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# THE CAPITAL-ASSET PRICING MODEL RECONSIDERED: TESTS IN REAL TERMS ON A SOUTH AFRICAN MARKET PORTFOLIO COMPRISING EQUITIES AND BONDS

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ABSTRACT
This paper extends previous work of the authors to reconsider the capital-asset pricing model (CAPM) in South Africa in real terms. As in that work, the main question this study aimed to answer remains: Can the CAPM be accepted in the South African market for the purposes of the stochastic modelling of investment returns in typical actuarial applications? To test the CAPM in real terms, conventional and index-linked bonds were included both in the composition of the market portfolio and in tests of the securities market line. For the investigation, quarterly total returns from the FTSE/JSE all-share index listed on the JSE Securities Exchange from 30 September 1964 to 31 December 2010 were used, together with yields on government bonds and consumer price indices over the same period. As expressed in the securities market line, the CAPM suggests that higher systematic risk, as measured by beta, is associated with higher expected returns, and that the relationship between expected return and beta is linear. In this investigation the above-mentioned predictions of the CAPM were tested for the South African market. Regression tests both of the zero-beta and standard versions of the CAPM were made, using both prior betas and in-period betas. Hotelling's test was also applied, as well as a regression analysis. These tests were made for individual periods as well as for all periods combined.

KEYWORDS
Beta; bonds; capital-asset pricing model; excess return; South Africa

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*The wrong sort of bees would make the wrong sort of honey.*
Winnie-the-Pooh

## 1. INTRODUCTION

1.1 The simplicity of the capital-asset pricing model (CAPM) has made it the central equilibrium model of financial economics (Ross, 1978). The attraction of the CAPM is that it offers powerful and intuitively pleasing predictions about how to explain the relationship between expected return and risk (Fama & French, 1992) and the variations in risk differentials on different risky assets (Friend et al., 1978). Ross (op. cit.) describes the CAPM as a paradigm, precisely because it is cast in terms of variables that are, at least in principle and with the usual exception of the *ex-ante–ex-post* distinction, empirically observable and statistically testable. As argued by Ross (op. cit.), expected returns are linear in beta if and only if the market portfolio is mean–variance efficient.

1.2 From an actuarial point of view, the CAPM provides a useful market-consistent pricing model for the stochastic modelling of investment returns. Typical actuarial applications of such modelling are:
– the market-consistent pricing of the liabilities of a financial institution;
– the determination of investment-performance benchmarks that are consistent both with its own liabilities and with market prices; and
– the determination of capital adequacy requirements.

1.3 The use of the CAPM for the development of a stochastic investment model for actuarial use is illustrated in Thomson & Gott (2009). This entails the stochastic determination of the expected return on each asset category during each future time interval, based on the expected return on the market portfolio, the risk-free rate and the beta of that asset category during that time interval. Such a model may then be used to determine an optimum portfolio for specified liabilities and to price those liabilities as explained in Thomson (2005) and as illustrated in Thomson (2011). The advantage of using the CAPM for this purpose is that it is a simple equilibrium model based on homogeneous expectations. This means that it provides a market-consistent price of the liabilities and it does not imply that, by optimising its investment portfolio, the institution can outperform the market. For the purposes of determining an investment-performance benchmark these are advantages. For the investment manager and for traders employed by the investment manager, on the other hand, a model that will assist in outperforming the benchmark is required. For that purpose the use of a CAPM-based stochastic model would not be appropriate.

1.4 For the purposes of the determination of capital adequacy requirements it should be borne in mind that the lower tails of the distribution of the returns on asset categories are particularly important. It is often erroneously thought that the CAPM presupposes normal distributions and that it can therefore not handle fat tails. In fact the CAPM may be used for any elliptically symmetric distribution, which includes fat-tailed distributions such as the *t*-distribution. What it cannot handle is skewness. For such purposes, before using a CAPM-based model, it should be ensured that the joint distribution of returns is elliptically symmetric.

1.5 In the case of life assurers and retirement funds, the time horizons involved are long-term and the monitoring intervals may be triennial, annual or quarterly. This study has been designed so as to address the needs of such applications. The time interval used should be dictated by the decision-making interval. Thus, if decisions are being made by traders in real time, continuous-time models are appropriate. On the other hand, if they are being made annually (e.g. when valuation results are available) then the modelling intervals should be annual. In this paper quarterly intervals are used. This choice is dictated by the maximum frequency with which actuaries may be expected to revisit the liability-driven portfolios that are used for benchmarking investment performance. The fact that the interval typically used for tests of the CAPM is weekly merely means that the rejection of the CAPM on the basis of typical tests has been premature as far as actuarial benchmarking is concerned. The use of longer intervals inevitably means that the power of the tests is reduced. However, if our decision-making interval is quarterly, we cannot reject the applicability of the CAPM on the basis of tests using weekly intervals. It is quite possible, for example, that, whilst outliers may cause skewness in weekly data, thanks to the central limit theorem they will not cause skewness in quarterly data.

1.6. As observed in a previous study (Reddy & Thomson, 2011), the market portfolio, which is central to testing the CAPM, is unobservable in practice. In that study the FTSE/JSE All Share Index was used as a market proxy. Roll (1977) and Bowie & Bradfield (1993) warned that the choice of a wrong proxy would reduce the predictive ability of the CAPM. It is possible that the results obtained in the previous study were misstated. In this study, in order to obtain a more realistic market portfolio, and in order to address the needs of actuarial applications, bonds were included both in the composition of the market portfolio and in tests of the securities market line. It is acknowledged that the inclusion of bonds does not solve the problem; it merely makes the results of tests more plausible than they would otherwise be. The inclusion of other assets would be of interest. This applies in particular to foreign assets, and indeed the CAPM should ultimately be tested on a multi-currency basis. On the other hand, it should be recognised that, whilst South African long-term institutions are strong participants in the pricing of

South African assets, they are weak participants in the pricing of foreign assets. Local tests of the CAPM therefore remain of interest. Tests including other assets are left for further research.

1.7 Furthermore, this paper reconsiders the CAPM in real terms. Although most tests of the CAPM are applied in nominal terms, it is preferable to measure returns in real terms. The reason for this is that investors' preferences must ultimately be expressed in terms of consumption of goods and services, not merely in terms of currency. In short-term applications the difference may not be material. Furthermore, since consumer price indices are calculated monthly in arrears, there are no short-term inflation-protected money-market instruments. However, in longer-term applications such as those typically used in actuarial modelling, the difference may be material and index-linked instruments exist.

1.8 From the literature it is clear that the CAPM cannot be accepted as applicable to all markets at all times. Whilst most of that literature tests the CAPM in nominal terms at relatively short time intervals, it is likely that, even in real terms and at longer time intervals, it would not be possible to accept the CAPM for all markets at all times. The purpose of this paper is not to test whether the CAPM can be accepted in real terms at quarterly intervals for all markets at all times. The purpose of this study was more modest. Following the previous study (Reddy & Thomson, op. cit.), the main question this study aimed to answer remains: Can the CAPM be accepted in the South African market for the purposes of the stochastic modelling of investment returns in typical actuarial applications? And again, in particular, does the CAPM explain expected excess return on the South African market and is the relationship between return and beta linear? In addressing these questions, the authors have extended the analysis in the previous study by including bonds in the market portfolio and by expressing returns in real terms. (Again, 'excess return' is the excess of the return over the risk-free rate; it is defined more formally in ¶4.4.2.) The intention was not to test whether the CAPM provides the best explanation of expected returns, but merely whether it provides an explanation. We have therefore not considered alternative models. Clearly the fate of the CAPM worldwide does not turn on the above questions. But the case of South Africa over the period for which data are available is of particular interest to readers of this journal.

1.9 The rest of the paper is organised as follows. In Section 2, literature on the inclusion of bonds in the CAPM market portfolio and on the expression of returns in real terms is briefly reviewed. In Section 3 the data used for this study are described. The method used is described and explained in Section 4. In Section 5 the results of the tests are presented and discussed. The results are summarised, and conclusions are drawn, and areas for further research are proposed, in Section 6.

## 2. LITERATURE REVIEW

This section extends the literature of the previous study, dealing only with the literature on the inclusion of bonds in the CAPM market portfolio and on the expression of returns in real terms. For more general literature relevant to this paper, the reader is referred to the literature review in that study.

### 2.1 THE INCLUSION OF GOVERNMENT BONDS

2.1.1 Although the market portfolio is unobservable, the closer the chosen proxy is to the market portfolio, the more reliable will be the predictions of the CAPM. Actuarial applications will generally require at least the inclusion of bonds in the market portfolio.

2.1.2 Reilly & Joehnk (1976) noted:

> Substantial progress has been made during the past several years in the application of portfolio theory to the valuation of risky assets. Unfortunately, most of these applications have been in the realm of common stock analysis and equity portfolios, with little consideration given to the use of the capital asset pricing model in either bond valuation or bond portfolio construction.

That paper attempted to rectify this deficiency, but its emphasis was on the default risk on corporate bonds, not on the holding-period risk on government securities.

2.1.3 Friend et al. (op. cit.) appear to have been the first to have included bonds in the market portfolio. They studied the US market during the period from 1964 to 1973. The risk–return relationship obtained in their study was significantly different from that using a market portfolio consisting of only equity. In particular, during the period from 1964 to 1968 the zero-beta returns on bonds and on equity were significantly lower than the risk-free rates. During the period from 1968 to 1973, the zero-beta return on bonds was lower than that on equity. They suggested that other periods needed to be analysed before any conclusion could be drawn. They also suggested that their findings might imply some segmentation between bond and equity markets.

2.1.4 Jarrow (1978) attempts to base holding-period returns on a stochastic model of the yield to redemption, and he finds that, on this basis, the beta of a bond is unstable over time. This analysis is applicable both to risky bonds and to government securities. Korn & Koziol (unpublished) use mean–variance analysis to investigate the risk–return profiles of government bonds. Like Jarrow (op. cit.), and notwithstanding his findings, they base holding-period returns on a stochastic model of the yield to redemption. They find that optimised bond portfolios exhibit “very attractive risk–return profiles”. They do not allow for the rebalancing of bonds to constant terms to maturity and they do not test a CAPM.

2.1.5 Gudikunst & McCarthy (1992) and Artikis (2001) investigate the application of the CAPM to bond mutual funds.

2.1.6 Yawitz & Marshall (1977) apply the CAPM to holding-period returns in the government bond market. Whilst they find that other measures of expected returns are better than the *ex-post* means of such returns, and that other measures of risk are better than individual bonds’ betas, they do not reject the CAPM.

2.1.7 In an attempt to quantify the effects of co-skewness on quarterly *ex-post* expected returns in a market portfolio comprising equities and corporate bonds in the USA over the period 1952 to 1976, Friend & Westerfield (1980) found that estimates of expected returns on the zero-beta portfolio were significantly higher than actual risk-free returns. For the purposes of these tests they grouped bonds so as to reduce measurement errors in the determination of betas.

2.1.8 Viceira (2012) presents evidence that "movements in both the short-term nominal interest rate and the yield spread are positively related to changes in the subsequent realized bond risk and bond return volatility." For this purpose he uses two measures of bond risk, one of which is the beta of the holding-period on the bond.

2.1.9 Applications of the CAPM to corporate bonds are of limited relevance to the inclusion of government bonds in the market portfolio, because the focus of such applications tends to be in credit risk. Applications of the CAPM to bonds that are based on stochastic models of the term structure are unnecessarily complicated. All that we need to do is to use holding-period returns for zero-coupon bonds of specified terms to redemption, rebalancing to those terms to redemption at the end of each time interval. This means that bond portfolios can be structured out of the zero-coupon bonds for the purposes of mean–variance analysis in general and the use of the CAPM in particular. The market portfolio can be similarly constructed, allowing for the market capitalisation of shares and of bonds at the various maturities selected. Because the range of maturities selected will generally be incomplete, this will inevitably be approximate. However, the extension of the range will allow arbitrarily close approximation. The use of mutual funds is also an unnecessary complication. Unless we know what bonds are in those mutual funds from time to time, we cannot apply the findings to other bond portfolios. Whilst the use of grouping to avoid measurement errors in the determination of betas is useful for equities, its benefit is of doubtful benefit in the case of bonds. None of these studies used real returns.

## 2.2 THE USE OF REAL RETURNS

2.2.1 In general, the preferences of individual agents in an economy are assumed to relate to the goods and services that money can buy, not to the money itself. For this reason the capital-asset pricing model of Sharpe (1964), Lintner (1965) and Mossin (1966) was adapted by Breeden (1979) to give a 'consumption CAPM' or 'CCAPM' (e.g. Mankiw & Shapiro, unpublished; Duffie, 2001), with betas expressed in terms of covariances of returns with consumption, rather than with the return on the market portfolio. However, from the point of view of an institutional investor, inter-temporal preferences may be expressed in terms of returns on investments, provided there is clarity as to what is meant by 'returns'. This avoids expressing expected returns in terms of consumption, a variable that is exogenous to the capital-asset market. Instead, we may merely define returns as real returns; i.e. as returns in excess of inflation, to give a 'real CAPM'.

2.2.2 The argument in the preceding paragraph is enhanced where, as in this paper, the emphasis is on institutional investors. The CCAPM is based on the assumption that investors' consumption is offset by demand for new government bonds (i.e. assets whose prices define risk-free returns to redemption dates corresponding to future consumption dates) and that this demand is in equilibrium with the supply of new government bonds. In practice, consumption decisions are made by households, whereas demand for new government bonds emanates from financial institutions. If the assumption of equilibrium between investors in secondary capital markets is questionable, the assumption of equilibrium between consumption and the supply of new bonds is more so. Findings by a number of authors (e.g. Mankiw & Shapiro, op. cit.) that the CAPM explains expected returns much better than the CCAPM may be due to this effect.

