A STOCHASTIC-PROGRAMMING APPROACH TO INTEGRATED ASSET AND LIABILITY MANAGEMENT OF INSURANCE PRODUCTS WITH GUARANTEES

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ABSTRACT

In recent years insurance products have become more complex by providing investors with various guarantees and bonus options. This increase in complexity has provided an impetus for the investigation into integrated asset- and liability-management frameworks that could realistically address dynamic portfolio allocation in a risk-controlled way. In this paper the authors propose a multi-stage dynamic stochastic-programming model for the integrated asset and liability management of insurance products with guarantees that minimises the down-side risk of these products. They investigate with-profit guarantee funds by including regular bonus payments while keeping the optimisation problem linear. The uncertainty is represented in terms of arbitrage-free scenario trees using a four-factor yield-curve model that includes macroeconomic factors (inflation, capacity utilisation and the repo rate). They construct scenario trees with path-dependent intermediate discrete yield-curve outcomes suitable for the pricing of fixed-income securities. The main focus of the paper is the formulation and implementation of a multi-stage stochastic-programming model. The model is back-tested on real market data over a period of five years.

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1. INTRODUCTION

1.1 In recent years multi-stage dynamic stochastic-programming models have become a popular tool for integrated asset and liability modelling. In contrast to the usual mean-variance approach (Markowitz, 1952) with a myopic view of managing investment risk over a single period, dynamic stochastic optimisation provides the asset manager with an integrated way to model both assets and liabilities in a flexible manner that takes into account multi-period dynamic asset allocation and the valuation of liabilities under future market conditions. In this approach the rebalancing of the asset portfolio is modelled explicitly. Examples of the use of dynamic stochastic-programming models in asset and liability management can be found in Kouwenberg (2001) and Mulvey *et al.* (2003). Dempster *et al.* (2003) show that these models will automatically hedge the current portfolio allocation against future uncertainties in asset returns and costs of liabilities up to the analysis horizon. They are also flexible enough to take into account multi-period horizons, portfolio constraints such as no short-selling, transaction costs and the investor's level of risk-aversion and utility.

1.2 Insurance products have become more complex by providing investors with various guarantees and bonus options. This increase in complexity has provided an impetus for the investigation into integrated asset- and liability-management frameworks that could realistically address dynamic portfolio allocation in a risk-controlled way. Examples of the use of dynamic portfolio-optimisation models for asset and liability management in the insurance industry are the Russell–Yasuda Kasai model by Cariño & Ziemba (1998), the Towers Perrin model by Mulvey & Thorlacius (1998) and the CALM model of Consigli & Dempster (1998). More recent contributions specifically in the area of insurance products with minimum guarantees using dynamic stochastic programming as an asset- and liability-management tool are Dempster *et al.* (2006) and Consiglio *et al.* (2006).

1.3 Dempster *et al.* (2006) propose an asset- and liability-management framework and give numerical results for a simple example of a closed-end guaranteed fund where no contributions are allowed after the initial cash outlay. They demonstrate the design of investment products with a guaranteed minimum rate of return focusing on the liability side of the product. Through back-testing they show that the proposed stochastic-optimisation framework reasonably addresses the risk created by the guarantee.

1.4 Consiglio *et al.* (2006) study the same type of problem by structuring a portfolio for with-profit guarantee funds in the United Kingdom. The optimisation problem is non-linear. They demonstrate how the model can be used to analyse the alternatives to different bonus policies and reserving methods. They investigate the asset and liability management of minimum-guarantee products for the Italian industry.

1.5 Inspired by the research of Dempster *et al.* (2006) and Consiglio *et al.* (2006), this paper proposes a multi-stage dynamic stochastic-programming model for the integrated

asset and liability management of insurance products with guarantees, that minimises the down-side risk of these products. As proposed in Dempster *et al.* (2006), this model also allows for portfolio rebalancing decisions over multiple periods, as well as for flexible risk-management decisions, such as the reinvestment of coupons at intermediate time steps. With-profits guarantee funds are investigated, as in Consiglio *et al.* (2006), by including regular bonus payments. Once these bonuses have been declared, the bonuses become guaranteed. To keep the optimisation problem linear, the way in which bonuses are declared is changed. The problem is kept linear, for two reasons. The first is that, by keeping the problem linear, the rebalancing of the portfolio can be modelled at future decision times. By doing so the dynamic stochastic-programming model automatically hedges the first-stage portfolio allocation against projected future uncertainties in asset returns (see Dempster *et al.*, 2003, 2006). The second reason is that the model is flexible enough to take into account portfolio constraints such as the prohibition of short-selling, transaction costs and coupon payments.

1.6   For the South African insurance market, Katz & Rosenberg (unpublished) uses a sample smoothed-bonus annuity contract to illustrate the weaknesses of traditional pricing, valuation and risk management tools used by life offices particularly in a low interest rate environment. Furthermore, they describe, illustrate and argue the merits of a coherent pricing, valuation and risk-management framework for managing smoothed-bonus contracts.

1.7   The pricing of contingent claims and the dynamic management of portfolios are two sides of the same coin. The main differences between them are highlighted by Consiglio *et al.* (2006). The literature on the pricing of products with guarantees assumes that the reference portfolio is given exogenously (e.g. equities 60% and bonds 40%), and does not address the problem of structuring this portfolio optimally. The possible upside potential is ignored. According to Dempster *et al.* (2006):

> "This is where the asset manager has a potential advantage. He or she can provide the protection while still exposing the client to high-risk markets through active asset allocation to potentially higher returns."

Consiglio *et al.* (2001) have shown that the financial institution could substantially increase shareholder value by structuring the reference portfolio. This can be done by viewing it as an integrated asset- and liability-management optimisation problem. Long-term options, which form the backbone of valuation methods, are in general available only as over-the-counter contracts. This adds a credit-risk component to the problem that is largely ignored. The replicating-portfolio approach used to value these products assumes continuous rebalancing. This assumption and the other assumptions of the Black–Scholes model are unrealistic.

1.8   Foroughi *et al.* (2003) explore the risks faced by South African life insurers arising from the provision of investment guarantees in these products. Furthermore,

they examine various forms of investment guarantees available in South Africa and the business issues created by writing these products. They compare existing methods used to value them and discuss practical issues around the building of asset and liability models for that purpose. They identify non-profit immediate annuities, participating immediate annuities, unit-linked saving products with a maturity guarantee and smoothed-bonus savings products with a maturity guarantee as the four main products with investment guarantees sold in South Africa.

1.9   It is worth noting that the with-profit guarantee funds discussed in this paper are operated by on a 90–10 basis: the policyholders benefit in 90% of the growth in asset share and the shareholders in 10%. These products occur to a lesser extent in the South African insurance market (see Foroughi *et al.*, 2003). The main goals of this paper are firstly to renew awareness of the concept of stochastic programming among the South African actuarial community and secondly to extend the work of Dempster *et al.* (2006) and Consiglio *et al.* (2006). This however provides wide scope for further research of these products in the South African insurance market.

1.10   The uncertainty is represented by scenario trees with a four-factor yield-curve model that includes macroeconomic factors (inflation, capacity utilisation and the repo rate). Scenario trees are constructed whose outcomes are path-dependent intermediate discrete yield curves suitable for the pricing of fixed-income securities (see section 3.1).

1.11   The rest of the paper is structured as follows. In Section 2 the formulation and implementation of the multi-stage stochastic-programming model is discussed. Section 3 presents back-testing results. The back-tests are done on real market data over a period of five years.

2.   SCENARIO OPTIMISATION FRAMEWORK

In this section a linear multi-stage dynamic stochastic programming model is proposed for the integrated asset and liability management of insurance products with guarantees, that minimises the down-side risk of these products.

