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Graphical exploration of two-dimensional functions — an aid to mastering fundamental calculus concepts

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Technology has become an integral part of all educational activities and can be viewed as a powerful lever to promote the understanding of fundamental mathematical concepts that underpin the study of calculus. This article reports on action research activities during 1993–1998 at the University of Pretoria, which focused on aspects that constitute the coherence between teaching, learning, mathematical conceptualisation and the use of computer graphing technology. Results identify some features of graphing utilities that are necessary to enhance fundamental concepts. The principle findings are that the meaningful combination of graphical exploration and graphical analysis according to a well thought-out didactical approach is necessary in order to incorporate technology successfully into mathematics instruction.

Background to the research

Prosperity for South Africa's people requires that they increasingly apply scientific and technological expertise. To ensure this, research and the effective training of students in the natural sciences, engineering and medical sciences are necessary. The number of students who, on strength of their scholastic achievements, are able to complete these courses successfully at university, is insufficient to satisfy the demands of a developing country such as South Africa.

During the early 1990s initiatives in the Faculty of Science at the University of Pretoria were implemented to identify and prepare students, who had the potential to pursue a career in science and applied science but who did not meet the entrance requirements for admission into the Faculty of Science, for tertiary study in the sciences. Failing to meet the entrance requirements is due to poor performance in mathematics (and science) in their final school examination. Maree (1997) points out that underachievement in mathematics is observable for all learners in South Africa. Statistical data reflecting on the pass rate in higher mathematics (Grade 12) for 1993 in South Africa reveal that only 1.58% black, 3.30% coloured, 12.54% Asian and 17.61% white learners passed (Maree, 1997:127). Results obtained from the Third International Mathematics and Science Study (TIMSS) in 1995 also reflect on the poor performance of South African learners and they were rated last amongst the 42 countries that participated in the study (Gray, 1997). In 1998 the TIMSS was repeated, and is referred to as TIMSS-Repeat (TIMSS-R). Once again South African pupils performed poorly when compared to other countries. The average score of 284 points out of 800 is well below the international average of 487 point for the 38 countries that participated in the TIMSS-R (Howie, 2001). Results like these portray a bleak picture of the mathematical competence level of South African students entering tertiary study. Poor performance in mathematics is, however, not restricted to South Africa. Addressing poor performance in mathematics, Marchese (1997) points out that mathematics educators should pursue greater depth to produce insight and intuition rather than mere rote learning to improve performance.

The challenge thus facing educators of first year mathematics students is to redress students' inadequate mathematical knowledge and at the same time foster mathematical conceptualisation. The fundamental concepts that underpin a study in calculus are embedded in a thorough understanding of functions of a single variable and their graphs that can be illustrated on a two-dimensional display. Technology can support this understanding by enabling the visualisation of functions through their graphs. The importance of visualisation as a powerful tool in mathematics education is well documented by various authors (Zimmerman & Cunningham, 1991). An obvious question that arises is how to incorporate graphing technology in mathematics education to ensure meaningful learning.

In the early eighties, Tall (1991) identified the possibility of using a computer graphing tool to visualise mathematics. Since then research studies have been reported on the use of graphing calculators in mathematics education (Beckman, 1988; Berger, 1997; Park & Travers, 1996; Rochowicz, 1996; Quesda & Maxwell, 1994; Tolias, 1993). Most of these studies have focused on a quantitative analysis of outcomes when the performance of users and non-users of graphing calculators were compared. These studies seem to point to a more positive than negative impact of technology on student performance (Dunham & Dick, 1994). Although these studies indicate *what* happened in the classroom, they do not indicate *why* it happened.

The research reported here differs from those quoted above in that it was not a quantitative comparative study conducted with two groups of subjects, but an action research study over an extended period of time with students enrolled for a first course in calculus. The study was of an exploratory nature rather than validational and aimed at developing new hypotheses instead of testing hypotheses.

Research questions

The aim of the research can be expressed through the following three questions:

1. What are the requirements for graphing technology to ensure meaningful visualisation of two-dimensional functions to promote better understanding of the mathematical concepts involved?
2. How can the exploration of two-dimensional functions through their graphs enhance conceptualisation of fundamental mathematical concepts that underpin a study in calculus?
3. How should instruction be structured to foster learner cognition and conceptualisation incorporating the visualisation of two-dimensional functions?

Method

The main research methodology followed was action research using the model plan-act-observe-reflect (Zuber-Skerritt, 1992; 1997). The action research activities during 1993–1998 comprised five main research cycles. The primary identifying principle in this research model is that the action research activities began in the observation phase. This was initiated by noticing the impact that graphing technology and instructional material can have on students' conceptualisation of mathematics. Reflection and planning during the first cycle (1993–1994) revealed that the graphing software, instructional material and instructional strategy then used were not conducive to mastering fundamental concepts of two-dimensional functions.