2.2.3 Samuelson (1969) shows that, for constant relative risk aversion, the consumption–saving problem can be separated from the asset-allocation problem, so that, for the latter, utility can be expressed in terms of wealth instead of consumption. For iso-elastic utility and elliptically symmetric distributions of returns, this justifies the use of the CAPM in terms of returns instead of consumption. It appears, though, that this separation presupposes constant prices of goods and services. Over time, wealth and investment returns must therefore be expressed in real terms. Again, this leads to the use of a real CAPM. For this case at least, the assumptions of the CCAPM would be unnecessary, even if there were equilibrium between consumption and the supply of new government bonds.

2.2.4 Ross (op. cit.) explains the real CAPM as follows:

> The basic argument is that returns are measured in a nominal accounting unit, say dollars, while preference is over real goods. This requires deflating returns by a price index to translate them into real returns. The CAPM given above now holds identically as before, with the understanding that all returns must be evaluated in real terms.

Kouri & de Macedo (1978) defend the real CAPM as follows:

> rational lenders and borrowers are presumably concerned with the real values of their assets and liabilities, and hence the purchasing power of a currency over goods and services available in the world economy is the appropriate standard of its value.

2.2.5 Despite its advantages, the real CAPM has not been widely dealt with in the literature. It appears that this is largely due to the following factors:

– Until relatively recently, not all currencies had index-linked bond markets (and notably the USA did not), so that real short-term risk-free rates were not observable.
– Even where such markets have been in existence for some time, the effects of lags between publication of price indices and the use of those indices in payments of coupons and redemption create difficulties in the establishment of real short-term risk-free rates.
– Whereas for long-term liability-driven investment, quarterly returns may constitute the short term, for traders, the short term may be measured in days. Much of the interest in the CAPM is driven by traders; there is no such thing as an overnight real rate of interest.

Nevertheless, the point is of particular importance in the context of the liability-driven investment of retirement funds, where members' reasonable expectations are justifiably formed in terms of the goods and services that money can buy rather than in terms of money *per se*.

2.2.6 It may appear to a reader that, because inflation is deducted from all returns, the net effect on expected returns will be zero. This objection overlooks the fact that the covariances of the returns will be different if real returns are used. The reason is that inflation may be more strongly correlated with the returns on some asset categories than with those on others. This will affect both the variances of those returns and the covariances between them, and therefore the betas.

2.2.7 It has been suggested to the authors that, whilst investors should consider real returns, they do not do so in practice. Whilst traders may not consider real returns, actuaries developing liability-driven benchmark portfolios may well be doing so. For example, for liabilities to the members and pensioners of pension funds, the trustees need to focus on real returns. In this way actuaries may be influencing prices. To the extent that traders are held accountable for their performance relative to such benchmarks, they may in effect be influenced by considerations of real return more than they themselves realise.

2.2.8 It has also been pointed out to the authors that, because inflation data are generally released some weeks after the month to which they refer, investors do not know with precision what the quarterly inflation rate will be in advance. This, it is suggested, means that allowance should be made for variations between investors' expectations. However, the CAPM assumes homogeneous expectations, so that the null hypothesis ignores such variations. More importantly, though, if one assumes semi-strong efficiency, the fact that the prices of goods and services are available in real time from the market means that the pricing of inflation-linked bonds incorporates that information correctly. Nevertheless, it is acknowledged that this necessitates a fairly heroic assumption, so the matter does deserve further attention. Furthermore, the observation in the preceding paragraph applies equally here.

2.2.9 It is quite speculative to argue that, as suggested in ¶2.2.7, actuaries may be influencing prices. It is also quite speculative to argue that, as suggested in ¶2.2.8, the pricing of inflation-linked bonds incorporates the prices of consumer goods and services efficiently. Nevertheless, if an empirical study finds that a CAPM based on real returns cannot be rejected, then it may have practical use, regardless of how traders' operations are affected by liability-driven benchmarks and of whether consumer-price information is reflected in inflation-protected bond prices.

## 3. DATA AND PERIODS CONSIDERED

### 3.1 DATA

Data for this study were obtained as explained in Appendix A. The time intervals used were calendar quarters. Particularly for earlier periods, the data were not generally available in the form required. The approximations and assumptions made are explained in that appendix, as well as the formulae used to obtain the values required.

## 3.2 PERIODS CONSIDERED

3.2.1 As explained in Appendix A, data were available from 30/9/1964 to 31/12/2010 (i.e. for quarters $t\in[1;185]$). However, this period has seen a number of changes, which may have affected the underlying assumptions of the CAPM or its testability. Whilst the choice of periods is inevitably subjective, an attempt was made to apply economic considerations. On the other hand, very short periods had to be avoided.

3.2.2 The onset of high inflation in the 1970s arguably created a discontinuity in expected returns on equities relative to bonds, as evidenced by the emergence of a reverse yield gap. In 1973, for the first time, inflation rates increased above the level of long-term interest rates (Thomson, 1996). The first period considered was therefore from 30/9/1964 to 31/12/1972 (i.e. for quarters $t\in[1;33]$).

3.2.3 Until 31/12/1985 the yield curve comprised only three points, which effectively represented yields on the primary market. The secondary market was not well developed and it was therefore not possible to determine a descriptive yield curve based on trades between investors. The yield curve reflected the yields at which the most recent issues had been made. Tests of the CAPM including bonds as well as equities prior to that date may be affected by the inefficiency of the bond market. The second period considered was therefore from 31/12/1972 to 31/12/1985 (i.e. for quarters $t\in[34;85]$).

3.2.4 Before 30/9/1989, the South African government imposed prescribed asset requirements on life offices, and even more exacting requirements on pension funds. This arguably created disequilibrium in bonds relative to equities. Whilst a cursory examination of the data before and after the abolition of prescribed assets reveals no obvious discontinuity either in the yields on or in the market capitalisation of government bonds, it was decided to treat this change as a discontinuity for the purposes of testing the CAPM. The third period considered was therefore from 31/12/1985 to 30/9/1989 (i.e. for quarters $t\in[86;100]$).

3.2.5 In South Africa inflation-linked bonds were first issued in 2000. Figure 1 shows the value of $z_{t,40}$, the continuously compounded quarterly spot yields on inflation-protected bonds with 40 quarters to maturity, for all times $t$ at which there were inflation-linked bonds in issue. As shown there, during the early years the yields were unsustainably high; investors were unfamiliar with these bonds and were reluctant to buy them. During the period from 2000 to 2002 (i.e. for quarters $t\in[143;152]$) yields decreased at a rapid rate from unreasonably high yields to more reasonable levels. Spot yields continued to decline after quarter 153, though at a much lower rate. A substantial decline occurred from quarter 161 onwards, but it levelled out relatively soon, from quarter 166. Under these circumstances it is not reasonable to suppose that the inflation-linked bond market was in equilibrium with the rest of the market. Furthermore, until 2002 there were fewer than four bonds in issue. Whilst the approximations suggested in ¶3.2.3 may be adequate for the purpose of the analysis illustrated in Figure 1, they are not ideal for the purposes of this paper. It was therefore decided to ignore inflation-linked bonds before a commencing quarter reflecting relative stability of spot yields. On the basis of the above discussion, two possible commencing quarters were considered: quarter 153 and quarter 166. For the following reasons it was decided to use quarter 153:

– On average the decline from quarter 143 to quarter 152 was steeper than the decline from 153 to 166.
– There was some evidence of stability during the latter period.
– The downward trend during the latter period was partially offset by subsequent increases.
– The use of a commencing quarter of 166 would have made the last period only 19 quarters.

It was therefore not possible to include such bonds in prior periods. The fourth and fifth periods were therefore from 30/9/1989 to 31/12/2002 (i.e. for quarters $t\in[101;152]$) and from 31/12/2002 to 31/12/2010 (i.e. for quarters $t\in[153;185]$) respectively.

3.2.6 Sample means and standard deviations of the real quarterly returns on each asset category during each period and for all periods combined are shown in Tables 1, 2 and 3 for equities, conventional bonds and inflation-protected bonds respectively.

3.2.7 It may be noted from these figures that, for all periods, the mean returns on equities and inflation-protected bonds were positive, though in some cases they were less than 0,01. For conventional bonds there were some negative mean returns in earlier periods and there were marked differences between different terms to redemption. As might be expected, the standard deviations on equities were greater than those on conventional bonds, which were in turn greater than those on inflation-protected bonds.

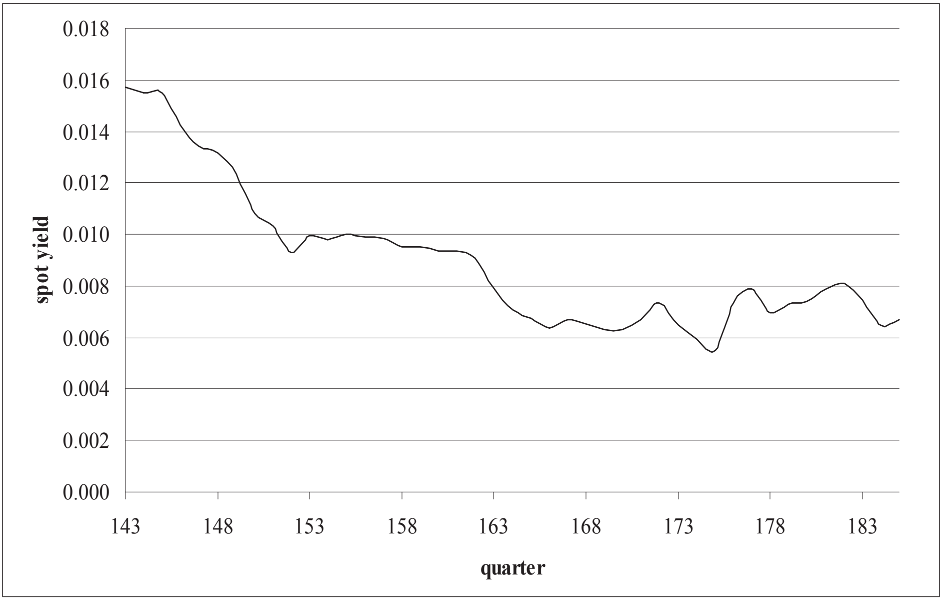


Figure 1. $z_{t,40}$: the continuously compounded quarterly spot yields on inflation-protected bonds with 40 quarters to maturity

Table 1. Sample statistics of the real returns on equities

| Period | Mean | Standard deviation |
|---|---|---|
| [1;33] | 0,022 | 0,110 |
| [34;85] | 0,020 | 0,141 |
| [86;100] | 0,025 | 0,141 |
| [101;152] | 0,009 | 0,108 |
| [153;185] | 0,031 | 0,097 |
| All | 0,020 | 0,119 |

Table 2. Sample statistics of the real returns on conventional bonds

| Term to redemption | 12 | | 40 | | 80 | |
|---|---|---|---|---|---|---|
| Period | Mean | Standard deviation | Mean | Standard deviation | Mean | Standard deviation |
| [1;33] | –0,001 | 0,009 | –0,004 | 0,024 | –0,010 | 0,106 |
| [34;85] | –0,017 | 0,034 | –0,030 | 0,079 | –0,023 | 0,165 |
| [86;100] | –0,006 | 0,033 | 0,004 | 0,048 | 0,036 | 0,298 |
| [101;152] | 0,013 | 0,038 | 0,025 | 0,098 | 0,036 | 0,198 |
| [153;185] | 0,012 | 0,033 | 0,017 | 0,079 | 0,019 | 0,153 |
| All | 0,001 | 0,034 | 0,001 | 0,079 | 0,008 | 0,178 |

Table 3. Sample statistics of the real returns on inflation-protected bonds for the period from quarter 153 to quarter 185

| Term to redemption | Mean | Standard deviation |
|---|---|---|
| 12 | 0,010 | 0,013 |
| 40 | 0,011 | 0,024 |
| 80 | 0,013 | 0,051 |

## 4. METHOD

### 4.1 REAL RETURNS

In this paper, returns are measured as forces of return. As discussed in ¶1.7, it is preferable to measure returns in real terms. Most tests of the CAPM, being more concerned about applications to trading, use relatively short intervals. Because of the difficulty of measuring real returns over such intervals—and because of the relative certainty of the level of inflation over such intervals—tests are usually applied to nominal returns. Those problems do not apply to long-term modelling. Conversely, the longer the term of the model, the greater is the uncertainty about levels of inflation. For the purposes of this research, real returns were therefore used.

### 4.2 THE RISK-FREE RATE

Because this study considered the CAPM in real terms, the risk-free return used was the real risk-free return that is the spot rate for an inflation-linked bond maturing one quarter hence. As mentioned in ¶3.2.5, inflation-linked bonds were first issued in

2000 and it was decided to ignore inflation-linked bonds before 31/12/2002. For earlier periods real risk-free returns were therefore not available. For those periods it was not possible to apply tests of the CAPM based on the risk-free return. However, tests based on zero-beta returns were made for each period considered and for all periods combined.

## 4.3 CONSTITUENTS OF THE MARKET PORTFOLIO

As explained in Appendix A, the market portfolio was assumed to comprise:

– listed equities included in the FTSE/JSE all-share index on the JSE Securities Exchange;
– zero-coupon conventional bonds with maturities of 3, 10 and 20 years; and
– with effect from 31/12/2002, zero-coupon inflation-linked bonds with the same maturities.

As further explained in that appendix, almost all the variability in the yields on bonds may be explained by the first three principal components of the yield curve, and therefore by those on three well-dispersed zero-coupon bonds. The reduction of the bond portfolio to maturities of 3, 10 and 20 years therefore results in no material loss in generality.

## 4.4 VARIABLES

4.4.1 From the data, the value of $R_{it}$, being the return on component $i$ for $i \in \{E, CB, ILB\}$ of the market portfolio during quarter $t$ was determined as explained in Appendix A for $i \in I^{-}$, $t \in [1;152]$, and for $i \in I^{+}$, $t \in [153;185]$;

where $I^{-}$ comprises:

– listed equities included in the FTSE/JSE all-share index on the JSE Securities Exchange; and
– zero-coupon conventional bonds with maturities of 3, 10 and 20 years;

and $I^{+}$ comprises:

– listed equities as above;
– zero-coupon conventional bonds with maturities as above; and
– zero-coupon inflation-linked bonds with the same maturities.