2.1   MODEL FEATURES

2.1.1   As in Consiglio *et al.* (2006) the optimal asset allocation of a with-profit guarantee fund is investigated by including regular bonuses. As explained above, the fund is operated by a proprietary insurer on a 90–10 basis. It is assumed that there is a cohort of policyholders, paying a single upfront premium and that no contributions are allowed thereafter. At maturity there is an underlying guarantee to pay a minimum annual rate of return of g on the initial premium. In addition to receiving a guaranteed rate of return on the initial premium, policyholders also receive several bonuses. Bonuses are meant to reflect the overall performance of the insurer's portfolio, and to correspond to policyholders' reasonable expectations. Two types of bonuses are received by the policyholder, namely regular bonuses (declared annually) and terminal bonuses

(awarded on maturity). Regular bonuses are 'vesting'; in other words they are guaranteed once declared and cannot be reduced (Consiglio *et al.*, 2006). The time horizon of the fund is T years. Two asset classes are used, namely, bonds bearing semi-annual coupons and equities modelled by means of indices.

2.1.2   To represent uncertainty, future yield curves and index movements are simulated and a scenario tree is constructed. A scenario tree is a discrete approximation of the joint distribution of random factors (yield curve and equity indices). To facilitate the mathematical formulation of the optimisation problem, the scenario tree is represented in terms of states (or 'nodes') s_t^v where $t = 0, \frac{1}{12}, \frac{2}{12}, \ldots, T$ and $v = 1, 2, \ldots, S_t$. The states at time t are denoted by:

$$\Sigma_t = \left\{ s_t^v \mid v = 1, 2, \ldots, S_t \right\}.$$

To enforce non-anticipativity, i.e. to prevent foresight of uncertain future events, we order the states in pairs $\left(s_t^{v(t)}, s_{t+1}^{v(t+1)} \right)$, where the dependence of the index v on t is explicitly indicated. The order of the states indicates that state $s_{t+1}^{v(t+1)}$ at time $t+1$ can be reached from state $s_t^{v(t)}$ at time t. $s_{t+1}^{v(t+1)}$ is a 'successor state' and $s_t^{v(t)}$ the 'predecessor state'. By using the superscripts "+" to denote successor states and "–" to denote the predecessors, we have:

$$s_t^{v(t)+} = s_{t+1}^{v(t+1)} \text{ and } s_{t+1}^{v(t+1)-} = s_t^{v(t)}.$$

Each state s_t^v has the associated probability p_t^s such that:

$$\sum\nolimits_{s \in \Sigma_t} p_t^s = 1.$$

Random factors in the scenario tree at time t are indexed by states $s \in \Sigma_t$.

2.1.3   The annual decision times $t_d = 0, 1, 2, \ldots, T-1$ are the times at which the fund will trade to rebalance its portfolio. The branching of the tree structure is represented by a 'tree-string', which is a string of integers specifying for each decision time t_d the number of branches for each state $s \in \Sigma_d$. This specification gives rise to a balanced scenario tree in which each sub-tree in the same period has the same number of branches. Figure 1 gives an example of a scenario tree with a (3, 2) tree-string, giving a total of 6 scenarios.

2.2   MODEL VARIABLES AND PARAMETERS

2.2.1   The notation used for variables and parameters of the model is set out in Tables 1 to 4, where the time index t takes values over the times $t = 0, \frac{1}{12}, \frac{2}{12}, \ldots, T$, and the states index s takes values from the set $\Sigma_t = \left\{ s_t^v \mid v = 1, 2, \ldots, S_t \right\}$. In this paper 'yield' means the yield to maturity on a zero-coupon bond.

Table 1: Time sets

$T^{total} = \left\{ 0, \frac{1}{12}, \frac{2}{12}, \ldots, T \right\}$	: set of all times considered in the stochastic program
$T^{d} = \left\{ 0, 1, 2, \ldots, T-1 \right\}$	: set of decision times
$T^{i} = T^{total} \setminus T^{d}$	: set of intermediate times
$T^{c} = \left\{ \frac{1}{2}, \frac{3}{2}, \ldots, T - \frac{1}{2} \right\}$	: set of coupon payment times between decision times

Table 2: Index sets

$\Sigma_{t} = \left\{ s_{t}^{v} \mid v = 1, 2, \ldots, S_{t} \right\}$	: set of states at period t
SI	: set of equity indices
$B = \left\{ B_{\tau} \right\}$	: set of government bonds with maturity τ
$\mathrm{I} = SI \cup B$	: set of all instruments

Figure 1: Graphic representation of a scenario tree
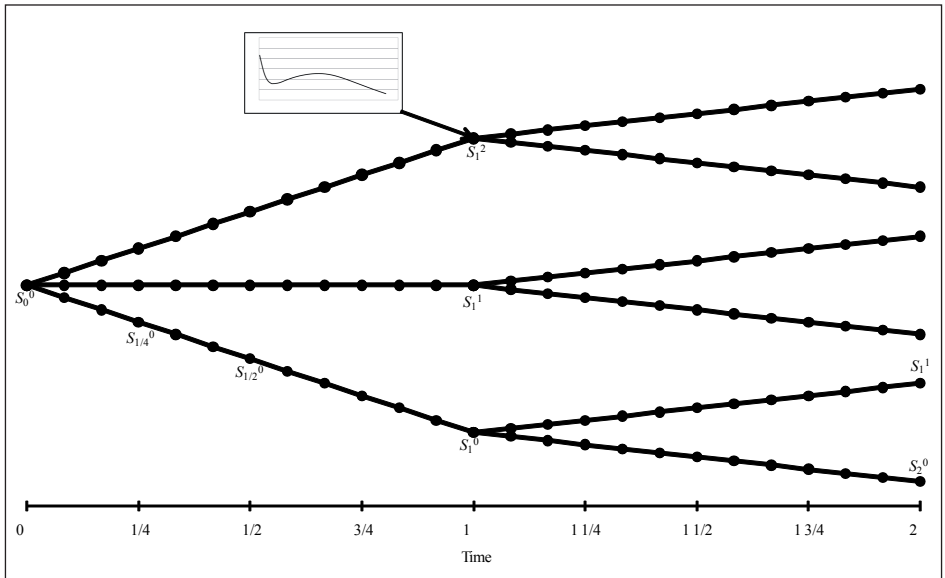
Table 3: Parameters

δ_{B_τ}	coupon rate of a government bond with maturity τ
F_{B_τ}	face value of a government bond with maturity τ
$r_{t,\tau}^s$	yield to maturity τ at time t in state s
g	minimum guaranteed rate of return
ρ	regulatory equity-to-debt ratio
$r_{t,b}^s$	benchmark rate at time t in state s
γ	policyholders' rate of participation in the profits of the insurer
β	target terminal bonus
$P_{t,i}^{a,s}$ or $P_{t,i}^{b,s}$	ask or bid price of asset $i \in I$ at time t in state s
f_a or f_b	proportional transaction costs on ask or bid transactions
p_t^s	probability of state s in period t

Table 4: Decision variables

$x_t^s = \left\{ x_{t,i}^s \right\}_{i \in I}$	quantities of assets bought at time t in state s
$y_t^s = \left\{ y_{t,i}^s \right\}_{i \in I}$	quantities of assets sold at time t in state s
$z_t^s = \left\{ z_{t,i}^s \right\}_{i \in I}$	quantities of assets held at time t in state s
A_t^s	value of assets account at time t in state s
L_t^s	value of liability account at time t in state s
E_t^s	value of equity account at time t in state s
c_t^s	amount of equity provided by shareholders at time t in state s
SF_t^s	amount of shortfall at time t in state s
RB_t^s	regular bonus payment declared at time t in state s
TB_T^s	policyholders terminal bonus at time T in state s