The main activities during the second cycle (1995) focused on the development of graphing software to address the shortcomings of available graphing software then in use. Action research activities in this developmental cycle was marked by successive cyclic repetitions during April to July 1995 concerning the development of the software. These activities resulted in the implementation of the graphing software, *Master Grapher for Windows* (Carr & Steyn, 1998) in July 1995. Further activities in cycle two focused on the format and structure of instructional material for use with the software and this became the main focus of the research activities in the third cycle during 1995 and 1996. These resulted in the compilation of instructional material for students comprising a workbook and answer sheets (formatted to correspond with questions in worksheets). A preliminary edition of the instructional package was implemented in the beginning of 1997. During the fourth cycle (1997) the instructional material as well as the software were edited and these versions (Greybe, Steyn & Carr, 1998) have been in use since the beginning of 1998. Research activities during 1998 mainly comprised observing students' use of the software and the instructional material. This research formally terminated in the reflection phase of the fifth cycle (1998). However, due to the nature of action research, the improvement of practice in some way or other (Cohen & Manion, 1994) is an ongoing process.

Encouraged by the layout of the Gold Fields Computer Centre for Education (Figure 1) where the action research activities took place, the observation of students, engaged in mathematics aided by technology, was the most frequently used methodology. Computers are spaced so that students can work on their own but interaction between students is not constrained and lecturers can easily move around among the students.

This observation not only contributed to the research activities but was inherently also part of the instructional activities. Discussions with the students, informal interviews and personal feedback from them pertaining to the software and instructional material contributed significantly to the insights gained during this research. During 1996 to 1998 a questionnaire was also administered at course end.

Participants

The research was done in collaboration with lecturers and tutors of the Gold Fields Computer Centre for Education and the Department of Mathematics and the participation of students was regarded as invaluable for the research. The participating students comprised science students on the Extended Programme in the Faculty of Science (N=642), main stream engineering students (N=123), and prospective engineering students (N=24). The distribution of the participating students during 1993–1998 is given in Table 1. Science students had participated in the research since 1993. These students had had inadequate schooling in mathematics and lacked conceptual understanding.

Table 1 Participating students enrolled for a first course in calculus

Students	1993	1994	1995	1996	1997	1998	Total
Science students (SciS)	28	35	160	120	155	144	642
Engineering students (EngS)	0	0	0	0	61	62	123
Prospective engineering students (PrEngS)	0	0	0	0	0	24	24
Total	28	35	160	120	216	230	789

Observations during 1993–1996 indicated that the use of the graphing utility seemingly had a positive effect on enhancing these students' conceptualisation. Therefore, the study was extended in 1997 and 1998 to also include a group of engineering students and prospective engineering students with presumed adequate schooling in mathematics.

All the students (N=789) were enrolled for a first course in calculus and they contributed, either directly or indirectly, to the outcomes of the study. This highlights, amongst others, the participatory and collaborative characteristics common to action research (Cohen & Manion, 1994; Zuber-Skerritt, 1992). The action research cycles were in essence research into teaching and contributed to the continuous improvement of practice and the extension of knowledge of the researchers.

The first research question: results and discussion

The following aspects concerning the requirements for graphing technology in mathematics instruction were identified during this research. These are based on observing the influence that the use of technology can have on a learner as well as the effect that computer aided instruction can have on a learner's mathematical conceptualisation.

Technical features of graphing utilities

Graphing technology for teaching and learning mathematics include pen and paper, computer graphing software, graphing calculators and the graphing capabilities of other software, such as symbolic manipulation programs and spreadsheets. In this study computer graphing software and pen and paper were used.

The use of graphing utilities in mathematics education can be categorised as exploratory and illustrative (Steyn, 1998). In the exploratory use of graphing technology, technology is regarded as an aid (tool) to support a learner in the thorough investigation and examination of mathematical concepts that are represented by a graphical (visual) image. In this sense exploring a graph entails much more than only looking at the visual image. The illustrative use of graphing technology is defined as merely 'have-a-look-at-it'. On the one hand this implies showing a learner the visual image in order to get an overall idea of the graph. On the other, a 'have-a-look-at-it' use presupposes proficiency in the use of graphing technology as well as the necessary knowledge to interpret a graphical image. The latter scenario implies a competence in mathematics that students, taking a first course in calculus, do not necessarily have. During this study the aim was to identify those features required of graphing technology for its exploratory use in learning mathematics.

The first technical aspect concerns the user interface of a graphing utility. The presentation of text and images on the screen should be conducive to learning in the sense that the utilisation of left and right brain functions are promoted and optimised if text is in the right visual field and visual images (graphics) are in the left visual field (Herrmann, 1995). The layout of the user interface of the graphing utility in Figure 2A is structured according to these principles but this is not generally a feature of graphing software. As the human brain has an inherent fast response to colour, shape and contrast (Jensen, 1996), the use of colour in the user interface of a graphing utility should also be presented in such a way so as to aid and not to impair learning. Furthermore, care should be taken when using colour as a code for classification. This may affect interpretation of images by users who are colour blind. A further aspect pertaining to the user interface and the exploratory use of graphing technology concerns the size of the visual image. The researchers are of the opinion that the size of a computer screen is more conducive to true exploration than that of a graphing calculator.