4.4.2 For $t \in [153, \ldots, 185]$ the excess return on component $i$ during quarter $t$ was determined as:

$$r_{it} = R_{it} - R_{Ft};$$

where:

$R_{Ft}$ is the return during quarter $t$ on an inflation-linked bond maturing at the end of that quarter.

4.4.3 The market capitalisation $m_{it}^{*}$ of component $i$ of the market portfolio at time $t$ was determined as explained in Appendix A. From these values the return on the market portfolio during quarter $t$ was determined as:

$$R_{Mt}=\begin{cases}\dfrac{\sum_{i\in I^-} m^*_{i,t-1}R_{it}}{\sum_{i\in I^-} m^*_{i,t-1}} & \text{for } t\in[1,152];\\[2ex] \dfrac{\sum_{i\in I^+} m^*_{i,t-1}R_{it}}{\sum_{i\in I^+} m^*_{i,t-1}} & \text{for } t\in[153,185].\end{cases}$$

The excess return on the market portfolio during quarter *t* was determined as:

$$r_{Mt}=R_{Mt}-R_{Ft}\ .$$

4.5 EXOGENOUS VARIABLES

4.5.1 While the variances and covariances of the variables to be modelled are endogenous to the requirements of long-term actuarial modelling, other explanatory variables would be exogenous, and would require separate modelling. As observed in Reddy & Thomson (op. cit.), many authors have investigated the effects of exogenous variables. However, as observed there, these effects add little value to long-term modelling and they are diluted by the aggregation of equities into sectors and portfolios.

4.5.2 As in Reddy & Thomson (op. cit.) the effects of exogenous variables have been ignored. As explained below, however, (cf. sections 5.1.2, 5.1.4 and 5.2.2) the effects of nonlinearity were tested.

4.6 THE ZERO-BETA VERSION OF THE CAPM

In terms of the zero-beta version of the CAPM, it is not necessary to refer to the risk-free asset. In order to test the zero-beta version of the CAPM, it is necessary to translate the *ex-ante* parameters of an equilibrium model into *ex-post* realisations. For that purpose it is necessary to assume the validity of some return-generating function. For any *ex-ante* model and *ex-post* realisations there is almost certainly some generating function that will link those realisations with the model (Blume & Husic, 1973). The following return-generating process for component *i* in quarter *t* may be used to test the zero-beta version of the CAPM.

$$R_{it}=\gamma_0+\beta_{it}\gamma_1+\varepsilon_{it}\ ; \qquad (1)$$

where:

$$\beta_{it}=\frac{\sigma_{iMt}}{\sigma_{MMt}};$$

$$\varepsilon_{it}\sim N\left(0,\sigma^2_{\varepsilon it}\right);$$

$$\sigma_{ijt}=\operatorname{cov}\left(R_{it},R_{jt}\right);$$

$$\sigma^2_{\varepsilon it}=\operatorname{var}\left(\varepsilon_{it}\right)\text{ ; and}$$

$$\operatorname{cov}\left(\varepsilon_{it},\varepsilon_{iu}\right)=0 \text{ for } t\neq u\ .$$

### 4.6.1 Prior Betas

The investigation was first carried out by calculating a prior beta for each component $i$ for each quarter $t=12,\ldots,185$. For this purpose, five years of quarterly prior real returns were used to give the prior beta as:

$$\hat{\beta}_{it}=\frac{\hat{\sigma}_{i\mathrm{Mt}}}{\hat{\sigma}_{\mathrm{MMt}}}; \tag{2}$$

where:

$$\hat{\sigma}_{ijt}=\frac{1}{19}\sum_{v=t-20}^{t-1}\left(R_{iv}-\overline{R}_{it}\right)\left(R_{jv}-\overline{R}_{jt}\right);\text{ and}$$

$$\overline{R}_{ut}=\frac{1}{20}\sum_{v=t-20}^{t-1}R_{uv}\ \text{ for } u=i,j.$$

### 4.6.2 In-period Betas

4.6.2.1 If the rational-expectations hypothesis (REH) holds, then it is not necessary to use prior betas. As explained by Blume & Husic (op. cit.), if *ex-ante* values of beta differ from *ex-post* values at the start of a period, then *ex-post* estimates derived from values during the period "may more accurately mirror investors' *ex-ante* expectations." If both *ex-ante* betas and *ex-post* in-period sample betas are unbiased estimates of population betas for the respective indices, then the in-period sample betas are unbiased estimates of the *ex-ante* betas. The in-period sample betas may then be used to test the joint hypothesis that both the CAPM and the REH hold. Such a test is useless as an operational test: it does not test whether the CAPM works, because in-period sample betas are not available *ex ante*. However, for the purpose of testing whether the CAPM can be used in long-term models it is relevant, since such a model can generate unbiased *ex-ante* betas.

4.6.2.2 Another investigation was carried out using in-period betas, which, for each component $i$, were estimated for each calendar year $Y=1965,\ldots,2010$ as:

$$\hat{\beta}^{\mathrm{Y}}_{i[Y]}=\frac{\hat{\sigma}_{i\mathrm{M}[Y]}}{\hat{\sigma}_{\mathrm{MM}[Y]}}; \tag{3}$$

where:

$$\hat{\sigma}_{ij[Y]}=\frac{1}{3}\sum_{t\in Y}\left(R_{it}-\overline{R}_{i[Y]}\right)\left(R_{jt}-\overline{R}_{j[Y]}\right);\text{ and}$$

$$\overline{R}_{u[Y]}=\frac{1}{4}\sum_{t\in[Y]}R_{ut}\ \text{ for } u=i,j.$$

The annual return for each calendar year $Y$ was calculated as:

$$R^{\mathrm{Y}}_{i[Y]}=\sum_{t\in[Y]}R_{it}=4R_{i[Y]}; \tag{4}$$

where:

$$R_{i[Y]} = \frac{1}{4} \sum_{t \in [Y]} R_{it}.$$

4.6.2.3 A further investigation was carried out using in-period betas for each period $[p] \in \{[1;33],[34;85],[86;100],[101;152],[153;185]\}$; where $[a,b]$ denotes the period from quarter $a$ to quarter $b$. The in-period beta for each component $i$ in period $p$ was estimated as:

$$\hat{\beta}_{i[p]} = \frac{\hat{\sigma}_{i\mathrm{M}[p]}}{\hat{\sigma}_{\mathrm{MM}[p]}}; \tag{5}$$

where:

$$\hat{\sigma}_{ij[p]} = \frac{1}{q_{[p]} - 1} \sum_{t \in [p]} \left(R_{it} - \overline{R}_{i[p]}\right)\left(R_{jt} - \overline{R}_{j[p]}\right); \text{ and}$$

$$\overline{R}_{u[p]} = \frac{1}{q_{[p]}} \sum_{t \in [p]} R_{ut} \text{ for } u = i, j; \text{ and}$$

$q_{[p]}$ is the number of quarters in $p$.

The quarterly return for each period $p$ was calculated as:

$$R_{i[p]} = \frac{1}{q_{[p]}} \sum_{t \in [p]} R_{it}. \tag{6}$$

4.6.2.4 A further investigation was carried out using all periods combined. For this purpose, for each component $i$, the in-period beta for all periods combined was estimated as:

$$\hat{\beta}_{i} = \frac{\hat{\sigma}_{i\mathrm{M}}}{\hat{\sigma}_{\mathrm{MM}}}; \tag{7}$$

where:

$$\hat{\sigma}_{ij} = \frac{1}{184} \sum_{t=1}^{185} \left(R_{it} - \overline{R}_{i}\right)\left(R_{jt} - \overline{R}_{j}\right); \text{ and}$$

$$\overline{R}_{u} = \frac{1}{185} \sum_{t=1}^{185} R_{ut} \text{ for } u = i, j.$$

The quarterly return for all periods combined was calculated as:

$$R_{i} = \frac{1}{185} \sum_{t=1}^{185} R_{it}. \tag{8}$$

## 4.7 THE STANDARD VERSION OF THE CAPM

As with the zero-beta version of the CAPM, in order to test the standard version of the CAPM it is necessary to assume the validity of some return-generating process. With the standard version of the CAPM it is necessary to refer to the risk-free asset. The following return-generating process may be used to test the zero-beta version of the CAPM for component $i$ in quarter $t$:

$$r_{it} = \beta_{it} r_{\mathrm{M}t} + \varepsilon_{it}\,; \tag{9}$$

where:

$$\beta_{it} = \frac{\sigma_{i\mathrm{M}t}}{\sigma_{\mathrm{MM}t}}\,;$$

$$\varepsilon_{it} \sim N\left(0, \sigma^2_{\varepsilon it}\right);$$

$$\sigma_{ijt} = \mathrm{cov}\left(r_{it}, r_{jt}\right);$$

$$\sigma^2_{\varepsilon it} = \mathrm{var}\left(\varepsilon_{it}\right); \text{ and}$$

$$\mathrm{cov}\left(\varepsilon_{it}, \varepsilon_{iu}\right) = 0 \text{ for } t \neq u\,.$$

### 4.7.1 Prior Betas

As for the zero-beta version, the investigation was first carried out by calculating a prior beta for each component $i$ for each quarter $t \in [173;185]$. Again, five years of quarterly prior returns were used to give the prior beta as in equation (2).

### 4.7.2 In-period Betas

4.7.2.1 Another investigation was again carried out using in-period betas which, for each component $i$, were estimated for each calendar year $Y = 2003, \ldots, 2010$ as in equation (3). The annual return for each calendar year $Y$ was calculated as:

$$r^{\mathrm{Y}}_{i[Y]} = \sum_{t \in [Y]} r_{it} = 4 r_{i[Y]}; \tag{10}$$

where:

$$r_{i[Y]} = \frac{1}{4} \sum_{t \in [Y]} r_{it}.$$

4.7.2.2 A further investigation was carried out using all quarters combined. For this purpose, for each asset class $i$, the in-period beta for all quarters $t \in [153;185]$ was estimated as in equation (7) where:

$$\hat{\sigma}_{ij} = \frac{1}{32} \sum_{t=1}^{33} \left(R_{it} - \bar{R}_i\right)\left(R_{jt} - \bar{R}_j\right); \text{ and}$$

$$\bar{R}_u = \frac{1}{33} \sum_{t=1}^{33} R_{ut} \text{ for } u = i, j.$$

The quarterly return for all quarters combined was calculated as:

$$r_i = \frac{1}{33}\sum_{t=1}^{33} r_{it} \ . \tag{11}$$

## 5. EMPIRICAL TESTS

### 5.1 THE ZERO-BETA VERSION OF THE CAPM

#### 5.1.1 A Test of the Explanatory Power of the CAPM using Prior Betas

5.1.1.1 To reduce the confounding effect of the REH and to allow extension to the zero-beta version of the CAPM, a less explicitly expressed statement may be tested, viz. that:

$$E\{R_i\} = \gamma_0 + \gamma_1 \beta_i \ . \tag{12}$$

The null hypothesis for the zero-beta version of the CAPM for a given quarter *t* may be expressed in terms of the equation:

$$R_{it} = \gamma_0 + \gamma_1 \beta_{it} + \varepsilon_{it} \ . \tag{13}$$

In the zero-beta version of the CAPM, equation (13) implies that:

$$\gamma_0 = R_{Zt} \text{; and}$$
$$\gamma_1 = E\{R_{Mt} - R_{Zt}\};$$

where $R_{Zt}$ is the return on the zero-beta portfolio *Z* for a given quarter *t*.

5.1.1.2 A major advantage of such tests is that they do not use *ex-post* expected values of $R_{Mt}$.

5.1.1.3 Using the above test to investigate whether the CAPM explains rates of return, the return for each component *i* of the market portfolio was regressed, for each quarter $t \in Y$ where $t=21,\ldots,185$ and $Y=1969,\ldots,2010$, against the corresponding prior beta estimate. The relationship examined is equation (13), expressed in terms of estimated prior betas as:

$$R_{it} = \gamma_{0[Y]} + \gamma_{1[Y]}\hat{\beta}_{it} + \varepsilon_{it}.$$

5.1.1.4 A similar regression analysis was applied to each period $[p]$. The return for each component *i* was regressed, for each period $[p] \in \{[21;33],[34;85],[86;100],[101;152],[153;185]\}$, against the corresponding prior beta estimate. The relationship examined, for each period *p*, expressed in terms of estimated prior betas is:

$$R_{it} = \gamma_{0[p]} + \gamma_{1[p]}\hat{\beta}_{it} + \varepsilon_{it}.$$

5.1.1.5 A further investigation involved a regression of the return on each component $i$ for each quarter $t$ for $t=21,\ldots,185$, against the corresponding prior beta estimate, which represents a regression analysis across all quarters. The relationship examined is again equation (13), expressed in terms of estimated prior betas as:

$$R_{it} = \gamma_0 + \gamma_1 \hat{\beta}_{it} + \varepsilon_{it}.$$

5.1.1.6 The results of the analysis for each period $p$ and for all periods combined are given in Table 4.

Table 4. Summary of regression analysis for the test of the explanatory power of the CAPM using prior betas

| Period | $R^2$ | Estimate | Value | $t$-value | $p$-value |
|---|---|---|---|---|---|
| [21;33] | 0,004 | $\gamma_{0[p]}$ | –0,014 | –0,841 | 0,202 |
| | | $\gamma_{1[p]}$ | 0,010 | 0,442 | 0,670 |
| [34;85] | 0,013 | $\gamma_{0[p]}$ | –0,025 | –2,254 | 0,013 |
| | | $\gamma_{1[p]}$ | 0,023 | 1,653 | 0,950 |
| [86;100] | 0,014 | $\gamma_{0[p]}$ | –0,012 | –0,319 | 0,375 |
| | | $\gamma_{1[p]}$ | 0,032 | 0,898 | 0,814 |
| [101;152] | 0,000 | $\gamma_{0[p]}$ | 0,024 | 1,731 | 0,958 |
| | | $\gamma_{1[p]}$ | –0,004 | –0,283 | 0,389 |
| [153;185] | 0,017 | $\gamma_{0[p]}$ | 0,010 | 1,200 | 0,884 |
| | | $\gamma_{1[p]}$ | 0,023 | 1,715 | 0,956 |
| All | 0,005 | $\gamma_0$ | –0,001 | –0,141 | 0,444 |
| | | $\gamma_1$ | 0,015 | 1,947 | 0,974 |

5.1.1.7 First, as indicated in Table 4, the values of $R^2$ for the tests for each period and for all periods combined are low, indicating that most of the risk is unsystematic. The zero-beta version of the CAPM predicts that $\gamma_{0[p]}, \gamma_0 \geq 0$. For this version of the CAPM the null hypothesis is that $\gamma_{0[p]}, \gamma_0 \geq 0$ and the alternative hypothesis is that $\gamma_{0[p]}, \gamma_0 < 0$. Again, the CAPM is rejected if the null hypothesis is rejected. The $p$-values for each period other than period [34;85] and for all periods combined with negative values of $\gamma_{0[p]}$ and $\gamma_0$ are greater than 5% (using a one-tailed test). The null hypothesis is rejected only for period [34;85].

