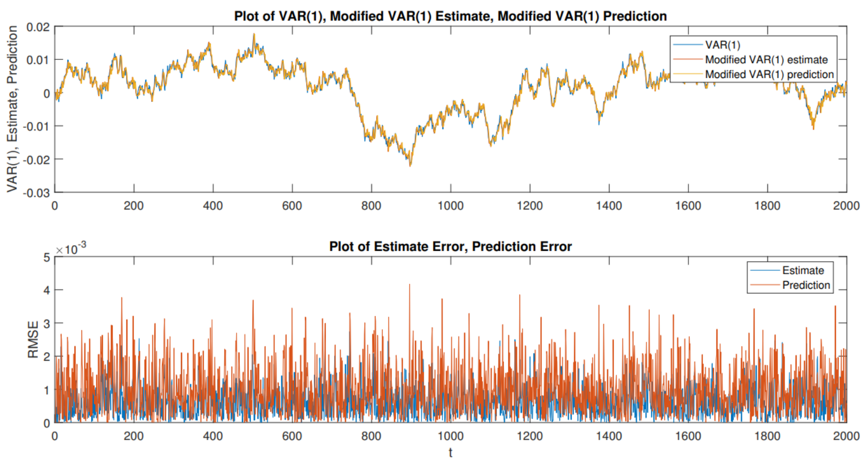

Figure 8: Pentavariate VAR(1) - Variable 3

The first subplot gives comparison of the VAR(1), modified VAR(1) estimate and modified VAR(1) prediction, denoted by the blue line, red line and the yellow line, respectively while the second subplot shows the errors between VAR(1) and the modified VAR(1) and between VAR(1) and the modified VAR(1) prediction for variable 3.

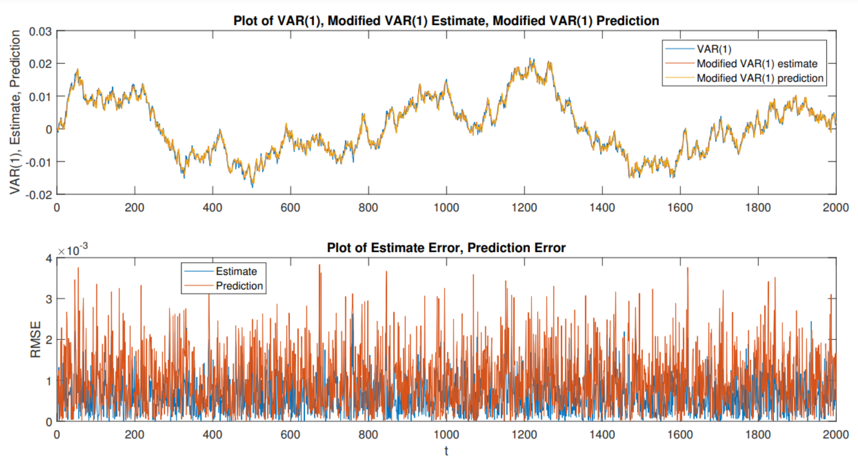

Figure 9: Pentavariate VAR(1) - Variable 4

The first subplot gives comparison of the VAR(1), modified VAR(1) estimate and modified VAR(1) prediction, denoted by the blue line, red line and the yellow line, respectively while the second subplot shows the errors between VAR(1) and the modified VAR(1) and between VAR(1) and the modified VAR(1) prediction for variable 4.

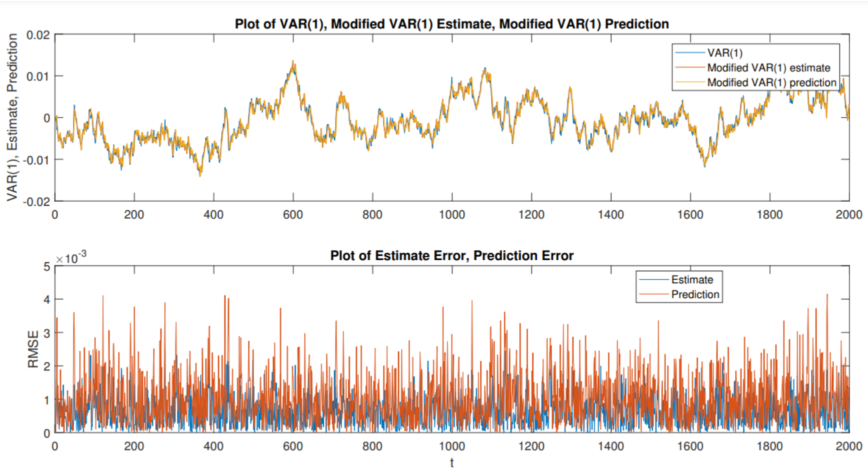

Figure 10: Pentavariate VAR(1) - Variable 5

The first subplot gives comparison of the VAR(1), modified VAR(1) estimate and modified VAR(1) prediction, denoted by the blue line, red line and the yellow line, respectively while the second subplot shows the errors between VAR(1) and the modified VAR(1) and between VAR(1) and the modified VAR(1) prediction for variable 5.