A second aspect required for the exploratory use of graphing technology concerns technical capabilities. An educational graphing tool must promote opportunities for authentic explorations and convincing observations. For example, entering functions must be easy. It ought to be possible to display more than one function simultane-

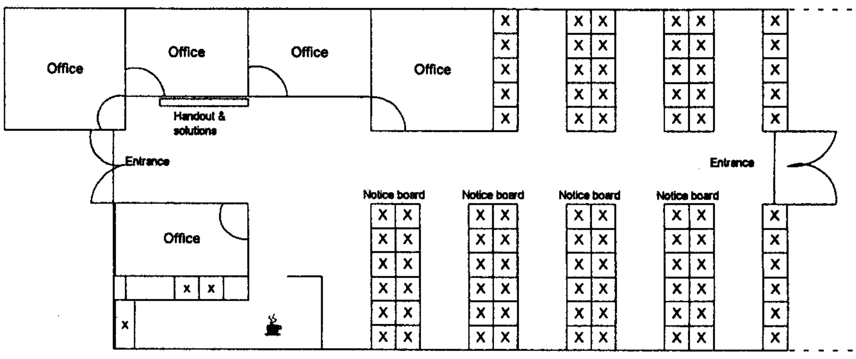

Figure 1 Layout of the Goldfields Computer Centre of Education

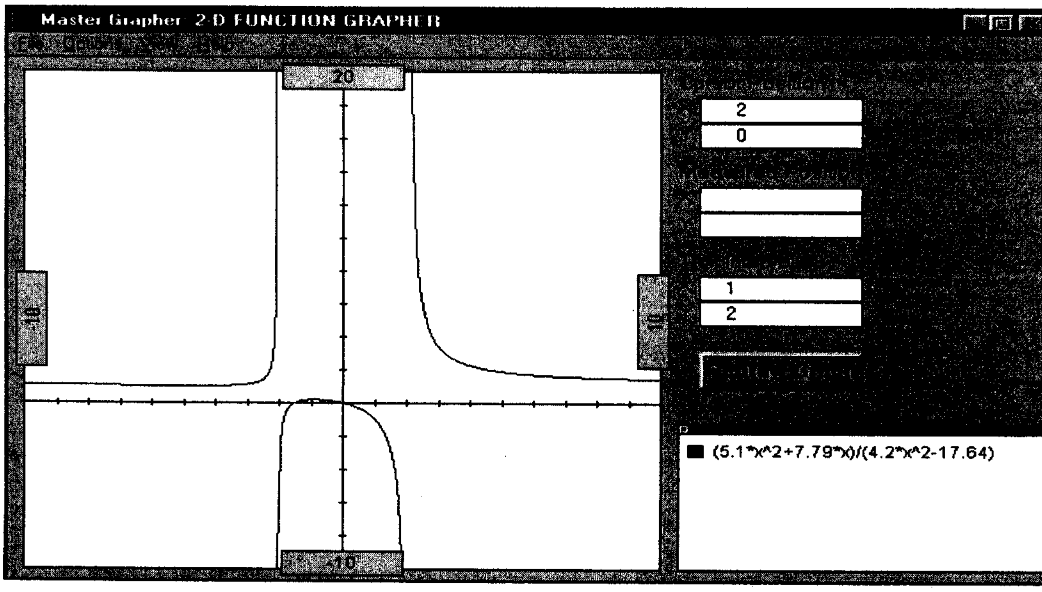

Figure 2A

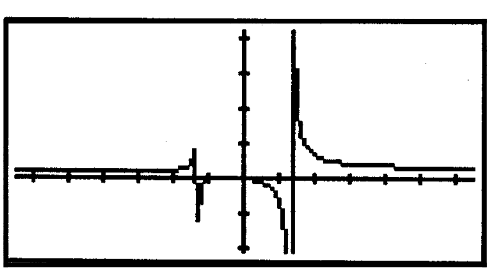

Figure 2B

Figure 2 The function $f(x)=\dfrac{5.1x^2+7.79x}{4.2x^2-17.64}$ displayed by two graphing utilities ously and functions must be distinguishable from each other. Real exploratory activities, such as physically following a curve with a pointing device (mouse), can enhance one's intuitive feeling for a graphical image. Changing the dimensions (minimum and maximum values on the X- and Y-axes) of the graph window (viewing rectangle), in which the graph is displayed, must be easy and toggling between consecutive graph windows should be possible. As it is not always possible to represent in only one window all the mathematical characteristics of a function one would like to visualise, an easily accessible zoom feature ought to be available. Very useful features, for the meaningful exploration of related changes in *x* and *y* values, are moving vertical and horizontal lines as well as the ability to add fixed vertical and horizontal lines to the image displayed by the graphing utility.

Didactical and pedagogical features of graphing utilities

Experiences gained during the research have indicated that when technology is incorporated into a mathematics tuition programme, the focus should not be on the technology. This means that the skill to use a graphing utility should be easily acquired and retained by the learners for whom the instruction is intended. Technology should add value to students' learning experiences and should facilitate rather than dictate learning.