5.1.1.8 The zero-beta version of the CAPM predicts that $\gamma_{1[p]}, \gamma_1 \geq 0$. The null hypothesis is that $\gamma_{1[p]}, \gamma_1 \geq 0$ and the alternative hypothesis is that $\gamma_{1[p]}, \gamma_1 < 0$. The CAPM is rejected if the null hypothesis is rejected. For some periods the value of $\gamma_{1[p]}$ is very low. In fact for period [101;152] it is negative. However, when it is borne in mind that we are working with real quarterly returns on the market portfolio including bonds, low market risk premiums are not unexpected. The null hypothesis is not rejected for any of the periods nor for all periods combined. The zero-beta form of the CAPM also predicts that $\gamma_{1[p]}, \gamma_1 > \gamma_{0[p]}, \gamma_0$ respectively. This prediction is not tested here, though by inspection it appears that this may hold true for most periods and all periods combined.

5.1.1.9 One would not necessarily expect all periods to be insignificant at the 5% level as the probability of that is $0{,}95^5 = 0{,}77$. The results reported in Table 4 do therefore not constitute grounds for rejection of the CAPM. However, it may be argued on the basis of that table that the CAPM did not necessarily apply in every period. Of the 42 years considered, there were three years during which one or both of the parameters were outside of their 95% confidence limits. The probability of having at least three years outside of their 95% confidence limits is about 79,0%, which is not significant. The CAPM cannot therefore be rejected on the grounds of these tests.

5.1.1.10 The zero-beta version of the CAPM does not only predict that $\gamma_{1[Y]}, \gamma_{1[p]}, \gamma_1 \geq 0$ but also that it equals the expected excess return on the market portfolio. For this reason, the differences between $\gamma_{1[Y]}$ and $R_{M[Y]} - R_{Z[Y]}$ are compared. The null hypothesis is that $\left(R_{M[Y]} - R_{Z[Y]}\right) - \gamma_{1[Y]} = 0$ and the alternative hypothesis is that $\left(R_{M[Y]} - R_{Z[Y]}\right) - \gamma_{1[Y]} \neq 0$. These differences have a mean of –0,003 and a standard deviation of 0,029. The $p$-value for this $t$-test for differences is 0,453, which, on the basis of a two-tailed test, is greater than 5%. Therefore, the null hypothesis is not rejected. A similar test is performed to compare the differences between $\gamma_{1[p]}$ and $R_{M[p]} - R_{Z[p]}$. These differences have a mean of –0,004 and a standard deviation of 0,006. The $p$-value for this $t$-test is 0,227 which is also greater than 5%. Again, the null hypothesis is not rejected. Therefore, neither of these tests changes the conclusion that the CAPM cannot be rejected for the zero-beta version of the CAPM using prior betas.

### 5.1.2 A Test for Nonlinearity using Prior Betas

5.1.2.1 In order to test for nonlinearity between the return for each component $i$, for each quarter $t \in Y$ where $Y = 1969,\ldots,2010$, and the corresponding prior beta estimate, the relationship examined, for each calendar year $Y$, expressed in terms of estimated prior betas is:

$$R_{it} = \gamma_{0[Y]} + \gamma_{1[Y]}\hat{\beta}_{it} + \gamma_{2[Y]}\hat{\beta}_{it}^2 + \varepsilon_{it}.$$

The estimate values, $t$-values and $p$-values are those for $\gamma_{2[Y]}$ only; other coefficients are not relevant to this test.

5.1.2.2 Similarly, in order to test for nonlinearity between return for each component $i$ for each quarter $t \in p$ where $[p] \in \{[1;33],[34;85],[86;100],[101;152],[153;185]\}$, and the corresponding prior beta estimate. The relationship examined, for each period $p$, expressed in terms of estimated prior betas is:

$$R_{it} = \gamma_{0[p]} + \gamma_{1[p]}\hat{\beta}_{it} + \gamma_{2[p]}\hat{\beta}_{it}^2 + \varepsilon_{it}.$$

5.1.2.3 Furthermore, in order to test for nonlinearity between the return on each component $i$ for all periods combined, and the corresponding prior beta estimate, the relationship examined expressed in terms of estimated prior betas is:

$$R_{it} = \gamma_0 + \gamma_1 \hat{\beta}_{it} + \gamma_2 \hat{\beta}_{it}^2 + \varepsilon_{it}.$$

5.1.2.4 The results of the analysis for each period $p$ and for all periods combined are given in Table 5.

Table 5. Summary of the regression analysis for the test for nonlinearity using prior betas

| Period | $R^2$ | Estimate | Value | $t$-value | $p$-value |
|---|---|---|---|---|---|
| [21;33] | 0,005 | $\gamma_{2[p]}$ | 0,058 | 0,197 | 0,845 |
| [34;85] | 0,022 | $\gamma_{2[p]}$ | 0,068 | 1,365 | 0,174 |
| [86;100] | 0,014 | $\gamma_{2[p]}$ | –0,008 | –0,124 | 0,902 |
| [101;152] | 0,005 | $\gamma_{2[p]}$ | –0,030 | –0,976 | 0,330 |
| [153;185] | 0,017 | $\gamma_{2[p]}$ | –0,001 | –0,029 | 0,977 |
| All | 0,005 | $\gamma_2$ | –0,003 | –0,184 | 0,854 |

5.1.2.5 For the relationship between the return and the beta estimate to be linear, one would expect that $\gamma_{2[p]}, \gamma_2 = 0$ . This is the null hypothesis; the alternative hypothesis is that $\gamma_{2[p]}, \gamma_2 \neq 0$ . As indicated in Table 5, $\gamma_{2[p]}$ and $\gamma_2$ are not significantly different from zero for any period nor for all periods combined.

5.1.2.6 However, of the 42 years considered, there were five years during which $\gamma_{2[Y]}$ was outside of its confidence limits. The probability of at least five such occurrences is 5,7%, which is not significant. It may be concluded that, considered annually, the relationship is linear.

### 5.1.3 A Test of the Explanatory Power of the CAPM using In-period Betas

5.1.3.1 A similar test to that used in section 5.1.1 was used to investigate the explanatory power of the CAPM using in-period betas. In this case, the in-period return for each component $i$ was regressed, for each calendar year $Y$ where $Y$= 1965,…,2010, against the corresponding in-period beta estimate for each component $i$ for each calendar year $Y$. The relationship examined is equation (13), expressed in terms of estimated in-period betas as:

$$R_{i[Y]}^{Y} = \gamma_{0[Y]}^{Y} + \gamma_{1[Y]}^{Y} \widehat{\beta}_{i[Y]}^{Y} + \varepsilon_{i[Y]}^{Y}.$$

5.1.3.2 A similar regression analysis was applied to each period $p$. The in-period return for each component $i$ was regressed, for each period $p$ where $[p] \in \{[1;33],[34;85],[86;100],[101;152],[153;185]\}$, against the corresponding in-period beta estimate for each component $i$ for each period $p$. The relationship examined is equation (13), expressed in terms of estimated in-period betas as:

$$R_{i[p]} = \gamma_{0[p]} + \gamma_{1[p]} \widehat{\beta}_{it} + \varepsilon_{i[p]}.$$

5.1.3.3 A further investigation involved a regression of the in-period return on each component $i$ for the entire period from $t=1,\ldots,185$, against the corresponding in-period beta estimate for each component $i$. The relationship examined is equation (13), expressed in terms of estimated in-period betas as:

$$R_i = \gamma_0 + \gamma_1 \widehat{\beta}_i + \varepsilon_i.$$

5.1.3.4 The results of the analysis for each period $p$ and for all periods combined are given in Table 6.

Table 6. Summary of regression analysis for the test of the explanatory power of the CAPM using in-period betas

| Period | $R^2$ | Estimate | Value | $t$-value | $p$-value |
|---|---|---|---|---|---|
| [1;33] | 0,886 | $\gamma_{0[p]}$ | –0,006 | –1,715 | 0,114 |
| | | $\gamma_{1[p]}$ | 0,019 | 3,947 | 0,971 |
| [34;85] | 0,735 | $\gamma_{0[p]}$ | –0,034 | –2,957 | 0,049 |
| | | $\gamma_{1[p]}$ | 0,033 | 2,356 | 0,929 |
| [86;100] | 0,422 | $\gamma_{0[p]}$ | 0,004 | 0,275 | 0,595 |
| | | $\gamma_{1[p]}$ | 0,021 | 1,208 | 0,825 |
| [101;152] | 0,044 | $\gamma_{0[p]}$ | 0,024 | 1,736 | 0,888 |
| | | $\gamma_{1[p]}$ | –0,005 | –0,303 | 0,395 |
| [153;185] | 0,763 | $\gamma_{0[p]}$ | 0,013 | 7,943 | 1,000 |
| | | $\gamma_{1[p]}$ | 0,015 | 4,015 | 0,995 |
| All | 0,368 | $\gamma_0$ | 0,006 | 2,107 | 0,956 |
| | | $\gamma_1$ | 0,009 | 1,705 | 0,926 |

5.1.3.5 First, as indicated in Table 6, the values of $R^2$ for the tests of the periods and all periods combined have increased. The zero-beta version of the CAPM predicts that $\gamma_{0[p]},\gamma_0 \geq 0$. For this version of the CAPM the null hypothesis is that $\gamma_{0[p]},\gamma_0 \geq 0$ and the alternative hypothesis is that $\gamma_{0[p]},\gamma_0 < 0$. Again, the CAPM is rejected if the null hypothesis is rejected. And again for some periods the value of $\gamma_{1[p]}$ is very low, and again negative for period [101;152]. Nevertheless, since the $p$-values for each period and for all periods combined with negative values of $\gamma_{0[p]}$ and $\gamma_0$ are all greater than 5% (using a one-tailed test), with the exception of period [34;85] (the same exception as shown in Table 4), the null hypothesis is not rejected for any of the periods, except that period.

5.1.3.6 The zero-beta version of the CAPM predicts that $\gamma_{1[p]},\gamma_1 \geq 0$. The null hypothesis is that $\gamma_{1[p]},\gamma_1 \geq 0$ and the alternative hypothesis is that $\gamma_{1[p]},\gamma_1 < 0$. The CAPM is rejected if the null hypothesis is rejected. The null hypothesis is not rejected for any period except period [34;85], nor for all periods combined. The zero-beta form of the CAPM also predicts that $\gamma_{1[p]},\gamma_1 > \gamma_{0[p]},\gamma_0$ respectively. This prediction is not tested here, though by inspection it appears that it may not hold true for period [101;152].

5.1.3.7 Again one would not necessarily expect all periods to be insignificant at the 5% level. The results reported in Table 6 do therefore not constitute grounds for rejection of the CAPM. However, it may be argued on the basis of that table that the CAPM did not necessarily apply in every period. Of particular concern here is the fact that it is the last period, for which inflation-linked bonds are included, that fails the test. Of the 46 years considered, there were 11 years during which one or both of the parameters were outside of their 95% confidence limits. The probability of having at least 11 outside of their 95% confidence limits is 0,4%, which is significant. Considered annually, using in-period betas, the zero-beta version of the CAPM is rejected on the grounds of these tests for all periods combined. However, most of the 11 years during which one or both of the parameters were outside of their 95% confidence limits occurred before 1986. During the 25 years from 1986 to 2010 there were only four years during which one or both of the parameters were outside of their 95% confidence limits. The probability of having at least four outside of their 95% confidence limits is 22,3%, which is not significant. It appears, therefore, that, using in-period betas, it would be reasonable to conclude that the zero-beta version of the CAPM cannot be rejected for periods since 1986.

5.1.3.8 On the basis of the tests using prior betas, the zero-beta version of the CAPM may be accepted for the period since 1985. However, it would be inconsistent to accept the zero-beta version of the CAPM using prior betas during the period from 1986 to 1989 and to reject it using in-period betas during that period. This would suggest that the prior betas are a better estimate of the *ex-ante* betas than the in-period betas. The best we can say for the zero-beta version of the CAPM is that it may be accepted for the period since 1989 using either prior or in-period betas.

5.1.3.9 As in the case of the prior betas (¶5.1.1.10), the zero-beta version of the CAPM does not only predict that $\gamma_{1[Y]}, \gamma_{1[p]}, \gamma_1 \geq 0$ but also that it equals the expected excess return on the market portfolio. For this reason, the differences between $\gamma_{1[Y]}^{Y}$ and $R_{M[Y]}^{Y} - R_{Z[Y]}^{Y}$ are compared. In the light of the observation in the preceding paragraph, only the periods since 1989 are considered. The null hypothesis is that $\left(R_{M[Y]}^{Y} - R_{Z[Y]}^{Y}\right) - \gamma_{1[Y]}^{Y} = 0$ and the alternative hypothesis is that $\left(R_{M[Y]}^{Y} - R_{Z[Y]}^{Y}\right) - \gamma_{1[Y]}^{Y} \neq 0$. These differences have a mean of –0,016 and a standard deviation of 0,116. The *p*-value for this *t*-test for differences is 0,517, which is greater than 5% (using a two-tailed test). Therefore, the null hypothesis is not rejected. A similar test is performed to compare the differences between $\gamma_{1[p]}$ and $R_{M[p]} - R_{Z[p]}$. These differences have a mean of 0,051 and a standard deviation of 0,036. The *p*-value for this *t*-test is 0,296, which is also greater than 5%. Again, the null hypothesis is not rejected. Therefore, for periods since 1989, neither of these tests changes the conclusion that the CAPM cannot be rejected for the zero-beta version of the CAPM using in-period betas.