2.3 BOND PRICING

It is assumed that all bonds pay semi-annual coupons of δ_{B_τ} and derive bid and ask prices by adding a spread, sp, to the yields. Let $P_{t,B_\tau}^{a,s}$ denote the ask price at time t of a coupon-bearing bond with maturity τ, so that:

$$P_{t,B_\tau}^{a,s} = F_{B_\tau} e^{-(\tau-t)\left(r_{t,\tau-t}^s + sp\right)} + \sum_{m=\left\{\frac{\lfloor 2t \rfloor}{2}+\frac{1}{2}, \frac{\lfloor 2t \rfloor}{2}+1, \ldots, \frac{\lfloor 2t \rfloor}{2}+\tau\right\}} \frac{1}{2} \delta_{B_\tau} F_{B_\tau} e^{-(m-t)\left(r_{t,(m-t)}^s + sp\right)}$$

for $t \in T^{total}$ and $s \in \Sigma_t$;

where the principal amount is discounted in the first term and the coupon-payment stream in the second term. Let $P_{t,B_\tau}^{b,s}$ denote the bid price at time t of the bond with maturity τ, so that:

$$P_{t,B_\tau}^{b,s} = F_{B_\tau} e^{-(\tau-t)\left(r_{t,\tau-t}^s - sp\right)} + \sum_{m=\left\{\frac{\lfloor 2t \rfloor}{2}+\frac{1}{2}, \frac{\lfloor 2t \rfloor}{2}+1, \ldots, \frac{\lfloor 2t \rfloor}{2}+\tau\right\}} \frac{1}{2} \delta_{B_\tau} F_{B_\tau} e^{-(m-t)\left(r_{t,(m-t)}^s - sp\right)},$$

for $t \in T^{total}$ and $s \in \Sigma_t$.

2.4 VARIABLE DYNAMICS AND CONSTRAINTS

2.4.1 The variable dynamics and constraints for the minimum-guarantee problem are as stated in ¶¶2.4.2–10.

2.4.2 The cash-balance constraints ensure that the amount of cash that is received from selling assets, coupon payments at decision times and equity supplied for a shortfall is equal to the amount of assets bought, as follows:

$$\sum_{i \in I} P_{0,i}^{a,s} x_{0,i}^s \left(1 + f_a\right) = A_0^s, \text{ for } t \in \{0\} \text{ and } s \in \Sigma_t; \text{ and}$$

$$\sum_{i \in I} P_{t,i}^{b,s} y_{t,i}^s \left(1 - f_b\right) + \sum_{i \in I \setminus \{SI\}} \frac{1}{2} \delta_i F_i y_{t,i}^s + c_t^s = \sum_{i \in I} P_{t,i}^{a,s} x_{t,i}^s \left(1 + f_a\right),$$

for $t \in T^d \setminus \{0\}$ and $s \in \Sigma_t$.

2.4.3 The short-sale constraints eliminate the short-selling of assets in each state during each time period, as follows:

$$x_{t,i}^s \geq 0, \text{ for all } i \in I, t \in T^{total} \setminus \{T\} \text{ and } s \in \Sigma_t;$$

$$y_{t,i}^s \geq 0, \text{ for all } i \in I, t \in T^{total} \setminus \{0\} \text{ and } s \in \Sigma_t; \text{ and}$$

$$z_{t,i}^s \geq 0, \text{ for all } i \in I, t \in T^{total} \setminus \{T\} \text{ and } s \in \Sigma_t.$$

2.4.4 The inventory constraints give the quantity invested in each asset for each state during each time period, as follows:

$$z_{0,i}^s = x_{0,i}^s, \text{ for } t \in \{0\} \text{ and } s \in \Sigma_t, \text{ and}$$

$$z_{t,i}^s = z_{t,i}^{s^-} + x_{t,i}^s - y_{t,i}^s, \ i \in I, t \in T^{total} \setminus \{0\} \text{ and } s \in \Sigma_t.$$

2.4.5 As the portfolio is rebalanced only at decision times, the information constraints ensure that portfolio cannot be changed between decision times, as follows:

$$x_{t,i}^s = y_{t,i}^s = 0, \text{ for } i \in I, t \in T^i \setminus T^c \text{ and } s \in \Sigma_t.$$

2.4.6 The coupon reinvestment constraints ensure that the coupons that are paid at the coupon times are reinvested in the same coupon-bearing bonds, as follows:

$$x_{t,i}^s = \frac{\frac{1}{2}\delta_i F_i z_{t,i}^{s-}}{P_{t,i}^{a,s}(1+f_a)}, \text{ for } i \in I \setminus \{SI\}, t \in T^c \text{ and } s \in \Sigma_t;$$

$$y_{t,i}^s = 0, \text{ for } i \in I \setminus \{SI\}, \text{ for } t \in T^c \text{ and } s \in \Sigma_t; \text{ and}$$

$$x_{t,SI}^s = 0, y_{t,SI}^s = 0, \text{ for } t \in T^c \text{ and } s \in \Sigma_t.$$

2.4.7   The asset-account constraints determine the value of the asset account for each state at each time. The value of the asset account is determined after rebalancing; i.e. any equity c_t^s that has been provided by shareholders to fund shortfalls, is taken into account by the cash-balance constraints, as follows:

$$A_0^s = L_0 + E_0^s, \text{ for } t \in \{0\} \text{ and } s \in \Sigma_t;$$

$$A_t^s = \sum_{i \in I} P_{t,i}^{a,s} z_{t,i}^s (1+f_a), \text{ for } t \in T^{total} \setminus \{T\} \text{ and } s \in \Sigma_t; \text{ and}$$

$$A_T^s = \sum_{i \in I} P_{T,i}^{b,s} z_{T-\frac{1}{12},i}^{s-} (1-f_b) + \sum_{i \in I \setminus \{SI\}} \frac{1}{2}\delta_i F_i z_{T-\frac{1}{12},i}^{s-}, \text{ for } s \in \Sigma_t.$$

2.4.8   The liability-account constraints determine the value of the liability account for each state at each time. The liability grows at the guaranteed rate of return and is increased by any regular bonus payments that are declared, as follows:

$$L_0^s = L_0, \text{ for } t \in \{0\} \text{ and } s \in \Sigma_t; \text{ and}$$

$$L_t^s = L_{t-\frac{1}{12}}^{s-} e^{\frac{1}{12}g} + RB_t^s, \text{ for } t \in T^{total} \setminus \{T\} \text{ and } s \in \Sigma_t.$$

2.4.9   The equity-account constraints determine the value of the equity account for each state at each time. The equity grows at the one-month return. The shortfall is funded by the shareholders by the infusion of additional equity, as follows:

$$E_0^s = c_0^s, \text{ for } t \in \{0\} \text{ and } s \in \Sigma_t; \text{ and}$$

$$E_t^s = E_{t-\frac{1}{12}}^{s-} e^{r_{t-\frac{1}{12}t}^{s-}} + c_t^s, \text{ for } t \in T^{total} \setminus \{T\} \text{ and } s \in \Sigma_t.$$

2.4.10   The regular-bonus constraints determine the amount of the regular bonus payment for each state at each decision time. To determine the amount of the regular bonus the approach described by Consiglio *et al.* (2006) is followed. This is based on that of Ross (1989), where the regular bonuses are determined by aiming for a target terminal bonus, i.e. the insurer wishes the policyholders' terminal benefit to be a fixed portion of the total benefit received. Regular bonuses are assumed to be declared at decision times only (i.e. annually).