In using graphing technology as a tool to explore mathematics, learners need to be able to get an intuitive feeling of the graph through the activity (for example, following the curve with the pointing device to get a feeling of a function increasing/decreasing). This highlights the fact that the screen size of a computer and the relative ease in manipulating a pointing device (mouse) in comparison with those of graphing calculators are beneficial to authentic experiential activities. Fuchsteiner (1997:14) points out that "intuition and concepts constitute the elements of all our knowledge ... no reform in the education of mathematics can be successful which does not focus on how we can strengthen intuition".

A further aspect that was distinctly noticed when a graphing approach is used for the teaching and learning of fundamental concepts related to two-dimensional functions, is that the images, displayed by the graphing utility, should be accurate representations of the functions. Figure 2A illustrates the graph of a rational function drawn by the computer graphing tool developed in this research (Carr & Steyn, 1998) and Figure 2B illustrates the same function drawn by a graphing calculator. The image in Figure 2B is, unfortunately, displayed by many graphing utilities. The authors feel that such an image cannot be used for authentic graphical exploration, as students believe that the graph displayed is the correct representation of the function. Images like these proved to be very confusing. For example, when students had to interpret the image on the screen (as in Figure 2B) and draw a freehand graph, it invariably occurred that they drew an exact copy of the image on the screen. When then asked to explain the graph, students reported that the graph had 'sharp' turning points for some *x* value(s) on the interval (–2, –1) and that the graph 'stopped' below and above the X-axis for *x* value(s) on the interval (1, 2). In an example such as this one concerning a rational function (Figure 2), the image as in Figure 2B cannot be used to enhance concepts, e.g. in this case, the concept of vertical asymptotes or the range of a function. This ambiguity forfeits the purpose of a visual image. Experiences with students in this research have undoubtedly shown that ambiguous graphical images give rise to mathematical misconceptions.

The second research question: results and discussion

The second research question addresses the significance that the exploration of two-dimensional functions through their graphs has for enhancing conceptualisation of fundamental mathematical concepts that underpin a study in calculus. The answer to this question is treated according to the convention illustrated in Figure 3, namely that:

> Graphical exploration is experiential and non-verbal and is mainly focused on the utilisation of functions of the *global* hemisphere and graphical analysis is verbal and structured and is mainly focused on the utilisation of functions of the *linear* hemisphere.

The convention illustrated in Figure 3 is based on research regarding the human mind in the fields of neuroscience, psychology and anthropology that resulted from the pioneering work of Sperry and colleagues, on split-brain patients at the California Institute of Technology in the 1960s (Herrmann, 1995). Since then ongoing research has reaffirmed that functionally the human mind can be divided into two hemispheres that control vastly different aspects of thought and action (Gazzaniga, 1998). The specialised functions associated with each hemisphere are listed in Table 2 (Trotter, 1976:219).

Table 2 Specialised functions associated with each brain hemisphere

Left hemisphere	Right hemisphere
Speech / verbal	spatial / music
logical, mathematical	holistic
linear, detailed	artistic, symbolic
sequential	simultaneous
controlled	emotional
intellectual	intuitive, creative
dominant	minor (quiet)
worldly	spiritual
active	receptive
analytic	synthetic, gestalt
reading, writing, naming	facial recognition
sequential ordering	simultaneous comprehension
perception of significant order	perception of abstract patterns
complex motor sequences	recognition of complex figures

It has been established that for most people linear, logical, analytical, quantitative and fact based knowledge are located in the left brain hemisphere. The right hemisphere predominantly supports and co-ordinates gestalt, intuition, emotion, spatial perception and kinaesthetic feelings. Herrmann (1995) combined the knowledge of hemispheric differentiation with theories of how the human brain is physiologically organised to develop a four quadrant whole brain model for thinking preferences. Steyn (1998) adapted Herrmann's model as a whole brain model to describe the manifestation of mathematical concepts (Figure 3) through graphical exploration and analysis.

A study by researchers from the Massachusetts Institute of Technology (MIT) and France has indicated that rote arithmetic takes place in an area of the brain usually reserved for verbal tasks whereas approximate calculations entails the brain's large scale network involving visual, special and analogical mental transformations (Halber, 1999). Although the MIT research findings are not directly linked to the research reported in this article, they show that different parts of the brain are used for different mathematical processing. If the MIT research is interpreted in terms of the location of specialised brain functions (Table 2) and the model in Figure 3, it means that rote arithmetic can be associated with the left brain hemisphere whereas approximations involves activities in both the right and left brain hemispheres and thus entails *whole brain* activity.