### 5.1.4 A Test for Nonlinearity using In-Period Betas

5.1.4.1 In order to test for nonlinearity between return for each component *i*, for each calendar year *Y* where *Y* = 1965,…,2010, and the corresponding in-period beta estimate for each component *i* for each calendar year *Y*. The relationship examined, for each calendar year *Y*, expressed in terms of estimated in-period betas is:

$$R^{Y}_{i[Y]} = \gamma^{Y}_{0[Y]} + \gamma^{Y}_{1[Y]}\widehat{\beta}^{Y}_{i[Y]} + \gamma^{Y}_{2[Y]}\widehat{\beta}^{Y\,2}_{i[Y]} + \varepsilon^{Y}_{i[Y]}.$$

The estimate values, $t$-values and $p$-values are those for $\gamma_{2[Y]}$ only; other coefficients are not relevant to this test.

5.1.4.2 Similarly, in order to test for nonlinearity between return for each component $i$ for each period $p$ where $[p] \in \{[1;33],[34;85],[86;100],[101;152],[153;185]\}$, and the corresponding in-period beta estimate for each component $i$ for each period $p$. The relationship examined, for each period $p$, expressed in terms of estimated in-period betas is:

$$R_{i[p]} = \gamma_{0[p]} + \gamma_{1[p]}\widehat{\beta}_{i[p]} + \gamma_{2[p]}\widehat{\beta}^{2}_{i[p]} + \varepsilon_{i[p]}.$$

5.1.4.3 Furthermore, it was necessary to test for nonlinearity between the in-period return on each component $i$ for the entire period from $t=1,\ldots,185$, and the corresponding in-period beta estimate for each component $i$. The relationship examined expressed in terms of estimated in-period betas is:

$$R_i = \gamma_0 + \gamma_1\widehat{\beta}_i + \gamma_2\widehat{\beta}^2_i + \varepsilon_i.$$

5.1.4.4 The results of the analysis for each period $p$ and for all periods combined are given in Table 7.

Table 7. Summary of regression analysis for the test for nonlinearity using in-period betas

| Period | $R^2$ | Estimate | Value | $t$-value | $p$-value |
|---|---|---|---|---|---|
| [1;33] | 0,996 | $\gamma_{2[p]}$ | 0,074 | 5,609 | 0,112 |
| [34;85] | 0,967 | $\gamma_{2[p]}$ | 0,057 | 2,658 | 0,229 |
| [86;100] | 1,000 | $\gamma_{2[p]}$ | –0,080 | –44,592 | 0,014 |
| [101;152] | 0,990 | $\gamma_{2[p]}$ | –0,084 | –9,548 | 0,066 |
| [153;185] | 0,875 | $\gamma_{2[p]}$ | 0,041 | 1,894 | 0,131 |
| All | 0,545 | $\gamma_2$ | 0,020 | 1,250 | 0,279 |

5.1.4.5 For the relationship between the return and the beta estimate to be linear, one would expect that $\gamma_{2[p]},\gamma_2 = 0$. This is the null hypothesis; the alternative hypothesis is that $\gamma_{2[p]},\gamma_2 \neq 0$. As indicated in Table 7, $\gamma_{2[p]}$ is significantly non-zero in period [86;100].

5.1.4.6 Again one would not necessarily expect all periods to be insignificant at the 5% level and the results reported in Table 7 do therefore not constitute grounds for rejection of the CAPM. However, it may again be argued on the basis of that table that the CAPM did not necessarily apply in every period. Of the 46 years considered, there were four years during which $\gamma_{2[Y]}$ was outside of its 95% confidence limits. The probability of having at least four outside of their 95% confidence limits is about 19,7%, which is not significant. On the basis of these tests the linearity of the CAPM cannot be rejected. However, these results are secondary to those of ¶5.1.3.7 above.

5.1.4.7 Table 7 suggests that, since quarter 100 (1989), it would be reasonable to accept that the relationship is linear.

### 5.1.5 Further Analysis of Tests of the Zero-Beta Version of the CAPM

5.1.5.1 Whilst the results reported above differed for different periods, overall, as explained in ¶¶5.1.1.9 and 5.1.3.7 for prior betas and in-period betas respectively, the CAPM is rejected for certain periods either because the results of the tests for those periods were significant or because the number of years during which there were significant results was itself significant. In this section these year-by-year results are analysed so as to identify the periods during which they were significant.

5.1.5.2 Table 8 shows, for periods approximately corresponding to those of Tables 4 to 7, the number of years during each period in which significant results were found for each of the parameters. The correspondence is approximate because, whereas the periods shown in Tables 4 to 7 are integral numbers of quarters, the analysis here is by calendar year. Table 8 also shows, for each period, the *p*-value of the binomial test; i.e. the probability that the number of years in which significant results were found would be greater than or equal to the number found. *P*-values less than or equal to 5% are highlighted. For $\gamma_{0[Y]}$ and $\gamma_{1[Y]}$ the number of significant results were taken from the tests of explanatory power described in sections 5.1.1 and 5.1.3. For $\gamma_{2[Y]}$ the number of significant results were taken from the tests for non-linearity described in sections 5.1.2 and 5.1.4. For the purposes of calculating the 'combined' number of significant results, a year was taken as showing a significant result if any one of the parameters was significant at the 5% level. On the null hypothesis the probability of this occurrence in any one year is $1-0{,}95^3=0{,}143$. Table 8 also shows the number of significant years for selected combinations of periods, and for all periods combined, with the corresponding *p*-values.

5.1.5.3 It may be seen from Table 8 that, for prior betas, the CAPM cannot be rejected for all periods combined. Only for the period from 2003 to 2010 is it rejected.

5.1.5.4 For in-period betas the CAPM is rejected for all periods combined. Of the periods considered, it is rejected only for the periods from 1965 to 1985. It appears from these observations that, whilst for in-period betas the CAPM must be rejected for earlier years, it may be accepted for periods from 1986 onwards.

## 5.2 THE STANDARD VERSION OF THE CAPM

### 5.2.1 A Test of the Explanatory Power of the CAPM using Prior Betas

5.2.1.1 As in equation (13) the null hypothesis for the standard version of the CAPM, for a given quarter *t* may be expressed in terms of the equation:

$$r_{it}=\gamma_0+\gamma_1\beta_{it}+\varepsilon_{it}\ . \qquad (14)$$

Table 8. Number of significant years in tests of the zero-beta version

| Year | | No. of years | No. of significant years | | | | $p$-value |
|---|---|---|---|---|---|---|---|
| start | end | | $\gamma_{0[Y]}$ | $\gamma_{1[Y]}$ | $\gamma_{2[Y]}$ | combined | |
| Prior betas | | | | | | | |
| 1969 | 1972 | 4 | 0 | 1 | 0 | 1 | 0,460 |
| 1973 | 1985 | 13 | 0 | 1 | 1 | 2 | 0,572 |
| 1986 | 1989 | 4 | 0 | 0 | 0 | 0 | 1,000 |
| 1990 | 2002 | 13 | 0 | 0 | 1 | 1 | 0,865 |
| 2003 | 2010 | 8 | 1 | 0 | 3 | 4 | 0,018 |
| 1969 | 1985 | 17 | 0 | 2 | 1 | 3 | 0,445 |
| 1986 | 2010 | 25 | 1 | 0 | 4 | 5 | 0,280 |
| 1969 | 2010 | 42 | 1 | 2 | 5 | 8 | 0,243 |
| In-period betas | | | | | | | |
| 1965 | 1972 | 8 | 0 | 1 | 3 | 4 | 0,018 |
| 1973 | 1985 | 13 | 6 | 1 | 0 | 6 | 0,006 |
| 1986 | 1989 | 4 | 0 | 0 | 0 | 0 | 1,000 |
| 1990 | 2002 | 13 | 2 | 1 | 0 | 3 | 1,000 |
| 2003 | 2010 | 8 | 0 | 1 | 1 | 2 | 0,319 |
| 1965 | 1985 | 21 | 6 | 2 | 3 | 10 | 0,000 |
| 1986 | 2010 | 25 | 2 | 2 | 1 | 5 | 0,280 |
| 1965 | 2010 | 46 | 8 | 4 | 4 | 15 | 0,001 |

In the standard form of the CAPM, equation (14) implies that:

$$\gamma_0 = 0 \text{ ; and}$$

$$\gamma_1 = E\{r_{\mathrm{M}t}\}$$

5.2.1.2 Using the above test to investigate whether the CAPM explains excess rates of return, the excess return for each component $i$ was regressed, for each quarter $t \in Y$ where $t = 173,\ldots,185$ and $Y = 2007,\ldots,2010$, against the corresponding prior beta estimate for each component $i$ for each quarter $t \in Y$. The relationship examined is equation (14), expressed in terms of estimated prior betas as:

$$r_{it} = \gamma_{0[Y]} + \gamma_{1[Y]}\hat{\beta}_{it} + \varepsilon_{it}.$$

5.2.1.3 A further investigation involved a regression of the return on each component $i$ for $\forall t$ where $t = 173,\ldots,185$, against the corresponding prior beta estimate for each component $i$ for $\forall t$. The relationship examined is equation (14), expressed in terms of estimated betas as:

$$r_{it} = \gamma_0 + \gamma_1\hat{\beta}_{it} + \varepsilon_{it}.$$

5.2.1.4 The results of the analysis for period [153;185] are given in Table 9.

Table 9. Summary of regression analysis for the test of the explanatory power of the CAPM using prior betas

| Period | $R^2$ | Estimate | Value | $t$-value | $p$-value |
|---|---|---|---|---|---|
| [153;185] | $9{,}82\times10^{-5}$ | $\gamma_0$ | –0,001 | –0,056 | 0,955 |
| | | $\gamma_1$ | 0,002 | 0,093 | 0,537 |

5.2.1.5 First, as indicated in Table 9, the value of $R^2$ for the test for all quarters combined is low, indicating that most of the risk is unsystematic. The standard version of the CAPM predicts that $\gamma_0=0$. For this version of the CAPM the null hypothesis is that $\gamma_0=0$ and the alternative hypothesis is that $\gamma_0 \neq 0$. Again, the CAPM is rejected if the null hypothesis is rejected. Since the $p$-value for all quarters combined is greater than 2,5% (using a two-tailed test), the null hypothesis is not rejected.

5.2.1.6 The standard version of the CAPM also predicts that $\gamma_1 \geq 0$. The null hypothesis is that $\gamma_1 \geq 0$ and the alternative hypothesis is that $\gamma_1<0$. The CAPM is rejected if the null hypothesis is rejected. Again the value of $\gamma_1$ is very low. But again the null hypothesis is not rejected.

5.2.1.7 The standard version of the CAPM does not only predict that $\gamma_{1[Y]},\gamma_1 \geq 0$ but also that it equals the expected excess return on the market portfolio. For this reason, the differences between $\gamma_{1[Y]}$ and $r_{M[Y]}$ are compared. The null hypothesis is that $r_{M[Y]}-\gamma_{1[\mathrm{Y}]}=0$ and the alternative hypothesis is that $r_{M[Y]}-\gamma_{1[\mathrm{Y}]}\neq 0$. These differences have a mean of –0,014 and a standard deviation of 0,031. The $p$-value for this $t$-test for differences is 0,443, which, on the basis of a two-tailed test, is greater than 5%. Therefore, the null hypothesis is not rejected. This result does not change the conclusion that the standard version of the CAPM cannot be rejected using prior betas.

5.2.1.8 Of the four years considered, not one showed parameters outside of its 95% confidence limits. The CAPM cannot be rejected on the basis of these tests. However, with only four years' observations, the power of this test is low.

### 5.2.2 A Test for Nonlinearity using Prior Betas

5.2.2.1 In order to test for nonlinearity between return for each component $i$, for each quarter $t\in Y$ where $t=173,\ldots,185$ and $Y=2007,\ldots,2010$, and the corresponding prior beta estimate for each component $i$ for each quarter $t\in Y$. The relationship examined, for each calendar year $Y$, expressed in terms of estimated prior betas is:

$$r_{it}=\gamma_{0[Y]}+\gamma_{1[Y]}\hat{\beta}_{it}+\gamma_{2[Y]}\hat{\beta}_{it}^2+\varepsilon_{it}\,.$$

The estimate values, $t$-values and $p$-values are those for $\gamma_{2[Y]}$ only; other coefficients are not relevant to this test.

5.2.2.2 Furthermore, it was necessary to test for nonlinearity between each component $i$ for $\forall t$ where $t=173,\ldots,185$, and the corresponding prior beta estimate for each component $i$ for $\forall t$. The relationship examined expressed in terms of estimated betas as:

$$r_{it} = \gamma_0 + \gamma_1 \hat{\beta}_{it} + \gamma_2 \hat{\beta}_{it}^2 + \varepsilon_{it}.$$

5.2.2.3 The results of the analysis for period [153;185] are given in Table 10.

Table 10. The results of the regression for the test for nonlinearity using prior betas

| Period | $R^2$ | Estimate | Value | $t$-value | $p$-value |
|---|---|---|---|---|---|
| [153;185] | 0,003 | $\gamma_2$ | –0,029 | –0,498 | 0,620 |

5.2.2.4 For the relationship between the return and the beta estimate to be linear, one would expect that $\gamma_2 = 0$. This is the null hypothesis; the alternative hypothesis is that $\gamma_2 \neq 0$. As indicated in Table 10, the null hypothesis is not rejected.

5.2.2.5 Of the four years considered, one year showed $\gamma_{2[Y]}$ outside its 95% confidence limits. The probability that, in four years, one or more is outside its confidence limits is 18,5%, which is not significant. Again, though, it must be recognised that the power of the test is low.