2.4.11   It is assumed that the asset account will grow at the current benchmark rate, $r_{t,b}^s$, up to termination, giving the terminal asset value as:

$$A_T^s = A_t^{bs} e^{r_{t,b}^s(T-t)};$$

where

$$A_t^{bs} = \sum_{i \in I} P_{t,i}^{b,s} z_{t-\frac{1}{12},i}^{s-} \left(1 - f_b\right) + \sum_{i \in I \setminus \{SI\}} \tfrac{1}{2} \delta_i F_i z_{t-\frac{1}{12},i}^{s-}$$

is the value of the asset account before transactions. It is further assumed that the liabilities will grow at the minimum guaranteed rate of return, g, up to termination. Furthermore, it is assumed that the regular bonus payment, RB_t^s, that is declared at time t, will stay constant throughout the remainder of the term and will be invested at the minimum guaranteed rate of return, g. Thus the terminal liability value is:

$$L_T^s = L_{t-\frac{1}{12}}^{s-} e^{g(T-t+\frac{1}{12})} + RB_t^s \left(\frac{e^{g(T-t)} - 1}{e^g - 1} + e^{g(T-t)} \right),$$

where

$$\left(\frac{e^{g(T-t)} - 1}{e^g - 1} + e^{g(T-t)} \right)$$

is the accumulated value of a constant annuity with payment of one cash unit from time t to time T invested at the minimum guaranteed rate of return, g.

2.4.12   The terminal bonus,

$$TB_T^s = \gamma \left(A_T^s - L_T^s \right),$$

received by the policyholders needs to constitute $100\beta\%$ of the total amount received by the policyholders, so that:

$$\frac{TB_T^s}{TB_T^s + L_T^s} = \beta.$$

Solving for RB_t^s yields:

$$RB_t^s = \frac{\gamma\left(1-\beta\right) A_t^{b,s} e^{r_{t,b}^s (T-t)} - \left(\beta + \gamma\left(1-\beta\right)\right) L_{t-\frac{1}{12}}^{s-} e^{g(T-t+\frac{1}{12})}}{\left(\beta + \gamma\left(1-\beta\right)\right)\left(\dfrac{e^{g(T-t)} - 1}{e^g - 1} + e^{g(T-t)} \right)}.$$

When the expected terminal asset amount exceeds the expected terminal liability amount, regular bonuses will be positive. Conversely, when the expected terminal liability amount exceeds the expected terminal asset amount the regular bonus will be negative. As the declaration of negative bonuses would be unfair towards policyholders, the following regular-bonus constraint is introduced:

$$RB_t^s \geq \frac{\gamma\left(1-\beta\right) A_t^{b,s} e^{r_{t,b}^s (T-t)} - \left(\beta + \gamma\left(1-\beta\right)\right) L_{t-\frac{1}{12}}^{s-} e^{g(T-t+\frac{1}{12})}}{\left(\beta + \gamma\left(1-\beta\right)\right)\left(\dfrac{e^{g(T-t)} - 1}{e^g - 1} + e^{g(T-t)} \right)},$$

for $t \in \left(T^d \setminus \{0\}\right) \cup \{T\}$ and $s \in \Sigma_t$;

where

$$RB_t^s \geq 0 \text{ and } RB_t^s = 0 \text{ for } t \in \left(T^i \cup \{0\}\right) \setminus \{T\} \text{ and } s \in \Sigma_t.$$

By enforcing the regular-bonus constraint the optimisation will determine the regular bonus amount RB_t^s at each decision period.

2.4.13   Consiglio *et al.* (2006) also consider the 'working party approach' based on Chadburn (unpublished), which in turn is based on work done by a working party of the Institute of Actuaries. This approach allows for the declaration of regular bonuses (in return form) to reflect the benchmark return, provided that the liability account remains lower than the value of the 'reduced asset account', which accumulates at 75% of the return on assets. Consiglio *et al.* (2006) test their model with both these features and find that bonus policies aiming for a target terminal bonus outperform bonus polices based on the working party's approach.

2.4.14   The shortfall constraints determine the regulatory shortfall of the portfolio for each state at each time. The shortfall is calculated by using the value of the asset account before transaction as follows:

$$SF_0^s + L_0 \geq (1+\rho)L_0^s, \text{ for } t \in \{0\} \text{ and } s \in \Sigma_t; \text{ and}$$

$$SF_t^s + A_t^{bs} \geq (1+\rho)L_{t-\frac{1}{12}}^s e^{g\frac{1}{12}} \text{ for } t \in (T^{total}\setminus\{0\}) \text{ and } s \in \Sigma_t;$$

where

$$A_t^{bs} = \sum_{i \in I} P_{t,i}^{b,s} z_{t-\frac{1}{12},i}^{s-}(1-f_b) + \sum_{i \in I\setminus\{SI\}} \tfrac{1}{2}\delta_i F_i z_{t-\frac{1}{12},i}^{s-}$$

is the value of the asset account before transactions; and

$$SF_t^s \geq 0 \text{ for } t \in T^{total} \text{ and } s \in \Sigma_t.$$

The shortfalls SF_t^s at decision times are funded by the shareholders' equity payment, c_t^s, thus:

$$c_t^s = SF_t^s \text{ for } t \in (T^d) \cup \{T\} \text{ and } s \in \Sigma_t, \text{ and zero at intermediate nodes; and}$$

$$c_t^s = 0 \text{ for } t \in T^i \setminus \{T\} \text{ and } s \in \Sigma_t.$$

By enforcing the shortfall constraints the optimisation will determine the amount of equity c_t^s to be provided by the shareholders at each decision time.

2.4.15   Portfolio-composition constraints can be introduced in order to reduce concentration risk and to consider the policyholder expectations on the underlying asset mix. The following portfolio constraints are applied:

$$\frac{\sum_{i \in \tilde{I}} P_{t,i}^{a,s} z_{t,i}^s}{W_t^s} \leq \pi_{\tilde{I}};$$

where $\tilde{I}$ may be some subset of I and $\pi_{\tilde{I}}$ the upper limit for the proportion of asset share invested in the subset of assets $\tilde{I}$.

2.5   OBJECTIVE FUNCTION

2.5.1   In the management of a minimum-guarantee fund there are two main goals to take into account. The first is the management of the investment strategies of the fund. The second is to maximise the shareholder value taking into account the minimum guarantee given to policyholders. The shareholders' final wealth is given as:

$$(1-\gamma)((A_T - E_T) - L_T) + E_T;$$

where

$$(1-\gamma)\left(\left(A_T - E_T\right) - L_T\right)$$

is the excess amount they receive after the liability and the equity have been paid.

2.5.2   The objective to consider is the maximum expected excess wealth of the shareholders and the minimum average expected shortfall over all periods. Dempster *et al.* (2006) have shown that the examination of shortfall at intermediate nodes improves the results. The objective function is given as:

$$\max_{\substack{\left\{x_{t,i}^s, y_{t,i}^s, z_{t,i}^s : \\ i \in I, t \in T^d \cup \{T\}, s \in \Sigma_t\right\}}} \left\{ \begin{array}{c} (1-\alpha) \displaystyle\sum_{s \in \Sigma_T} p^s (1-\gamma)\left(\left(A_T^s - L_T^s\right) - E_T^s\right) \\[2ex] -\alpha \displaystyle\sum_{t \in T^{total}} \sum_{s \in \Sigma_t} p^s \frac{SF_t^s}{\left|T^{total}\right|} \end{array} \right\};$$

where $\alpha \in [0,1]$ sets the level of risk-aversion and can be chosen freely. If the value of α is closer to 1, more importance is given to the shortfall of the portfolio and less to the expected excess wealth of the shareholders and hence a more risk-averse portfolio allocation strategy will be taken and vice versa. In the extreme case where $\alpha = 1$, only the shortfall will be minimised and the expected excess wealth will be ignored, and $\alpha = 0$, the unconstrained case, only maximises the expected excess wealth of the shareholders.