Figs. 6-10 shows that the updated model gives precise estimates as seen from the small value of root mean square error in the update and in the prediction for each variable. The RMSE values range from 0 to 0.005 in the five variables.

3.1 Application to Real Data

We consider testing the performance of the updated model by considering fitting a VAR(p) model to some secondary data. We consider secondary quarterly data for the contribution of the manufacturing, wholesale and retail, and transport and communication sectors to the national GDP obtained from the Kenya National Bureau of statistics (KNBS), quarterly GDP reports from 2009 quarter 1 to 2022 quarter 3. We fitted the VAR(2) model given by

$$\begin{aligned}\begin{pmatrix} y_{1,t} \\ y_{2,t} \\ y_{3,t} \end{pmatrix} &= \begin{bmatrix} -0.5 & 0.09 & 0.01 \\ -0.27 & -0.21 & 0.13 \\ 0.1 & 0.05 & -0.21 \end{bmatrix} \begin{pmatrix} y_{1,t-1} \\ y_{2,t-1} \\ y_{3,t-1} \end{pmatrix} + \\ &\quad \begin{bmatrix} -0.59 & 0.25 & -0.13 \\ -0.1 & -0.34 & 0.04 \\ 0.23 & -0.04 & 0.08 \end{bmatrix} \begin{pmatrix} y_{1,t-2} \\ y_{2,t-2} \\ y_{3,t-2} \end{pmatrix} + \begin{pmatrix} u_{1,t} \\ u_{2,t} \\ u_{3,t} \end{pmatrix}\end{aligned} \tag{3.3}$$

We test the performance of the fitted model by associating it with measurement equation given by

$$\begin{pmatrix} x_{1,t} \\ x_{2,t} \\ x_{3,t} \end{pmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} y_{1,t} \\ y_{2,t} \\ y_{3,t} \end{pmatrix} + \begin{pmatrix} \eta_{1,t} \\ \eta_{2,t} \\ \eta_{3,t} \end{pmatrix} \tag{3.4}$$

Using **Algorithm 1** and MATLAB software we obtain the plots in Figs. 11-13 which represent the manufacturing, wholesale and retail, and transport and communication respectively. The first subplot in Figs. 11-13 represents the output for VAR(2), modified VAR(2) estimate and modified VAR(2) prediction denoted by the blue line, red line and the yellow line, respectively. The second subplots in Figs. 11-13 represents the RMSE in the estimate and prediction denoted by the blue and the red lines, respectively. From Figs. 11-13, it is observed that the updated model performs well due to the small RMSE values obtained in the estimate and the prediction. For instance, in the variables, manufacturing and wholesale and retail, the RMSE values in both the estimate and the prediction are less than 0.005 which can be considered to be low. In the transport and communication variable, the RMSE values are less than 0.004 which we may consider to be low thus good performance from the processes of prediction and update.

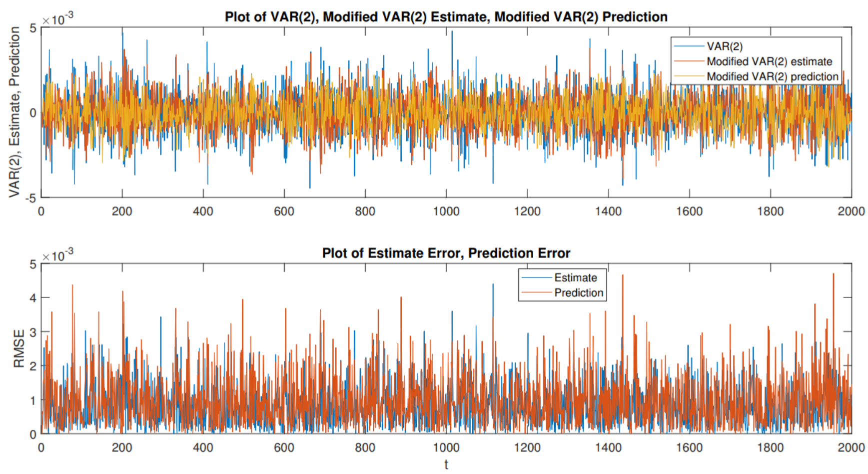

Figure 11: Trivariate VAR(2) - Variable 1 (manufacturing)

The first subplot gives comparison of the VAR(2), modified VAR(2) estimate and modified VAR(2) prediction, denoted by the blue line, red line and the yellow line, respectively while the second subplot shows the errors between VAR(2) and the modified VAR(2) and between VAR(2) and the modified VAR(2) prediction for variable 1.