5.2.3 A Test of the Explanatory Power of the CAPM using In-period Betas

5.2.3.1 A similar test to that used in section 5.1.3 was used to investigate the explanatory power of the CAPM. In this case, the in-period return for each component $i$ was regressed, for each calendar year $Y$ where $Y = 2003,\ldots,2010$, against the corresponding in-period beta estimate for each component $i$ for each calendar year $Y$. The relationship examined is equation (14), expressed in terms of estimated in-period betas as:

$$r_{i[Y]}^{\mathrm{Y}} = \gamma_{0[Y]}^{\mathrm{Y}} + \gamma_{1[Y]}^{\mathrm{Y}} \widehat{\beta}_{i[Y]}^{\mathrm{Y}} + \varepsilon_{i[Y]}^{\mathrm{Y}}.$$

5.2.3.2 A further investigation involved a regression of the in-period return on each component $i$ for the entire period from $t = 153,\ldots,185$, against the corresponding in-period beta estimate for each component $i$. The relationship examined is equation (14), expressed in terms of estimated in-period betas as:

$$r_i = \gamma_0 + \gamma_1 \widehat{\beta}_i + \varepsilon_i.$$

5.2.3.3 The results of the analysis for period [153;185] are given in Table 11.

Table 11. Summary of regression analysis for the test of the explanatory power of the CAPM using in-period betas

| Period | $R^2$ | Estimate | Value | $t$-value | $p$-value |
|---|---|---|---|---|---|
| [153;185] | 0,77 | $\gamma_0$ | 0,007 | 4,207 | 0,008 |
| | | $\gamma_1$ | 0,015 | 4,134 | 0,995 |

5.2.3.4 The value of $R^2$ is low indicating that most of the risk is unsystematic. The standard version of the CAPM predicts that $\gamma_0=0$. For this version of the CAPM the null hypothesis is that $\gamma_0=0$ and the alternative hypothesis is that $\gamma_0 \neq 0$. Again, the CAPM is rejected if the null hypothesis is rejected. Since the $p$-value for all quarters combined is less than 2.5% (using a two-tailed test), the null hypothesis is rejected.

5.2.3.5 The standard version of the CAPM also predicts that $\gamma_1 \geq 0$. The null hypothesis is that $\gamma_1 \geq 0$ and the alternative hypothesis is that $\gamma_1 < 0$. The CAPM is rejected if the null hypothesis is rejected. The null hypothesis is not rejected.

5.2.3.6 The parameters were outside of their 95% confidence limits in three of the eight years considered. The probability that at least one of the two parameters is outside of its 95% confidence limits at least three years is 3,6%, which is not significant. However, this test does not recognise the high significance of the rejection of the test of $\gamma_0$ over the period [153;185]. On that basis the standard version of the CAPM using in-period betas must be rejected. For this reason no test for nonlinearity is made for the nonlinearity of the zero-beta version of the CAPM using in-period betas.

5.2.3.7 Because of the limited period over which the standard version of the CAPM could be tested, no further analysis could be made.

## 5.3 HOTELLING'S TEST OF THE SECURITIES MARKET LINE USING IN-PERIOD BETAS

5.3.1 Let us suppose that both *ex-ante* assumptions at the start of a period and estimates based on *ex-post* observations during that period are unbiased estimates of the underlying values during that period, and therefore that the latter are unbiased estimates of the former. Whilst this does not imply perfect foresight, it does imply greater correspondence than might reasonably be expected. Nevertheless, as explained in the previous study, for the purpose of testing whether the CAPM can be used in long-term models it is relevant, since such a model can generate unbiased *ex-ante* betas.

5.3.2 The method used for the test described in this section follows Shanken (1985). Because the covariances and betas are estimated in-period, the quadratic form does not follow a multivariate $\chi^2$ distribution. Instead, with Shanken's (op. cit.) notation, assuming that the distribution of returns is multivariate normal, the regression statistic

$$Q_C = T\boldsymbol{e}'\hat{\boldsymbol{\Sigma}}^{-1}\boldsymbol{e} \tag{15}$$

follows a Hotelling's $T^2$ distribution with $N-2$ and $T-2$ degrees of freedom, where:

$N$ is the number of asset classes and $n=N-2$

$T$ is the length of the time series and $m=T-2$

$\boldsymbol{e} = \hat{\boldsymbol{\mu}} - \hat{\boldsymbol{X}}'\hat{\boldsymbol{\Gamma}}_C$;

$\hat{\boldsymbol{\mu}} = \begin{pmatrix} \hat{\mu}_1 \\ \vdots \\ \hat{\mu}_N \end{pmatrix}$;

$$\hat{\boldsymbol{X}} = \begin{pmatrix} 1 & \hat{\beta}_1 \\ 1 & \vdots \\ 1 & \hat{\beta}_N \end{pmatrix};$$

$$\hat{\mu}_i = R_{i[p]} \text{ ; and}$$

$$\hat{\beta}_i = \widehat{\beta}_{i[p]}.$$

For in-period betas, this test is a more powerful test than the standard regression tests used in sections 5.1 and 5.2.

5.3.3 For the zero-beta version of the CAPM this test was performed on each period $p$ for which $m>n$. It was not possible to apply this test to the period [153;185] either for the zero-beta version or for the standard CAPM, as the number of components in the market portfolio was too great in comparison with the length of the time series. It was also not possible to apply the test to the period [34;85] as the covariance matrix was virtually singular.

5.3.4 For each period the test was constructed as follows. The linear regression parameters are defined as:

$$\boldsymbol{\Gamma}_C = \begin{pmatrix} \hat{\gamma}_0 \\ \hat{\gamma}_1 \end{pmatrix} = \left( \hat{\boldsymbol{X}}' \hat{\boldsymbol{\Sigma}}^{-1} \hat{\boldsymbol{X}} \right)^{-1} \hat{\boldsymbol{X}}' \hat{\boldsymbol{\Sigma}}^{-1} \hat{\boldsymbol{\mu}};$$

where:

$$\hat{\boldsymbol{\Sigma}} = \begin{pmatrix} \hat{\sigma}_{11} & \cdots & \hat{\sigma}_{1N} \\ \vdots & \ddots & \vdots \\ \hat{\sigma}_{N1} & \cdots & \hat{\sigma}_{NN} \end{pmatrix};$$

$$\hat{\sigma}_{ij} = \frac{1}{n_p - 2} \sum_{t \in p} \left( R_{it} - \overline{R}_i \right) \left( R_{jt} - \overline{R}_j \right); \text{ and}$$

$$\overline{R}_u = \frac{1}{n_p} \sum_{t \in p} R_{ut}; u = i, j.$$

The value of $Q_C$ may now be found from equation (15). Following Shanken (op. cit.) this is then adjusted as follows:

$$Q_C^A = \frac{Q_C}{\left( 1 + \frac{\hat{\gamma}_1^2}{\hat{\sigma}_{\mathrm{MM}}} \right)}; \text{ and}$$

$$Q^* = \frac{T}{T-2} Q_C^A.$$

Then the likelihood ratio test statistic defined in Shanken (op. cit.) is:

$$T\ln\left(1+\frac{Q^*}{T}\right);$$

which means that the test statistic

$$F=\frac{T(m-n+1)}{mn}\ln\left(1+\frac{Q^*}{T}\right)$$

is distributed $F_{n,m-n+1}$.

5.3.5 The results of these tests are summarised in Table 12. It may be seen from that table that none of the tests shows significant results at the 5% level. On the basis of these tests the zero-beta version of the CAPM cannot be rejected for in-period betas. However, this does not override the finding in section 5.1.3.

Table 12. Summary of results for Hotelling's test

| Period | $Q^*$ | $F$ | $p$-value |
|---|---|---|---|
| Excluding inflation-linked bonds | | | |
| [1;33] | 0,957 | 0,456 | 0,638 |
| [34;85] | 3,840 | 1,815 | 0,174 |
| [86;100] | 0,652 | 0,295 | 0,750 |
| [101;152] | 1,613 | 0,778 | 0,465 |
| [153;185] | 0,200 | 0,096 | 0,908 |
| All | 0,154 | 0,076 | 0,074 |
| Including inflation-linked bonds | | | |
| [153;185] | 0,306 | 0,147 | 0,864 |

## 6. SUMMARY AND CONCLUSION

6.1 In section 5.1 it was shown using prior betas that, except for the period [34;85] (i.e. from 1973 to 1985) the zero-beta version of the CAPM cannot be rejected. This was confirmed by further analysis of year-by-year results. In that analysis the period from 2003 to 2010 was significant, but this did not affect the acceptance of the zero-beta version of the CAPM using prior betas for the period from 1986 to 2010.

6.2 It was shown that, using in-period betas, the zero-beta version of the CAPM must be rejected for all periods combined. Again, the period [34;85] was significant. Furthermore, it appeared that, for some periods, $\gamma_1<\gamma_0$, which is contrary to the prediction of the CAPM. Further analysis of year-by-year results for in-period betas indicated that, whilst the zero-beta version of the CAPM must be rejected for periods up to 1985, it may be accepted for later periods. Tests for nonlinearity suggested that the period for which the CAPM could be accepted should be reduced to the period [101;185] (i.e. since 1989).

6.3 In section 5.2 it was shown using prior betas that, whilst the explanatory power of the CAPM was low, the standard version of the CAPM could not be rejected. Tests of its linearity could also not be rejected. However, the power of these tests was low.

6.4 Using in-period betas it was found that the standard CAPM must be rejected for the period [153;185] on the grounds of tests for the parameter $\gamma_0$ for that period as a whole.

6.5 Using in-period betas, Hotelling's test for the residuals of the zero-beta version of the CAPM was applied to each period and to all periods combined. On the basis of these tests the CAPM could not be rejected. However, this does not override the results summarised in ¶6.2.

6.6 In summary, using in-period betas, the zero-beta version of the CAPM may be accepted for the period since 1989. On the basis of the tests using prior betas, the zero-beta version of the CAPM may be accepted for the period since 1985. However, it would be inconsistent to accept the zero-beta version of the CAPM using prior betas during the period from 1986 to 1989 and to reject it using in-period betas during that period. This would suggest that the prior betas are a better estimate of the *ex-ante* betas than the in-period betas. The best we can say for the zero-beta version of the CAPM is that it may be accepted for the period since 1989 using either prior or in-period betas.

6.7 Whilst the standard version of the CAPM was accepted for prior betas, it was rejected for in-period betas. Once again, it would be inconsistent to accept the standard version of the CAPM using prior betas and to reject it using in-period betas. The standard version of the CAPM must therefore be rejected.

6.8 All the tests made in this paper presuppose the REH: it is implicitly assumed that *ex-ante* expectations are unbiased and that they can be represented either by in-period data or by prior data. Rejections of the tests may be rejections of the REH rather than rejections of the CAPM. Tests of pricing models against true *ex-ante* expectations would require continual monitoring of such expectations, and of prices, over time. On the other hand, where the null hypotheses are accepted, neither the REH nor the null hypothesis can be rejected.

6.9 The South African market is small in comparison with the major markets of the world. It would be of interest to apply tests similar to those in this paper to larger markets. It would also be of interest to apply them to a multi-currency CAPM. The exploration of more complete market proxies and of other relaxations of the assumptions of the CAPM is also matter for further research, as is the assumption that the pricing of inflation-linked bonds correctly incorporates information regarding consumer prices before publication of the index.

6.10 In summary, this paper shows that, assuming normal distributions and the REH, the standard version of the CAPM must be rejected For real quarterly returns on a South African market portfolio comprising equities and bonds, the zero-beta version may be accepted for the period since 1989 using either prior betas or in-period betas. If it can be assumed that later years represent the status quo, it would be reasonable to use the zero-beta version of the CAPM for the stochastic modelling of real returns on investments.

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REFERENCES

Artikis, PG (2001). Risk analysis of bond mutual funds: a case study in the Greek financial market, *Managerial Finance* **27**(6), 42–59

Blume, ME &Husic, F (1973). Price, beta and exchange listing. *Journal of Finance* **28**(2), 283–99

Bowie, DC & Bradfield, DJ (1998). Robust estimation of beta coefficients: evidence from a small stock market. *Journal of Business Finance and Accounting* **25**(3), 439–54

Breeden, D (1979). An intertemporal asset pricing model with stochastic consumption and investment. *Journal of Financial Economics* **7**, 265–96

Duffie, D (2001). *Dynamic Asset Pricing Theory*, 3rd edn. Princeton University Press, Princeton

Fama, EF & French, KR (1992). The cross-section of expected stock returns. *The Journal of Finance* **47**(2), 427–65

Frankel, J & Engel, CM (1984). Do asset-demand functions optimize over the mean and variance of real returns? A six-currency test. *Journal of International Economics* **17**, 309–23

Friend, I & Westerfield, R (1980). Co-skewness and capital asset pricing. *Jounal of Finance* **35**(4), 897–913

Friend, I, Westerfield, R & Granito, M (1978). New evidence on the capital asset pricing model. *Journal of Finance* **33**(3), 903–17

Gudikunst, A & McCarthy, J (1992). Determinants of bond mutual fund performance. *Journal of Fixed Income* **2**(1), 95–101

Jarrow, RA (1978). The relationship between yield, risk, and return on corporate bonds. *Journal of Finance* **33**(4), 1235–40

Korn, O & Koziol, C (unpublished). Bond portfolio optimization: A risk-return approach. CFR Working Paper 06-03, Centre for Financial Research (CFR), Universität Köln, 2006

Kouri, P & de Macedo, J (1978). Exchange rates and the international adjustment process. Brookings Papers on Economic Activity **1**, 111–50. Cited in Frankel & Engel (1984)

Lintner, J (1965). Valuation of risk assets and the selection of risky investments in stock portfolios and capital budgets. *Review of Economics and Statistics* **47**, 13–37

Maitland, AJ (2001). An empirical approach to immunization in South Africa. *South African Actuarial Journal* **1**, 119–38

Maitland, AJ (2002). Interpolating the South African yield curve using principal-components analysis: a descriptive approach. *South African Actuarial Journal* **2**, 129–45