2.5.3   The resulting linear program is presented in Appendix A.

3.   RESULTS

In this section the performance of the model is discussed. The first two parts explain the scenario-generation algorithm that was used to generate scenario trees, which is the input to the mathematical optimisation problem. In the third part back-tested results are presented for different levels of the guarantee rate and different levels of risk-aversion.

3.1   SCENARIO GENERATION

3.1.1   The yield-curve dynamics are estimated with the four-factor yield curve representation of Svensson (1994). The four unobserved factors, level, slope and the two curvature factors, which provide a good representation of the yield curve, are linked to the macroeconomic variables by means of a state-space model. The following three variables are included as measures of the state of the economy: manufacturing capacity utilisation, which represents the level of real economic activity relative to potential; the annual percentage change in the inflation index, which represents the inflation rate; and the repo rate, which represents the monetary-policy instrument. According to Diebold *et al.* (2006) these three macro-economic variables are considered to be the minimum set of fundamentals needed to capture the basic macroeconomic dynamics. The model parameters are estimated using a Kalman-filter approach. For a complete description of the model and the calibration of its parameters, see Raubenheimer & Kruger (2010).

3.1.2   Raubenheimer & Kruger (*op. cit.*) also propose a parallel simulation and clustering approach to create the scenario tree as described in section 2.1. As explained in ¶2.1.3, a T-period scenario tree is represented as a tree-string. Let

$$k_0, k_1, \ldots, k_d, \ldots, k_{T-1}$$

denote a typical tree-string. Then the branching factor for decision time t_d, is given by k_d. For the example in Figure 1, $k_0 = 3$ and $k_1 = 2$.

3.1.3 In order to group the scenarios, a measure of relative position is used to calculate the 'distance' between the discounting factors of the yield curve and that of the average, viz.:

$$D = \sum_\tau \left(\frac{1}{\left(1 + r_{t,\tau}^n\right)^\tau} - \frac{1}{\left(1 + r_{t,\tau}^M\right)^\tau} \right);$$

where $r_{t,T}^n$ is the yield to maturity τ and $r_{t,\tau}^M$ the average yield to maturity τ. Note that the relative distance D can be negative or positive, which means that a yield curve can be positioned to the left or to the right of the average yield curve. It is necessary to represent each group of scenarios with a single 'representative' or 'centroid', which becomes the data in the scenario tree. The mean of the group is used here as the notion of the centre.

3.1.4 The main steps of the algorithm can be outlined as follows:

Step 1: At $s = 0$ create a root node group containing N scenarios. Generate all the scenarios using Monte Carlo simulation and the four-factor yields-macro model. (See Raubenheimer & Kruger (2010) for details of this model.) Each scenario is equally likely and consists of $T+1$ sequential yield curves with the same starting point, the current yield curve (in total $(T+1) \times N$ yield curves are generated).

Step 2: Set $s = s + 1$ and for each group in the previous stage, calculate the average scenario and calculate the relative position of each scenario with respect to the average.

Step 3: For each group, sort the scenarios in descending distance order and group them into k_s equal-sized groups.

Step 4: For each new group, find the scenario closest (in absolute value) to average of the group, and designate it as the centroid. To each centroid assign the probability:

$$\left(\prod_{i=1}^{s-1} k_i \right)^{-1}.$$

Step 5: If $s < T$, go to step 2, otherwise stop.

3.1.5 The scenarios generated are not arbitrage-free (see Klaassen, 2002 and Filipović, 1999). Raubenheimer & Kruger (2010) propose the following method to eliminate the arbitrage opportunities:

Step 1: At the root node create a group of N scenarios. Generate all the scenarios using Monte Carlo simulation and the four-factor yields-macro model (as for the scenario tree). Each scenario is equally likely and consists of T sequential yield curves.

Step 2: At each decision time of the scenario tree calculate the average of the N generated scenarios (at the root node the current yield curve is used).

Step 3: Then, for each average yield curve and the corresponding one-period ahead scenarios, solve:

$$\frac{1}{N} \sum_{n=1}^{N} e^{-(\tau-1)\left(r_{t+1,\tau-1}^{n}+c_{t+1,\tau}\right)} = \frac{e^{-\tau r_{t,\tau}}}{e^{-r_{t,1}}}$$

for all maturities, to obtain the yield curve shifts $c_{t+1,\tau}$.

Step 4: Add the spread $c_{t+1,\tau}$ to the original scenario-tree yield curves.

3.1.6   The described method removes most of the arbitrage opportunities in the scenario tree, with a few opportunities left in sub-trees. For scenario trees with a short horizon all opportunities may be removed. This reduction of arbitrage opportunities is considered sufficient, since portfolio constraints in optimisation problems, such as the restriction of short-selling and the inclusion of bid–ask spreads, will eliminate the effect of the remaining arbitrage opportunities.

3.2   DATA AND CALIBRATION INSTRUMENTS

3.2.1   Six different assets were used, namely bonds, bearing semi-annual coupons, with maturities 5, 7, 10, 15 and 19 years, and the FTSE–JSE Top-40 equity index. In order to generate scenarios for the Top-40 index, the index is modelled using a simple linear regression model incorporating the three macroeconomic variables. The 'perfect fit bond curves', one of the five BEASSA 'zero coupon yield curve' series of yield curves,[1] with maturities 1, 2, 3, 6, 9, 12, 15, 18, 21, 24, 36, 48, 60, 72, 84, 96, 108, 120, 132, 144, 156, 168, 180, 192, 204, 216 and 228 months were used. The curves are derived from government bond data.[2] End-of-month data were used from August 1999 to February 2009 and a tree structure with approximately the same number of scenarios. The tree structure used in back-testing is displayed in Table 5.

Table 5: Tree structure used for back-testing

Year	Tree-string
April 2003	5.5.5.5.5 = 3 125
April 2004	8.8.8.8 = 4 096
April 2005	15.15.15 = 3 375
April 2006	56.56 = 3 136
April 2007	3 125

3.2.2   The scenario generation approach outlined above was used to generate the input scenarios for the optimisation problem. The four-factor yields-macro model is fitted to market data up to an initial decision time t and scenario trees are generated

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1   An introduction to the BEASSA zero coupon yield curves. Bond Exchange of South Africa, Johannesburg, 2003
2   The BEASSA zero coupon yield curves: Technical specifications. Bond Exchange of South Africa, Johannesburg, 2003

from time t to some chosen horizon $t+T$. The optimal first-stage or root-node decisions are then implemented at time t. The success of the portfolio strategy is measured by its performance with historical data up to time $t+1$. This whole procedure is rolled forward for T trading times. At each decision time t, the parameters of the four-factor yields-macro model are re-estimated using the historical data up to and including time t.