Although hemispheric differentiation and associated brain functions are commonly referred to as left-brained and right-brained, it should be pointed out that hemispheric differentiation for some people are transposed. In these cases, for example, linear functions are processed in the right hemisphere and global functions in the left hemisphere. In order to distinguish the two hemispheres by function, the terms *linear* and *global* are used in this research. Hannaford (1997) uses the terms *logic* and *gestalt* to distinguish between the hemispheres. The authors prefer the term *linear* to *logic* as graphical exploration in mathematics, described in this research, also includes logical processes. The term *global* (defined as 'embracing a group of items') is preferred to *gestalt* as the latter term is used (in this research) in the context of conceptualisation. Although a differentiation between

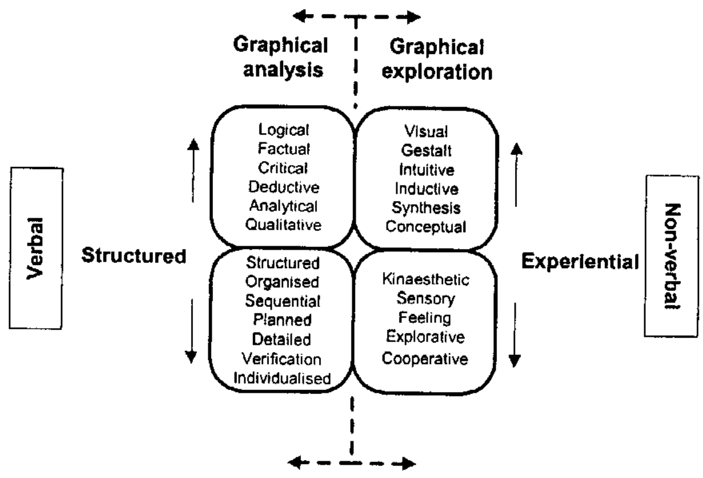

Figure 3 Manifestation of mathematical concepts through graphical analysis and exploration: a whole brain experience

linear and *global* is made in explaining the research findings, the following discussion distinctly exposes a *whole brain* functionality including both hemispheres.

In the research reported in this article the predominant categories that emerged in determining how the exploration of two-dimensional functions through their graphs can enhance mathematical conceptualisation are *visualisation*, *gestalt (wholeness)*, a combination of an *inductive* and *deductive* approaches and *verbalisation*.

A major contribution that a graphing utility can make in revealing mathematical concepts lies in the *visualisation* of functions through their graphs. Interpreted from a whole brain perspective (as in Figure 3) this implies utilisation of a predominantly global hemisphere functions. Graphical representations further enhance conceptualisation through wholeness. This means that a mere glance at the graphical representation of a function (see Figure 2A) conveys more information regarding the features of a function than is the case when an equation alone, for example $f(x) = \dfrac{5.1x^2 + 7.79x}{4.2x^2 - 17.64}$ representing the function in Figure 2A, is considered.

In their responses to a questionnaire, 93.2% of the students (N=425, Table 3), indicated that the visual image contributed to their understanding of mathematical concepts and 97.5% (N=403, Table 4) indicated that the concept of a graph was more meaningful after the exploration activities in the course than before.

Table 3 Responses of students to end of course questionnaire 1996–1998

Do you think that using visual images made the mathematical concepts more clear?

	1996	1997		1998			Total
	SciS	SciS	EngS	SciS	EngS	PrEngS	425
Yes	58	114	43	113	49	19	396
N	7	10	2	5	0	5	29
P	<0.0000	<0.0000	<0.0000	<0.0000	<0.0000	=0.0067	<0.0000

Using the binomial test and comparing the proportion 'yes' versus 'no' responses to the assumption that 50% would respond 'yes', the results were that for all student groups and years the 'yes' response was significantly high ($p < 0.5$ for all groups).

Table 4 Responses of students to end of course questionnaire 1996–1998

Is the concept of a graph now more meaningful to you than before you used the computer graphing tool?

	1996	1997		1998		Total
	SciS	SciS	EngS	SciS	EngS	403
Yes	65	121	45	113	49	393
N	2	3	0	4	0	9
No response	0	0	0	1	0	1
P	<0.0000	<0.0000	<0.0000	<0.0000	<0.0000	<0.0000

When students learn mathematical concepts by exploring examples graphically and intuitively (mainly global hemisphere utilisation, see Table 2) and discovering key aspects themselves, this approach can be described as *inductive*. The study revealed that the structured exploration of a variety of examples on the same topic leads to logical analytical deductions (mainly linear hemisphere utilisation, Table 2) that should ideally lead to synthesis and conceptualisation (mainly global hemisphere utilisation). A *deductive* approach to graphical exploration occurs when students already know an analytical theorem/rule (mainly linear hemisphere utilisation) and then study a graph (mainly global hemisphere utilisation) illustrating the theorem/rule. The ensuing analysis of graphical images compels a learner to formulate his/her conceptualisation of the mathematics involved.

Experiences with students during this study have also shown that *verbalisation* (orally and in writing) with regard to the mathematics content, eventually helps learners to become sensitive to the degree of correctness or incorrectness of their own mathematical formulation and conceptualisation. In their responses to a questionnaire, 93.2% of the students (N=426, Table 5) indicated that the communications during the practical graphing sessions helped them to improve their ability to formulate mathematical concepts and express themselves in the language of mathematics. Furthermore, 93.4% of the students (N=427, Table 6) reported that writing up their answers improved their skill in writing down mathematics correctly.

Again using the binomial test and comparing the proportion 'yes' versus 'no' responses to the assumption that 50% would respond 'yes', the results were that for all student groups and years the 'yes' response was significantly high ($p < 0.5$ for all groups).