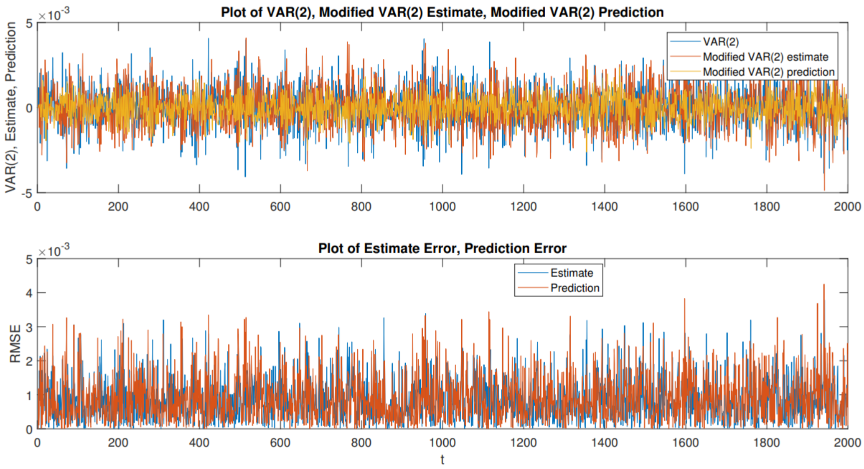

Figure 12: Trivariate VAR(2) - Variable 2 (wholesale and retail)

The first subplot gives comparison of the VAR(2), modified VAR(2) estimate and modified VAR(2) prediction, denoted by the blue line, red line and the yellow line, respectively while the second subplot shows the errors between VAR(2) and the modified VAR(2) and between VAR(2) and the modified VAR(2) prediction for variable 2.

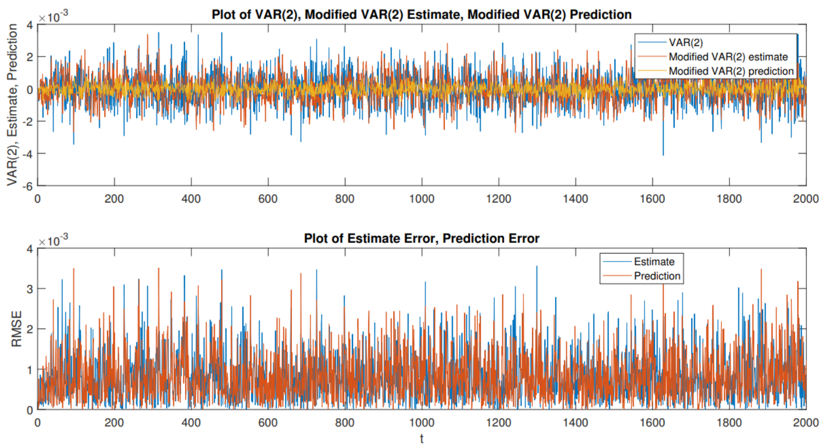

Figure 13: Trivariate VAR(2) - Variable 3 (transport and communication)

The first subplot gives comparison of the VAR(2), modified VAR(2) estimate and modified VAR(2) prediction, denoted by the blue line, red line and the yellow line, respectively while the second subplot shows the errors between VAR(2) and the modified VAR(2) and between VAR(2) and the modified VAR(2) prediction for variable 3.

4 Conclusion

In this paper we developed an updated Vector Autoregressive (VAR(p)) model using the Bayesian approach. The existing Vector Autoregressive (VAR) model is taken to be the prior while new information (measurements) that is gotten is used as the likelihood to update the existing VAR model. After incorporating new information, the proposed updated Vector Autoregressive model of order p is obtained (refer to **Algorithm 1**). The performance of the updated VAR(p) model is then compared with corresponding vector autoregressive models. It is found that the errors between VAR and the modified VAR estimate are less in the models considered. Furthermore, the errors between VAR and the modified VAR prediction are as well low. The plots of the RMSE show that the updated model performs well as indicated by small values of RMSE in the update and in the prediction.

Conflicts of Interest
The authors have no competing interests to declare that are relevant to the content of this article.

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