Mankiw, NG & Shapiro, MD (unpublished). Risk and return: Consumption versus market beta. Working paper no. 1399, National Bureau of Economic Research, Cambridge, Massachusetts

Mossin, J (1966). Equilibrium in a capital asset market. *Econometrica* **34**(4), 768–83

Reddy, TLR & Thomson, RJ (2011). The capital-asset pricing model: the case of South Africa. *South African Actuarial Journal* **11**, 43–84

Reilly, FK. & Joehnk, MD (1976). The association between market-determined risk measures for bonds and bond ratings. *Journal of Finance* **31**(5), 1387–403

Roll, R (1977). A critique of the asset pricing theory's tests part I: on past and potential testability of the theory. *Journal of Financial Economics* **4**(4), 129–76

Ross, SA (1978). The current status of the capital asset pricing model. *Journal of Finance* **33**(3), 885–901

Samuelson, PA (1969). Lifetime portfolio selection by dynamic stochastic programming. *Review of Economics and Statistics* **51**, 239–46

Shanken, J (1985). Multivariate tests of the zero-beta CAPM. *Journal of Financial Economics* **14**, 327–48

Sharpe, WF (1964). Capital asset prices: A theory of market equilibrium under conditions of risk. *Journal of Finance* **19**(3), 425–42

Thomson, RJ (1996). Stochastic investment models: the case of South Africa, *British Actuarial Journal* **2**(3), 765–801

Thomson, RJ (2005). The pricing of liabilities in an incomplete market using dynamic mean–variance hedging, *Insurance: Mathematics and Economics* **36**, 441–55

Thomson. RJ & Gott, DV (2009). Stochastic models for actuarial use: the equilibrium modelling of local markets. *ASTIN Bulletin* **39**(1), 339–70

Thomson, RJ (2011). The arbitrage-free equilibrium pricing of liabilities in an incomplete market: application to a South African retirement fund. *South African Actuarial Journal*, **11**, 1–41

Viceira, LM (2012). Bond risk, bond return volatility, and the term structure of interest rates. *International Journal of Forecasting* **28**(1), 97–117

Yawitz, JB & Marshall, WJ (1977). Risk and return in the government bond market. *Journal of Portfolio Management* **3**(4), 48–52

# APPENDIX A

## CALCULATION OF RETURNS AND THE EFFECTIVE MARKET PORTFOLIO

In this appendix the calculation of the quarterly returns on equities and bonds is explained, as well as the determination of the effective market portfolio at quarterly intervals, during the period from 30/9/1964 to quarter 185 at 31/12/2010. The meaning of 'effective' becomes clear below. We let $t$ denote the quarter-end, or if the context so requires, the quarter, from 0 at 30/9/1964 to 185 at 31/12/2010.

### A.1 INFLATION

For $t=101,\ldots,185$ the data for inflation comprised values of the consumer price index.[1] For earlier quarters these values comprised only two significant digits, and over some periods these did not change for up to three quarters. To avoid the resulting errors, earlier consumer price indices[2] were used, after rebasing them to correspond to the current series. The force of inflation during quarter $t$ was calculated as:

$$i_t = \ln\left(\frac{p_t}{p_{t-1}}\right);$$

where $p_t$ is the consumer price index, recalculated as explained above, at time $t$.

### A.2 EQUITIES

#### A.2.1 Data

The data for equities comprised the following:

– quarterly from 30/9/1964 to 31/12/2001: the JSE-Actuaries all-share price index and the dividend yield on that index;[3]
– quarterly from 31/3/1970 to 31/12/1981[4] and from 31/3/1989 to 31/12/2001[5]: the total market capitalisation of equities included in the JSE-Actuaries all-share price index; and
– quarterly from 31/12/2001 to 31/12/2010: the FTSE-JSE all-share price index, the dividend yield on that index, the corresponding total-return index and the market capitalisation of equities included in that index.[6]

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1 Source: INet Bridge; code ECPI
2 Source: South African Reserve Bank. Quarterly reports
3 Source: INet Bridge; codes CI01[CL] and CI01[DY]
4 Source: Actuarial Society of South Africa & Johannesburg Stock Exchange (1982). *The JSE Actuaries Index*. Old Mutual, Cape Town
5 Source: INet Bridge, code CI01[MC]
6 Source: INet Bridge; code J203[CL], J203[DY] and J203[MC]

### A.2.2 Returns

Let $S_t$, $D_t$ and $T_t$ denote the all-share price index, the dividend yield on that index and the corresponding total-return index respectively. We identify the JSE–Actuaries series and the FTSE–JSE series by means of the superscripts A and F respectively. For the period from 1/1/2002 ($t=150,\ldots,185$) returns on equities were calculated as:

$$R_{\mathrm{E}t} = \ln\left(\frac{T_t^{\mathrm{F}}}{T_{t-1}^{\mathrm{F}}}\right) - i_t \,.$$

For earlier periods it was necessary to calculate returns from price indices and dividend yields. For this purpose it was assumed that:

$$S_{t-1}^{\mathrm{A}}\left(1+h_t\right) = S_t^{\mathrm{A}} + d_t\left(1+\tfrac{1}{2}h_t\right);$$

where:

$$h_t = e^{R_{\mathrm{E}t}+i_t} - 1;$$

and $d_t$ is the amount of dividends paid on the all-share index. Since the dividend yield $D_t$ is annually retrospective, we assume the formula:

$$d_t = \frac{1}{16}\left(S_t^{\mathrm{A}}D_t^{\mathrm{A}} + S_{t+1}^{\mathrm{A}}D_{t+1}^{\mathrm{A}} + S_{t+2}^{\mathrm{A}}D_{t+2}^{\mathrm{A}} + S_{t+3}^{\mathrm{A}}D_{t+3}^{\mathrm{A}}\right);$$

where each of the terms in parentheses reflects one-quarter of the annual dividends that became payable in quarter $t$, together with noise from various other quarters reduced by averaging.

The formula used was therefore:

$$R_{\mathrm{E}t} = \ln\left(\frac{S_t^{\mathrm{A}} + \frac{1}{2}D_t^{\mathrm{A}}}{S_{t-1}^{\mathrm{A}} - \frac{1}{2}D_t^{\mathrm{A}}}\right) - i_t \,.$$

### A.2.3 Market Capitalisation

The market capitalisation of equities at time $t$ is $m_{\mathrm{E}t}^{*}$. For the periods for which values of the market capitalisation of the all-share index were not available, these values were estimated as follows. The ratio of market capitalisation to the all-share index was calculated for the period for which market-capitalisation values were available and these ratios were extrapolated or interpolated for the remaining periods. The resulting ratios were then applied to the all-share index values for those periods.

## A.3 CONVENTIONAL BONDS

### A.3.1 Data

The data for conventional bonds, on which coupons are payable half-yearly, comprised the following:

– quarterly from 30/9/1964 to 31/12/1964: the yield to redemption on government bonds for terms to redemption of 3 years and 20 years;

- quarterly from 31/3/1965 to 31/12/1985: the yield to redemption on government bonds for terms to redemption of 3 years, 10 years and 20 years;
- quarterly from 31/3/1985 to 31/12/2010: the yield to redemption on government bonds for terms to redemption of 1 year to 25 years at annual maturity intervals;
- on 30/6/1986 and 30/9/1986: the total loan debt of national government;
- quarterly from 30/9/1964 to 30/9/2010: the loan debt of national government by term to redemption for maturity intervals not exceeding 1 year, exceeding 1 but not 3 years, exceeding 3 but not 10 years and exceeding 10 years;
- quarterly or annually over various ranges from 31/3/1980 to 30/9/1990: the total nominal value of domestic marketable bonds issued by public-sector bodies other than national government; and
- quarterly from 30/9/1990 to 30/9/2010: the total nominal value of domestic marketable bonds issued by the public sector.

Yields to redemption are annual yields convertible half-yearly.[7] Loan debt is the nominal amount in issue.[8] From the data available for the period from 31/3/1980 to 30/9/1990 it was possible to estimate the total nominal value of domestic marketable bonds issued by the public sector.

### A.3.2 Returns

A.3.2.1 Let $y_{t,q}$ denote the yield to redemption at time $t$ for $q$ quarters to redemption. First we need to interpolate these yields between the maturity intervals available to give yields to redemption for half-yearly intervals. For $t=86,\ldots,185$ we have $y_{t,4}, y_{t,8}, \ldots y_{t,100}$ from which to interpolate. For this purpose the third-order difference formula:

$$y_{t,q_0+d} = y_{t,q_0} + \left(\frac{d}{4}\right)\Delta y_{t,q_0} + \frac{1}{2}\left(\frac{d}{4}\right)\left(\frac{d}{4}-1\right)\Delta^2 y_{t,q_0} + \frac{1}{6}\left(\frac{d}{4}\right)\left(\frac{d}{4}-1\right)\left(\frac{d}{4}-2\right)\Delta^3 y_{t,q_0}$$

was used. This formula uses the range of annual points $q_0, q_0+4, q_0+8, q_0+12$. Where possible, for the purpose of calculating $y_{t,q}$, the range of differencing was selected so as to straddle $q$; i.e. $q_0=q-6$, so that $d=6$. At the extremes of maturity, $d$ was defined so that the available data were used.

A.3.2.2 For $t=2,\ldots,85$ (i.e. from 31/3/1965 to 31/12/1985) we have just three values from which to interpolate: $y_{t,12}$, $y_{t,40}$ and $y_{t,80}$. We assume for the purpose of interpolation that the yield curve follows:

$$y_{t,q} = a_t + b_t \exp\left\{-\left(\frac{q-c_t}{100}\right)^2\right\}; \qquad \text{(A.1)}$$

where $a_t>0$ and $a_t+b_t>0$. Unlike most descriptive yield-curve formulae—apart from the Ayres–Barry yield-curve formula—this formula comprises only three parameters.

7 Source: INet Bridge; codes JAYC01 to JAYC20

8 Source: South African Reserve Bank, www.sarb.gov.za; codes KBP 4086, 4140–3 and 4564

Unlike the Ayres–Barry yield-curve formula, however, this formula allows for humped yield curves if $c_t > 0$.

A.3.2.3 Equation (A.1) may be solved by means of Newton's formula. Suppressing the subscripts $t$ and $q$, and instead denoting the $j$th values of $q$ and $y_{t,q}$ as $q_j$ and $y_j$ respectively for $j=1,2,3$, we obtain:

$$c = \lim_{n\to\infty} c_n\,;$$

$$b = \lim_{n\to\infty} b_n \text{ ; and}$$

$$a = \lim_{n\to\infty} a_n\,;$$

where:

$$c_{tn} = c_{n-1} - \frac{w_{n-1}}{w'_{n-1}}\,;$$

$$b_n = \frac{y_1 - y_3}{r_{1n} - r_{3n}}\,;$$

$$a_n = y_1 - br_{1n}\,;$$

$$w_n = (y_1 - y_3)(r_{3n} - r_{2n}) - (y_3 - y_2)(r_{1n} - r_{3n})\,;$$

$$w'_n = \frac{2}{\lambda^2}\left\{(y_1 - y_3)(s_{3n} - s_{2n}) - (y_3 - y_2)(s_{1n} - s_{3n})\right\};$$

$$r_{jn} = \exp\left\{-\left(\frac{q_j - c_n}{\lambda}\right)^2\right\} \text{ ; and}$$

$$s_{jn} = (q_j - c_n)\,r_{jn}\,.$$

For the purposes of this paper, it was assumed that:

$$\lambda = 100.$$

A.3.2.4 A problem arises where $y_{t,40} = y_{t,80}$. This is clearly intended to be an approximation due to scarcity of data. An exact fit would generally place a hump between $y_{t,40}$ and $y_{t,80}$, which would not reflect the intention. It is therefore appropriate to assume that $c_t = c_{t+1}$ and hence find $a_t$ and $b_t$ from equations (A.2) and (A.3) respectively. For $t=0,1$ (i.e. for 30/9/1964 and 31/12/1964) we have just two values from which to interpolate: $y_{t,12}$ and $y_{t,80}$. At these dates we again assume that $c_t = c_{t+1}$ and hence find $a_t$ and $b_t$ from equations (A.2) and (A.3).

A.3.2.5 We now have $y_{t,q}$ for $t = 0,\ldots,185,\ q = 2,4,\ldots,100$. Next we need to calculate continuously compounded quarterly spot yields $z_{t,q}$ for the same ranges of $t$ and

$q$. For this purpose we assume, for the sake of simplicity, that the yields to redemption $y_{t,q}$ are for bonds at par, so that:

$$\tfrac{1}{2}y_{t,q}e^{-2z_{t,2}}+\tfrac{1}{2}y_{t,q}e^{-4z_{t,4}}+\ldots+\left(1+\tfrac{1}{2}y_{t,q}\right)e^{-qz_{t,4q}}=1;$$

giving:

$$z_{t,q}=\begin{cases}\tfrac{1}{2}\ln\left(1+\tfrac{1}{2}y_{t,2}\right) \text{ for } q=2;\\[2ex] \dfrac{1}{q}\ln\left(\dfrac{1+\tfrac{1}{2}y_{t,q}}{1-\tfrac{1}{2}y_{t,q}\sum_{n=1}^{\frac{q}{2}-1}\exp\left(-2nz_{t,2n}\right)}\right) \text{ for } q=4,6,\ldots,100.\end{cases}$$

A.3.2.6 Now, for each $t$, we need to interpolate between $z_{t,2},z_{t,4},\ldots,z_{t,100}$ to find $z_{t,1},z_{t,3},\ldots,z_{t,99}$. Here we use the interpolation formula:

$$z_{t,q_0+d}=z_{t,q_0}+\left(\frac{d}{2}\right)\Delta z_{t,q_0}+\frac{1}{2}\left(\frac{d}{2}\right)\left(\frac{d}{2}-1\right)\Delta^2 z_{t,q_0}+\frac{1}{6}\left(\frac{d}{2}\right)\left(\frac{d}{2}-1\right)\left(\frac{d}{2}-2\right)\Delta^3 z_{t,q_0}.$$

This gives us $z_{t,q}$ for $t=0,\ldots,185, q=1,2,\ldots,100$.