3.3   BACK-TESTED RESULTS

3.3.1   Back-tests were performed over the period of five years from February 2004 to February 2008, for different levels of minimum guarantee and for different levels of risk-aversion. For each of these back-tests, at different levels, the annual expected excess return on equity is reported. This is taken to be

$$\sqrt[T]{\sum_{s \in \Sigma_T} \left(\frac{(1-\gamma)\left(A_T^s - L_T^s\right) + \gamma E_T^s}{E_T^s} \right)} - 1$$

at each decision time. The annual actual excess return on equity is also reported. This is taken to be

$$\sqrt[T]{\frac{(1-\gamma)\left(A_T - L_T\right) + \gamma E_T}{E_T}} - 1 \,.$$

The expected cost of the guarantee is also reported. This is taken to be the expected present value of the final equity, after deduction of the regulatory equity or equity at the start, viz.:

$$\sum_{s \in \Sigma_T} \left(\frac{E_T^s}{\prod_{t=1}^{T}\left(1 + r_{t,t+1/12}^{s^{v(t)}}\right)} - E_0 \right) p^s;$$

where

$$\left(s_t^{v(t)}, s_{t+1/12}^{v(t+1/12)} \right)$$

is a predecessor- and successor-state pair as defined in section 2.1. The actual cost of the guarantee, is taken to be

$$\left(\frac{E_T}{\prod_{t=1}^{T}\left(1 + r_{t,t+1/12}\right)} - E_0 \right).$$

3.3.2   Figure 2 shows the expected return on equity at decision times and the actual return on equity for different levels of the minimum guarantee, where $\alpha = 0,5$. The model underestimates the return on equity, but the expected return on equity improves as more data become available for model estimation after the first decision time. The actual return on equity increases as the minimum guarantee increases up to 13%; after 13% it decreases as the minimum guarantee increases.

3.3.3   Figure 3 shows the expected cost of the guarantee at the decision times and the actual cost of the guarantee, where $\alpha = 0,5$. The model firstly overestimates the cost of the guarantee and as more data become available for model estimation, after

the first decision time, the expected cost of the guarantee improves. For a minimum guarantee of less than 9% a year the model requires no additional equity. As the minimum guarantee increases above 13% a year, the amount of equity required increases.

Figure 2: Shareholders' annual excess return on equity for different levels
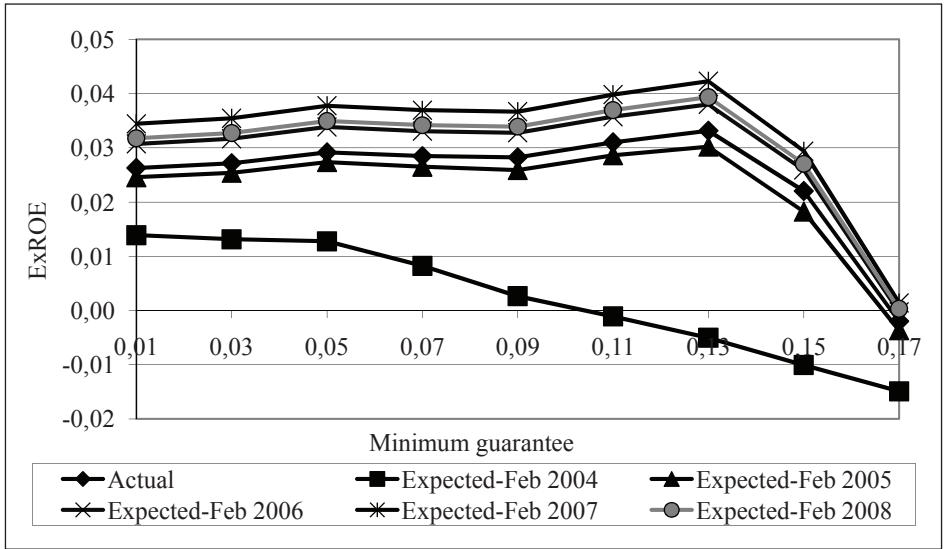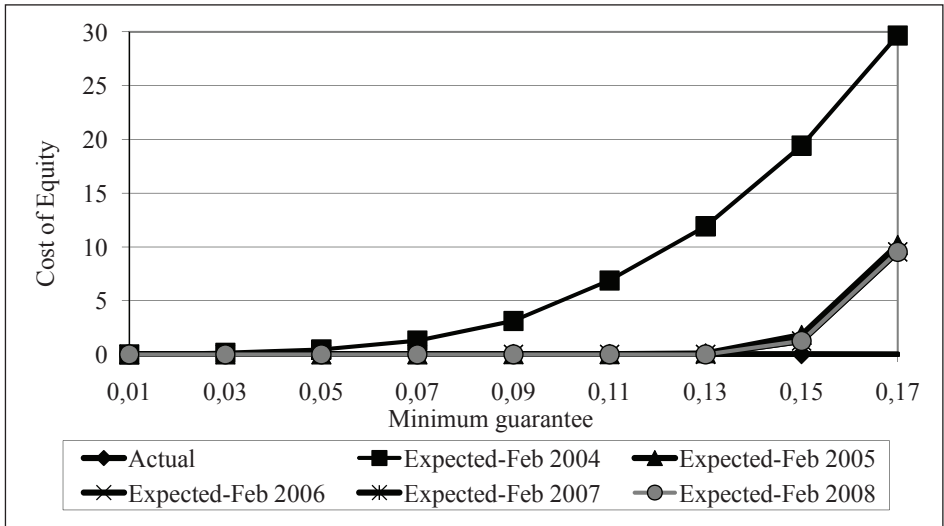of minimum guarantee

Figure 3: Cost of the guarantee for different levels of minimum guarantee

3.3.4   It is expected that the expected return on equity will decrease as the minimum guarantee increases, which is in contrast to the results. The increase in the expected return on equity if the minimum guarantee increases to 13% a year can be explained from the zero cost of the guarantee for minimum guarantees of less than 13% a year. Therefore a higher excess return on equity can be achieved for higher levels of minimum guarantee if this comes at no extra cost to the shareholders. If a cost is incurred to ensure a certain level of minimum guarantee, the expected return on equity will decrease if the minimum guarantee increases.

3.3.5   Figure 4 shows the performance of the asset and liability accounts at minimum guarantees of 1%, 9% and 15% a year for $\alpha = 0,5$. The asset level stays above the liability level over the entire period. Regular bonuses are paid up to a minimum guarantee of 15% a year, and regular bonuses decrease as the level of minimum guarantee increases. This is because the average benchmark rate, the rate used to determine the regular bonus payments, is about 8% a year. Thus for lower levels of minimum guarantee, the amount of regular bonus payments declared will be higher.

3.3.6   Consiglio *et al.* (2006) specify regular bonuses in return form. That is more realistic than the formulation of discrete annual payments, which is used here in order to keep the problem linear. Consiglio *et al.* (2006) assume that the bonus return, RB_t^s, that is declared at time t will stay constant throughout the remainder of the term, giving the terminal liability value as:

$$L_T^s = L_t^s e^{g(T-t)} e^{RB_t^s(T-t)}.$$

With all other assumptions staying constant the regular bonus yields:

Figure 4: Asset and liability account at minimum guarantees of 1%, 9% and 15% a year
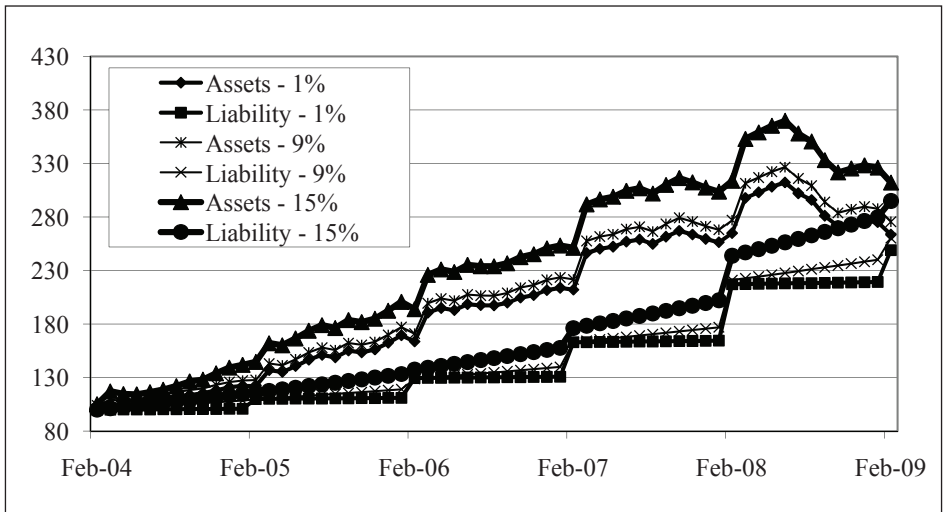
$$RB_t^s = \max\left[\frac{1}{(T-t)}\ln\left(\frac{\gamma(1-\beta)A_t^{b,s}e^{r_{t,b}^s(T-t)}}{(\beta+\gamma(1-\beta))L_t^s e^{g(T-t)}}\right),0\right].$$

3.3.7   The liability process proposed by Consiglio *et al.* (2006) has also been implemented here and included in the back-testing performance results. Figure 5 shows that the discrete approximation of bonuses mimics the more realistic approach of Consiglio *et al.* (2006). Recall that the approach adopted here has the added advantage of keeping the overall problem linear, which allows the inclusion of more realistic portfolio-management constraints.