As a learner becomes proficient in graphical exploration, linear and global hemisphere activities are seemingly utilised congruently. However, it cannot be assumed that students automatically have the expertise to do graphical exploration in such a way that global and linear hemisphere functions are utilised. Competency in whole brain utilisation in teaching and learning (in mathematics) should be facilitated through training (teaching), motivation and practice. This can ideally be accomplished when instruction follows a well-structured tuition approach. The third research question addresses this aspect.

Table 5 Responses of students to end of course questionnaire 1996–1998

Do you think that the practical graphing sessions helped you to improve your ability to formulate mathematical concepts and express yourself in the language of mathematics?

	1996	1997		1998			Total
	SciS	SciS	EngS	SciS	EngS	PrEngS	426
Yes	66	115	42	108	46	20	397
N	1	9	3	10	2	4	29
P	<0.0000	<0.0000	<0.0000	<0.0000	<0.0000	<0.0015	<0.0000

Table 6 Responses of students to end of course questionnaire 1996–1998

Do you think that completing the worksheets improved your skill in writing down the mathematics correctly?

	1996	1997		1998			Total
	SciS	SciS	EngS	SciS	EngS	PrEngS	427
Yes	65	114	44	106	48	22	399
N	2	10	1	12	1	2	28
P	<0.0000	<0.0000	<0.0000	<0.0000	<0.0000	<0.0000	<0.0000

The third research question: results and discussion

The third research question concerns an instructional strategy incorporating graphing technology to foster mathematical conceptualisation in a first course in calculus. The traditional instructional approach in university mathematics education is aimed at the utilisation of mainly linear hemisphere functions. In this regard Sperry remarks that "our educational system, as well as science in general, tends to neglect the non-verbal form of intellect" (as quoted by Herrmann, 1995:9). The first two research questions have indicated that modern graphing technology poses the possibility to utilise linear and global functions in the tuition of introductory calculus and by so doing enhance mathematical conceptualisation. This, however, necessitates a rethinking of the way in which learning is facilitated. The meaningful and thoughtful combination of graphical exploration and graphical analysis can be regarded as a condition for the manifestation of mathematical concepts in a teaching and learning approach aimed at whole brain utilisation.

The instructional model (Steyn, 1998) in Figure 4 presupposes an interdependence between teaching, learning, instructional media and instructional content when technology is used in (tertiary) mathematics education. In this model the tertiary student of the early 2000s is viewed as a developing learner, neither child nor adult in a technologically rich environment. The use of only traditional theories of learning to describe the cognitive activity of such a learner of matheatics is no longer sufficient. The concept of whole brain utilisation can be used as a paradigm for elucidating the teaching and learning of tertiary mathematics in the early 2000s. The facilitation of learning in this model is based on research and experiences related to student learning (Cross & Steadman, 1996) as well as on the concept of a whole brain approach to teaching.

This study revealed that students do not know what is expected of them when they have to use a graphing utility to investigate or explore the behaviour of two-dimensional functions. Students need to be taught how to explore graphs (mainly global hemisphere utilisation) and make meaningful interpretations (mainly linear hemisphere utilisation). In order to structure learners' exploration activities, when a graphing utility is used, detailed step-by-step guidelines for the exploration activity should be given. The following example illustrates this point.

The function $f(x)=\dfrac{5.1x^2+7.79x}{4.2x^2-17.64}$ in Figure 2 is used to illustrate the type of instructions needed, after the graph has been drawn, to facilitate the activities to explore the concepts of infinite limits and vertical asymptotes.

a) Use a vertical moving line and position it between the left and right curves of the graph of $f(x)$ so that it touches neither. Fix this vertical line with a right click. Write down the x value representing this line.

b) Use the cross hair cursor to trace the curve of $f(x)$ to determine what happens to the values of $f(x)$ as $x \to a^-$ and as $x \to a^+$ where a is the value you determined in Question (a).

These step-by-step instructions imply that students have to read, comprehend and use this instructional information to do the exploration activities. This promotes self-exploration by a learner and is self-paced. In such activities, linear hemisphere utilisation is promoted through the structured format and verbal information. This is followed by global hemisphere utilisation in the actual exploration activity, which is then again followed by linear hemisphere utilisation in analysing, formulating and writing down the solutions to the problem.

Ideally, for mastering fundamental mathematical concepts, graphical interpretation should be accompanied by algebraic verification (orally and/or in writing) and algebraic results should be illustrated graphically. For example, the concepts of horizontal asymptote and limits at infinity can be explored with similar instructions as above for the function $f(x)=\dfrac{5.1x^2+7.79x}{4.2x^2-17.64}$ in Figure 2. In this case a horizontal moving line will be used and positioned so that the exploration leads to discovering that $f(x)$ tends to 'some value' as $x \to -\infty$ and as $x \to +\infty$.

Students should be encouraged to 'speak mathematics' and explain the visual images on the screen. Lecturers (tutors) should listen and prompt students to (re-)formulate mathematically correctly. Having formulated the concept verbally, they then need to write it down in mathematical symbols. Once infinite limits, limits at infinity and asymptotic behaviour have been explored adequately, it can be followed by determining the vertical asymptote numerically and $\lim\limits_{x\to\pm\infty} f(x)=\dfrac{51x^2+7.79x}{4.2x^2-17.64}$ algebraically. This coherence between exploration and verification promotes the utilisation of linear as well as global hemisphere functional activities and promotes deeper understanding of the concepts involved.