A.3.2.7 As shown by Maitland (2001), the first three principal components of the JSE–Actuaries yield curve over the period from February 1986 to May 2000 were sufficient to explain 98,45% of its variability. For the purposes of this paper it was therefore assumed that the conventional bonds market consisted in a portfolio comprising three zero-coupon bonds. The terms chosen for this purpose were 20, 40 and 80 quarters. For the purpose of modelling risk-free short-term interest rates, a one-quarter bond was also modelled, but this was not included in the market capitalisation.

A.3.2.8 Finally, then, we calculate the real return on conventional bonds with terms to redemption of ¼, 5, 10 and 20 years as:

$$R_{\mathrm{CB}t(q)}=qz_{t-1,q}-(q-1)z_{t,q-1}-i_t \text{ for } t=1,\ldots,185,\ q=1,20,40,80.$$

A.3.3 Effective Market Capitalisation of Zero-Coupon Bonds by Term to Redemption

A.3.3.1 Let $m_t$ denote the total nominal value of conventional domestic marketable bonds issued by the public sector at time $t$. This is obtained by deducting from the total nominal value of domestic marketable bonds issued by the public sector the total nominal value of inflation-linked domestic marketable bonds issued as specified in section A.4.1 below. Then we estimate the total nominal value of conventional domestic marketable bonds issued by the public sector at time $t$ for maturity group $g$ as:

$$m_{tg}=\frac{m_{tg}^{\mathrm{N}}}{\sum_{g=1}^{4}m_{tg}^{\mathrm{N}}}m_t\,;$$

where $m_{tg}^{\mathrm{N}}$ is the loan debt of national government for maturity group $g$; and:

$$g=\begin{cases}1 \text{ for maturity intervals not exceeding 4 quarters;}\\ 2 \text{ for maturity intervals exceeding 4 but not 12 quarters;}\\ 3 \text{ for maturity intervals exceeding 12 but not 40 quarters;}\\ 4 \text{ for maturity intervals exceeding 40 quarters.}\end{cases}$$

A.3.3.2 We now assume that, in each group, the bonds are at par with terms to redemption uniformly distributed across its maturity interval. We also assume that the upper limit of group 4 is 100 quarters. At time $t$ the amounts payable during each subsequent maturity interval may then be estimated as:

$$\pi_{tg}=\begin{cases}\left(1+0{,}5y_{t,2}\right)m_{t1}+\left(y_{t,8}m_{t2}+y_{t,26}m_{t3}+y_{t,70}m_{t4}\right) \text{ for } g=1;\\ \left(1+y_{t,8}\right)m_{t2}+2\left(y_{t,26}m_{t3}+y_{t,70}m_{t4}\right) \text{ for } g=2;\\ \left(1+3{,}5y_{t,26}\right)m_{t3}+7\left(y_{t,70}m_{t4}\right) \text{ for } g=3;\\ \left(1+7{,}5y_{t,70}m_{t4}\right) \text{ for } g=4.\end{cases}$$

A.3.3.3 We now need the market portfolio of conventional bonds of the selected maturities. It would be possible to calculate this using Maitland (2002). However, it was decided to adopt a more heuristic approach, particularly since Maitland's method relates to yields to redemption rather than spot yields. Let $k_{tj}$ denote the market value at maturity $j$.

A.3.3.4 We denote the price of a zero-coupon bond with term to redemption $q$ at time $t$ as:

$$P(t,q)=\exp\left(-qz_{tq}\right).$$

Let:

$$\kappa_{\mathrm{CB}t}^{(p)}=\sum_{g=1}^{4}\frac{\pi_{tg}}{f_g-f_{g-1}}\left\{\begin{array}{l}\tfrac{1}{2}f_{g-1}^{p}P\left(t,f_{g-1}\right)+\left(f_{g-1}+1\right)^{p}P\left(t,f_{g-1}+1\right)+\ldots\\ +\left(f_g-1\right)^{p}P\left(t,f_g-1\right)+\tfrac{1}{2}f_g^{p}P\left(t,f_g\right)\end{array}\right\}$$

for $p=0,1,2$;

where $f_0=0$; $f_1=4$; $f_2=12$; $f_3=40$ and $f_4=100$; so that, in particular, $\kappa_{\mathrm{CB}t}^{(0)}$ is the market capitalisation of bonds at time $t$. Let $m_{tj}^{*}$ be the exposure of the market at time $t$ to a zero-coupon bond with term to redemption $q_j$ such that:

$$\boldsymbol{q} \equiv \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} = \begin{pmatrix} 5 \\ 10 \\ 20 \end{pmatrix}.$$

In other words,

$$\boldsymbol{m}^*_{\mathrm{CB}t} \equiv \begin{pmatrix} m^*_{1t} \\ m^*_{2t} \\ m^*_{3t} \end{pmatrix}$$

gives the effective market capitalisations at that time of zero-coupon bonds of the selected terms to redemption.

A.3.3.5 Then:

$$\boldsymbol{Q}\boldsymbol{m}^*_{CBt} = \boldsymbol{\kappa}_{\mathrm{CB}t};$$

where:

$$Q \equiv \begin{pmatrix} 1 & 1 & 1 \\ q_1 & q_2 & q_3 \\ q_1^2 & q_2^2 & q_3^2 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 12 & 40 & 80 \\ 144 & 1600 & 6400 \end{pmatrix}; \text{ and}$$

$$\boldsymbol{\kappa}_{\mathrm{CB}t} = \begin{pmatrix} \kappa^{(0)}_{\mathrm{CB}t} \\ \kappa^{(1)}_{\mathrm{CB}t} \\ \kappa^{(2)}_{\mathrm{CB}t} \end{pmatrix}.$$

This gives:

$$\boldsymbol{m}^*_{\mathrm{CB}t} = \boldsymbol{Q}^{-1}\boldsymbol{\kappa}_{\mathrm{CB}t}.$$

## A.4 INFLATION-LINKED BONDS

### A.4.1 DATA

The data for inflation-linked bonds comprised:

- for each bond issued: the date of issue, the amount originally issued, the coupon and the redemption date; and
- at quarterly intervals from 30/6/2000 (when the first bond was issued), for each bond issued, the yield to redemption;
- at quarterly intervals from 30/6/2009 for each bond issued, the cumulative amount issued and the total market capitalisation.

### A.4.2 RETURNS

A.4.2.1 The price of a zero-coupon bond, per unit of the amount issued, inflated to the date of calculation, is taken to be:

$$P = \frac{1}{2}c\left\{\frac{1-\left(1+\frac{y}{2}\right)^{-[q/2]}}{\frac{y}{2}}\right\}\left(1+\frac{y}{2}\right)^{-q/2+[q/2]} + \left(1+\frac{y}{2}\right)^{-q/2};$$

where:

$q$ is the term to redemption in quarters;
$c$ is the half-yearly coupon;
$y$ is the annual yield to redemption, convertible half-yearly; and
$[x]$ is the integral portion of $x$.

A.4.2.2 This gives:

$$\begin{aligned} P &= \frac{c}{y}\left\{\left(1+\frac{y}{2}\right)^{-q/2+[q/2]} - \left(1+\frac{y}{2}\right)^{-q/2}\right\} + \left(1+\frac{y}{2}\right)^{-q/2} \\ &= g^{-q/2}\left\{1-\frac{c}{y}\left(1-g^{h}\right)\right\}; \end{aligned} \tag{A.4}$$

where:

$$g = \left(1+\frac{y}{2}\right); \text{ and} \tag{A.5}$$

$$h = [q/2].$$

A.4.2.3 The duration of a zero-coupon bond in quarters is defined as:

$$D = -\frac{1}{P}\frac{\partial P}{\partial z}; \tag{A.6}$$

where $z$ is the quarterly continuously compounded yield, such that:

$$z = \frac{1}{2}\ln g\ .$$

Now:

$$\begin{aligned} \frac{\partial z}{\partial y} &= \frac{\partial z}{\partial g}\frac{\partial g}{\partial y} \\ &= \frac{1}{2g}\frac{\partial g}{\partial y}; \end{aligned}$$

and from equation (A.5):

$$\frac{\partial g}{\partial y} = \frac{1}{2}. \tag{A.7}$$

Thus:

$$\frac{\partial z}{\partial y} = \frac{1}{4g}\ . \tag{A.8}$$

It follows from equation (A.6) that:

$$\begin{aligned} D &= -\frac{1}{P}\frac{\partial P}{\partial y}\frac{\partial y}{\partial z} \\ &= -\frac{4g}{P}\frac{\partial P}{\partial y}. \end{aligned}$$

Also, from equation (A.4):

$$\frac{\partial P}{\partial y}=-\frac{q}{2}g^{-q/2-1}\frac{\partial g}{\partial y}\left\{1-\frac{c}{y}\left(1-g^{h}\right)\right\}+g^{-q/2}\left\{\frac{c}{y^{2}}\left(1-g^{h}\right)-\frac{c}{y}\left(-hg^{h-1}\frac{\partial g}{\partial y}\right)\right\};$$

and hence from equation (A.7):

$$\frac{\partial P}{\partial y}=-\frac{Pq}{4g}+\frac{cg^{-q/2}}{y}\left\{\frac{1}{y}\left(1-g^{h}\right)+\frac{1}{2}\left(hg^{h-1}\right)\right\}.$$

Thus:

$$D=q-\frac{2cg^{-q/2}}{Py}\left\{\frac{2g}{y}\left(1-g^{h}\right)+hg^{h}\right\}. \tag{A.9}$$

A.4.2.4 For a zero-coupon bond we have $c=0$, so that the yield to redemption for a term to redemption of $D$ quarters may be taken as that of a coupon-paying bond with $q$ quarters to redemption. In other words, we may plot the yield to redemption as a function of $D$ and read the spot yield off that curve. In order to obtain spot yields at integral values of $q$ we may use divided differences to give:

$$\begin{aligned}z_q&=z_a+(D-a)\Delta_{a,b}+(D-a)(D-b)\Delta^2_{a,b,c}+(D-a)(D-b)(D-c)\Delta^3_{a,b,c,d}\\&=z_a+(D-a)\left[\Delta_{a,b}+(D-b)\left\{\Delta^2_{a,b,c}+(D-c)\Delta^3_{a,b,c,d}\right\}\right];\end{aligned} \tag{A.10}$$

where:

$$\Delta_{a,b}=\frac{z_b-z_a}{b-a}\text{ etc.; }\Delta^2_{a,b,c}=\frac{\Delta_{b,c}-\Delta_{a,b}}{c-b}\text{ etc.; and }\Delta^3_{a,b,c,d}=\frac{\Delta^2_{b,c,d}-\Delta^2_{a,b,c}}{d-a};$$

and $z_a$, $z_b$, $z_c$ and $z_d$, are quarterly continuously compounded yields to redemption for durations $a$, $b$, $c$ and $d$ respectively. Where possible (i.e. except at the extremes), the bonds were chosen so that $a<b<D\le c<d$. Where there were fewer than four bonds, the higher-order differences were ignored.

A.4.2.5 In some cases the method explained in the preceding paragraph resulted in extrapolation to unacceptable levels at the extremes. In these cases reasonable values were assumed for $D=0$ and $D=100$.

A.4.2.6 Finally we calculate the real return on inflation-linked bonds with terms to redemption of ¼, 5, 10 and 20 years as:

$$R_{\mathrm{ILB}t(q)}=qz_{t-1,q}-(q-1)z_{t,q-1}\text{ for }t=144,\ldots,185,\ q=1,20,40,80.$$

Here it is not necessary to subtract the force of inflation as the return itself is expressed in real terms.

### A.4.3 Effective Market Capitalisation of Zero-Coupon Bonds by Term to Redemption

A.4.3.1 For the purpose of calculating the total nominal value of conventional domestic marketable bonds issued by the public sector at time $t$ (¶A.3.3.1), the amount of inflation-linked domestic marketable bonds so issued must be deducted from the total nominal value of domestic marketable bonds so issued. From 30/6/2009 onwards the amounts of inflation-linked domestic marketable bonds so issued were available. At earlier times it was necessary to estimate them. For this purpose, for each bond issued before 30/6/2009, the amount issued at times between the date of issue and that date was interpolated between the amounts issued at those dates.

A.4.3.2 In order to calculate the effective market capitalisation of zero-coupon bonds by term to redemption $q$ at a particular date $t$, the payment during each subsequent quarter was calculated and discounted at the mean spot rate for that quarter, to give the total payment during quarter $q$:

$$\pi_{tq}=\sum_{j\in J}\frac{m'_{jt}}{P_{jt}}\left(\tfrac{1}{2}C_j\delta_{t+q\in Q_j}+\delta_{t+q=q_j^*}\right);$$

where:

$$P_{jt}=\sum_{q\in Q_j}\left(\tfrac{1}{2}C_j+\delta_{q=q_j^*}\right)\exp\left(-qy_{tq_j^*}\right);$$

$$\delta_B=\begin{cases}1 \text{ if } B;\\ 0 \text{ otherwise;}\end{cases}$$

$J$ is the set of inflation-linked bonds;

$C_j$ is the annual coupon on bond $j$;

$Q_j$ is the set of quarters in which coupons are payable on bond $j$;

$q_j^*$ is the quarter in which bond $j$ is redeemable; and

$m'_{jt}$ is the market capitalisation of bond $j$ at time $t$.

A.4.3.3 Following ¶A.3.3.4, let:

$$\kappa_{\mathrm{ILB}t}^{(p)}=\sum_{q=1}^{100}\pi_{tq}q^pP(t,q) \text{ for } p=0,1,2;$$

where:

$$P(t,q)=\exp\left(-qz_{tq}\right);$$

so that, in particular, $\kappa_{\mathrm{ILB}t}^{(0)}$ is the market capitalisation of bonds at time $t$. As before:

$$m_{\mathrm{ILB}t}^{*}=\boldsymbol{Q}^{-1}\boldsymbol{\kappa}_{\mathrm{ILB}t}.$$