3.3.8   Figure 6 shows the first-stage optimal asset allocation at the (forward-rolling) rebalancing times for different levels of the minimum guarantee with α = 0,5. The asset allocation for February 2004 seems consistent. At reasonable levels of minimum guarantee the portfolio is less aggressive and allocates lower proportions of asset share to the risky asset. As the level of minimum guarantee increases, more asset share is allocated to the risky asset, up to 30% of the portfolio wealth. This is a result of the portfolio composition constraints, which are set to restrict the proportion of asset share invested in the equity index to 30%. Also, at higher levels of guarantee, more asset share is allocated to long-term bonds. After the first stage the portfolio invests more asset share in the risky asset for lower levels of minimum guarantee. This is due to the higher benchmark rate, and is necessary in order to pay bonuses. Again the proportion of asset share invested in the risky asset is restricted to 30%. If this restriction is lifted, greater proportions of asset share will be allocated to the risky asset. The asset allocation also does not change dramatically from one year to the next.

3.3.9   Figure 7 shows the expected return on equity at rebalancing times and the actual return on equity for different levels of risk-aversion at minimum guarantees

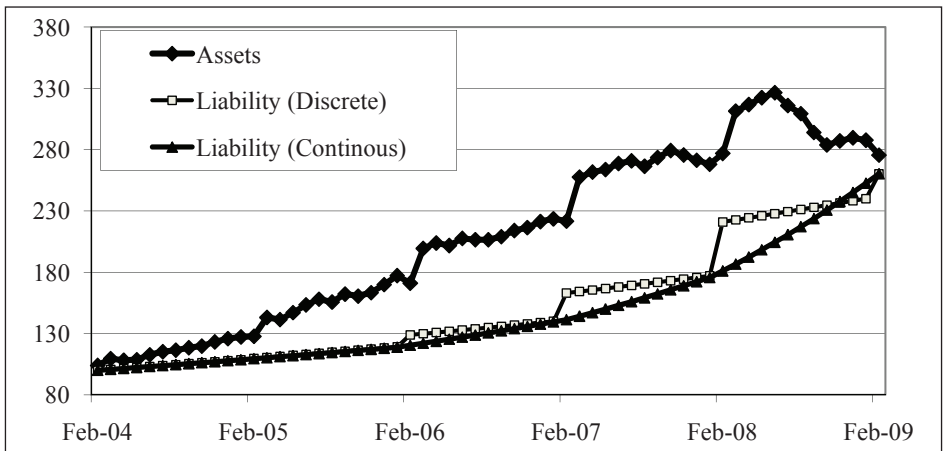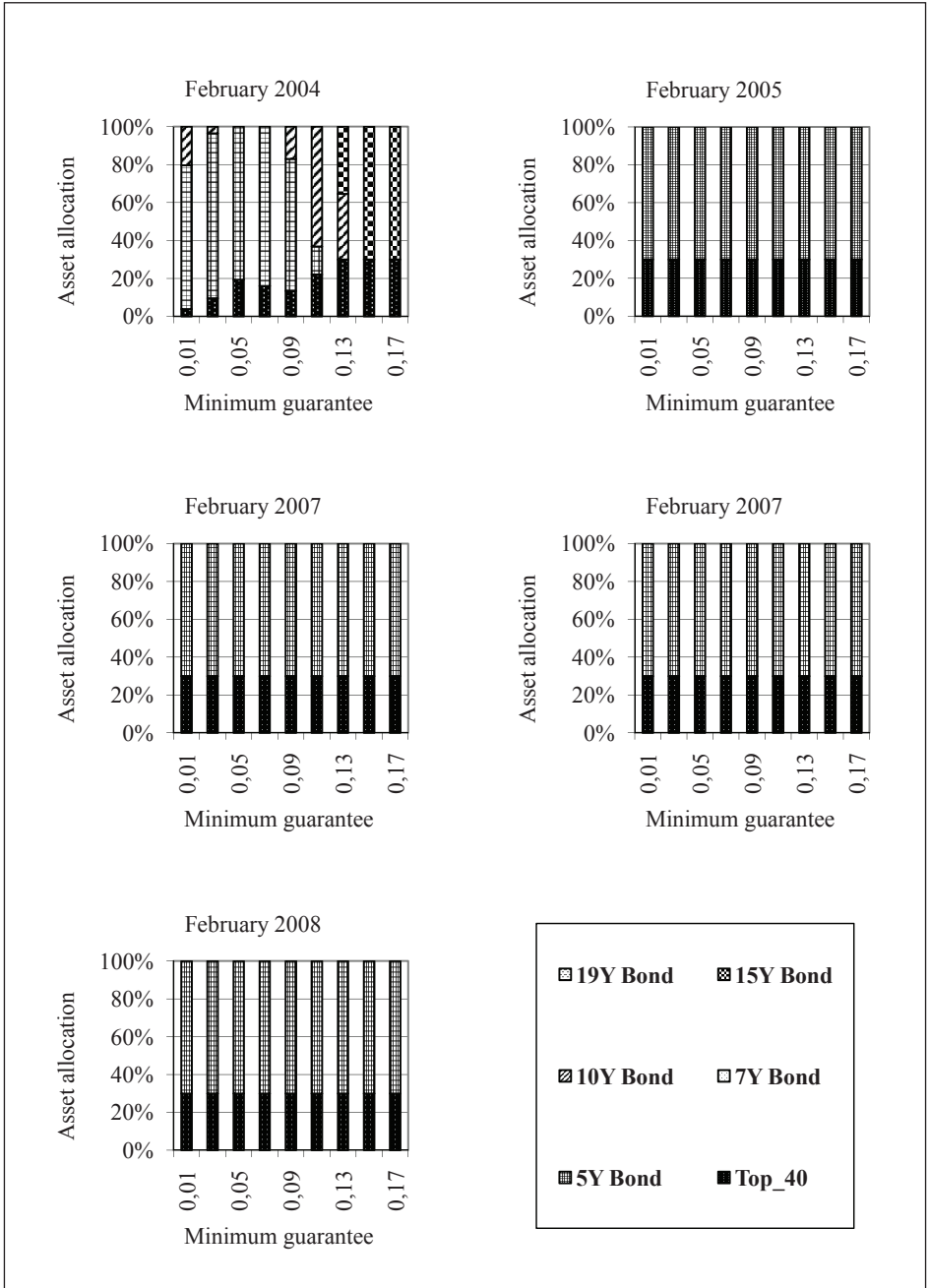Figure 5: Liabilities with different bonus options at a minimum guarantee of 1% a year

Figure 6: Asset allocation for different levels of minimum guarantee

of 9% and 15% a year. As before, the model underestimates the return on equity and the expected return on equity improves as more data become available. The expected return on equity decreases as the level of risk-aversion increases. For a minimum guarantee of 9% a year the expected return on equity remains constant as the risk-aversion level moves from 0 to 0,8 and then suddenly drops in the most risk-averse case. For a minimum guarantee of 15% a year the expected return on equity remains constant as the risk-aversion level moves from 0 to 0,4 and then decreases.