The following ten principles (Steyn, 1998), aimed at optimising learning when a graphing utility is used, are based on experiences with students using technology during the action research activities from 1993 to 1997.

- A *whole brain approach* encompasses a combination of graphical exploration and graphical analysis. The incorporation of both exploration and analysis contribute to the mastering of fundamental mathematical concepts when a graphing utility is used.
- *Structured exploration and interpretation.* Graphical explorations as an aid to mastering fundamental mathematical concepts need to be well structured. Students also need to be facilitated how to

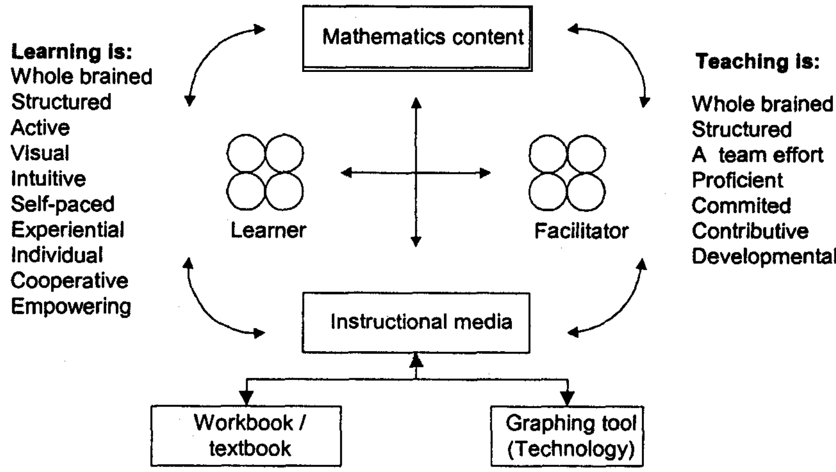

Figure 4 An instructional model for graphing technology aided mathematics tuition

explore graphs and make meaningful interpretations. Disregarding this principle may result in graphical images becoming 'nice to look at' and learners perceiving it as 'so what about it'.

- *Authentic representation.* The visual image displayed by a graphing utility must be vivid and unambiguous so that mathematical concepts can easily be explored, deduced or enhanced.
- *The left-to-right principle.* Graphical exploration and interpretation should be done starting from the left side of the screen (following the curve displayed on the screen) and working towards the right side. This is important, as analytical mathematical theorems that reflect on two-dimensional functions postulate concepts for increasing values of x. There is one exception to this principle, namely, when limits from the right, $\lim_{x \to a^+} f(x)$, are explored.
- *The principle of a complete graph.* Graphical exploration of two-dimensional functions should always commence with a visual representation of a graph that displays all the important features of the function. Such a visual representation can be described by the term 'complete graph'.
- *The principle of dimensions.* In order to represent a suitable complete graph, different graph windows (viewing rectangles) ought to be considered so that the most appropriate one can be chosen. This, however, necessitates that the graphing tool used, should enable easy changes in the dimensions.
- *Verification and illustration.* Ideally, for the mastering of fundamental mathematical concepts, graphical interpretation should be accompanied by algebraic verification (orally and/or in writing) and algebraic results should be illustrated graphically.
- *Meaningful numerical answers.* In an attempt to give the learner a feeling for the magnitude of decimal numbers in a real world situation, such as using a graph to determine solutions to a problem, it is suggested that numerical answers should be given to two decimal places. Students tend to get carried away by technology displaying (answers with) multiple decimal places that have no significance for real world applications.
- *The principle of non-assumption.* Facilitators of learning should never assume that learners observe and deduce from graphical images what teachers expect students to observe and deduce. "We must start [our teaching] where students are rather than where we wish they were" (Fuchsteiner, 1997:16).
- *The tool principle.* Technology (such as a graphing utility) should be viewed as a tool to facilitate learning and enhance teaching. It should neither intimidate a learner nor dominate instruction.

Additional research results

Students' attitude

Although the assessment of students' attitudes towards mathematics was not included in the research, the instructional strategy seemingly had a positive effect on their attitudes towards mathematics. There was a positive change in attitude from disliking to liking mathematics. 16.5% of the students (N=423, Table 7) indicated that they disliked mathematics at school compared to only 2.4% (N=404, Table 8) who disliked the calculus course. 37.6% (N=423, Table 6) indicated that they enjoyed mathematics at school whilst 55.7% (N=404, Table 7) indicated that they enjoyed the calculus course.

Table 7 Responses of students to end of course questionnaire 1996–1998

What was your feeling towards mathematics while you were at school?

	1996	1997		1998			Total
	SciS	SciS	EngS	SciS	EngS	PrEngS	423
Enjoyed it/it was OK	51	92	41	98	47	23	352
Didn't like it	17	28	4	18	1	1	70
No response	0	0	0	1	0	0	1
P	<0.0000	<0.0000	<0.0000	<0.0000	<0.0000	<0.0000	<0.0000

Comparing the proportion 'Enjoyed it/ It was OK' versus 'Didn't like it' responses to the assumption that 50% would respond 'Enjoyed it/ It was OK', the results were that for all student groups and years the 'Enjoyed it/ It was OK' response was significantly high ($p < 0.5$ for all groups).

Co-operative learning

It should also be pointed out that students grouped spontaneously when they felt the need to discuss the work with their peers. Informal co-operative learning usually occurred among students working in groups of two. Students also worked in groups of three or more. Groups consisting of more than three students seemed to engage in short discussions spontaneously, after which students continued to work on their own. 80.4% of students (N=404, Table 9) indicated that

Table 8 Responses of students to end of course questionnaire 1996–1998

What was your feeling towards mathematics as experienced in the calculus course?

	1996	1997		1998		Total
	SciS	SciS	EngS	SciS	EngS	404
Enjoyed it / it was OK	64	118	45	103	48	378
Didn't like it	1	3	0	4	1	9
No response	2	3	0	12	0	17
P	<0.0000	<0.0000	<0.0000	<0.0000	<0.0000	<0.0000

Table 9 Responses of students to end of course questionnaire 1996–1998

Which one of the descriptions below best fits the way in which you did the graphing sessions?

	1996	1997		1998		Total
	SciS	SciS	EngS	SciS	EngS	404
Alone, or alone with occasional help from lecturer/tutor	9	24	4	21	9	67
Discussion with other students or did not do them	67	123	40	110	49	325
P	<0.0000	<0.0000	<0.0000	<0.0000	<0.0000	<0.0000

they mostly worked co-operatively during the computer graphing sessions. It often occurred that enthusiastic discussions and difference of opinion regarding mathematical concepts evolved from the graphical exploration activities.

Comparing the proportion 'Alone' versus 'Discussion with other students' responses to the assumption that 50% would respond 'Alone', the results were that for all student groups and years the 'Alone' response was significantly high ($p < 0.5$ for all groups).

Contributive learning

During the research a further aspect regarding learning and teaching was identified and defined as *contributive learning* (Steyn, 1998*),* which differs from the known concepts of co-operative learning and collaborative learning. Whereas in the last two approaches the focus is on learning via the interaction between learners, *contributive learning* indicates the involvement of *the learner as a learner* and the *facilitator as a learner* in a mutual process of learning. Participants in the learning teaching activities thus contribute jointly to each other's learning. For the facilitator this learning is not confined to mathematical subject content. It can be diverse and include aspects of student learning as well as successes and pitfalls of instructional activities as was the case in this study. *Contributive learning* links learning and teaching and can serve as an invaluable contribution to the professional development of the facilitator (teacher/lecturer).

Conclusion

If educators of introductory calculus (in South Africa) are faced with results from research surveys such as the TIMSS and TIMSS-R, it inevitably necessitates a rethinking of the way in which learning facilitation of introductory calculus is done. Although this study can be viewed as limited with regard to quantitative data pertaining to students' academic performance in mathematics *per se*, this was not the focus of the research activities described in this article. The research was aimed, on the one hand, at investigating the prerequisites for graphing technology when used as an educational tool and, on the other hand, at implementing this technology in an instructional approach to enhance the fundamental concepts underpinning a study in calculus.

The study distinctly identified the features that graphing utilities should have when used as a tool for the exploration of two-dimensional functions. One of the most significant findings during this research project has been the realisation of the importance of the interdependence between teaching, learning, instructional media and instructional content when technology is used in tertiary mathematics education. This is supported by the quantitative and qualitative analysis of the students' feedback.

The insights and results gained in treating the research questions pave the way for further research. The seemingly positive experience that the students had had using graphical exploration as an integral part of their first course in calculus could be followed up in further studies to determine if this positive experience is also manifested in better grades. Preliminary findings of an action research study (Steyn, 2001) with first-year engineering students, where the principles reported in this article are implemented, reaffirm the results and also indicate that the academic performances of the students, using graphing technology in an instructional approach as put forth in this article, indeed improve.

Furthermore, the instructional approach and principles that resulted from the study reported in this article are not restricted to a tertiary environment. Given that an appropriate graphing tool as described above is used, the principles and pedagogy discussed in this article may very well be implemented at a level where learners encounter functions of a single variable and their graphs for the first time.

Limitations of the study

The authors realise that the critical reader may feel that, by providing a centre with computers and graphing tools and then expecting 50% to respond 'no' is not taking the so-called 'pleasing factor' in questionnaire responses into consideration. We also fully realise that self-reports often are notoriously unreliable and, in deep qualitative research, would have benefited from regular and in-depth interviewing with students to search for and grapple with concept understandings which, in any event, are very difficult to grasp even in the conventional interview. The reader may feel that s/he is left with a strong dependency on the observations of the researchers over a long period of time. The authors realise that the exploratory nature of our work does not fully compensate for these shortcomings and we accept the fact that the issues addressed here will have to be addressed if we want to raise the level of reliability of this limited, local study. We therefore recommend that the action research (over an extended period of time) mentioned here should be replicated on a much wider scale, taking into account the considerations mentioned here.

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