3.3.10   Figure 8 shows the expected cost of the guarantee at the decision times and the actual cost of the guarantee. The model firstly overestimates the cost of guarantee and as more data become available the expected cost of guarantee improves. For a minimum guarantee of 9% a year the expected cost of the guarantee decreases as the level of risk-aversion increases. For a minimum guarantee of 15% a year, the expected cost of the guarantee increases as the level of risk-aversion increases from 0,4 to 1.

Figure 7: Shareholders' annual excess return on equity for
different levels of risk-aversion
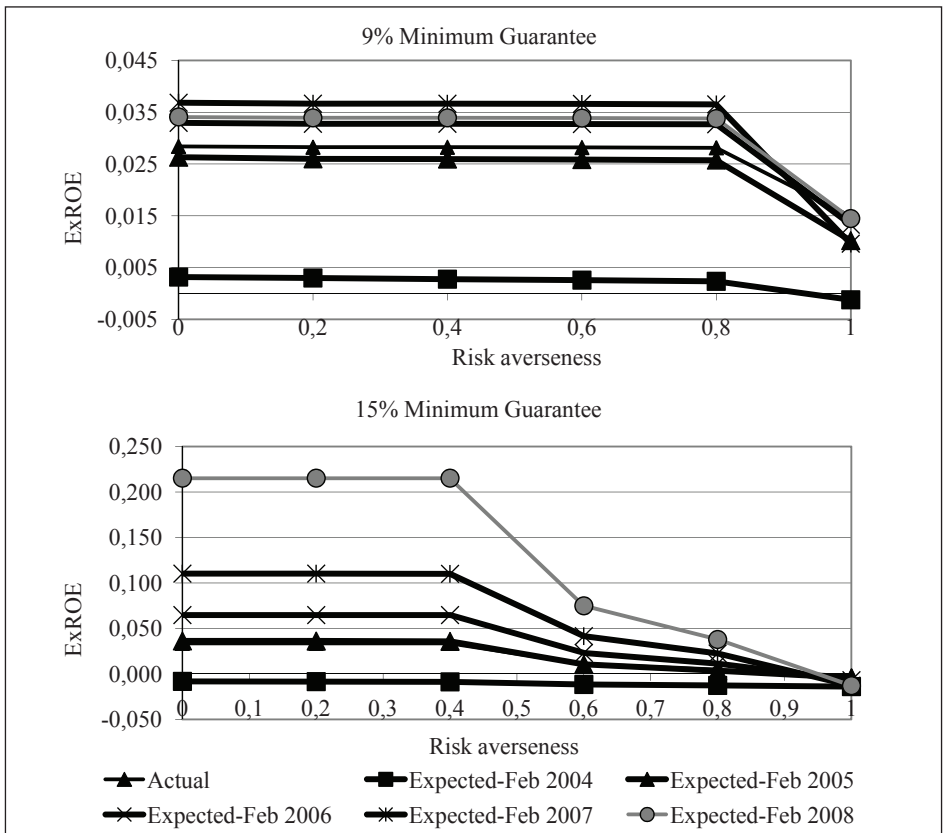
3.3.11   As extra equity is now required to achieve a minimum guarantee of 9% a year, the expected return on equity will remain constant if the expected cost of the guarantee is included in the objective function (i.e. $\alpha < 1$). If only the shortfall is minimised (i.e. $\alpha = 1$) a lower expected return on equity will be achieved. This is because, at a minimum guarantee of 9% a year, the actual cost of the guarantee is zero for all levels of risk-aversion. As a cost is incurred for a minimum guarantee of 15% a year if the level of risk-aversion increases from 0,4 to 1, the expected return on equity will decrease. A more risk-averse portfolio at much higher levels of minimum guarantee will require extra equity, as more importance is given to the shortfall of the portfolio.

3.3.12   Figure 9 shows the performance of the asset and liability accounts at risk-aversion levels of 0, 0,6 and 1 for minimum guarantees of 9% and 15% a year. The asset level stays above the liability level over the entire period. At risk-aversion levels of 0 and 0,6 the model tends to be more aggressive and at the level of 1 the model is more conservative. As mentioned above, for a minimum guarantee of 9% a year, a zero cost

Figure 8: Cost of the guarantee for different levels of risk-aversion
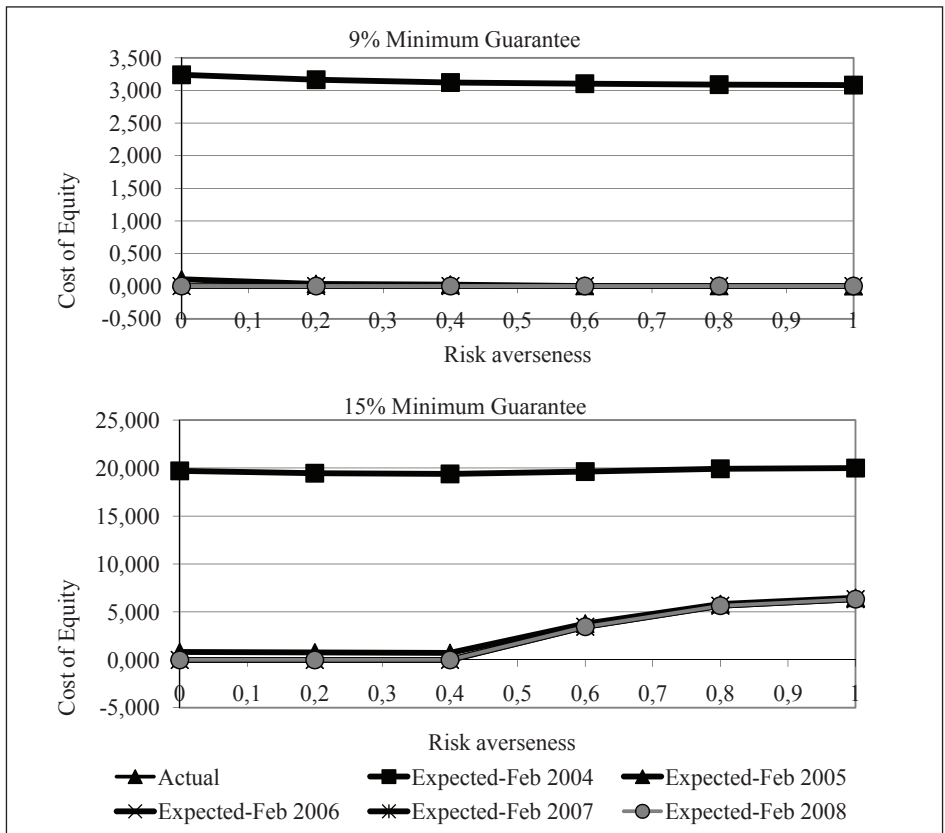
of the guarantee is achieved for all levels of risk-aversion. Because of the zero cost of the guarantee, the performance of the asset and liability accounts, as well as the payment of bonuses, is constant for levels of risk-aversion less than 1. If only the shortfall is minimised (i.e. $\alpha = 1$), the portfolio pays lower bonuses and achieves a lower level of assets at the final time horizon. The portfolio, however, still achieves the minimum guarantee of 9% a year at no extra cost. For a minimum guarantee of 15% a year, extra equity is needed for higher levels of risk-aversion and a lower expected return on equity is achieved. This is seen in Figure 9, where it is shown that the model tends to be more conservative from the risk-aversion level of 0,6.

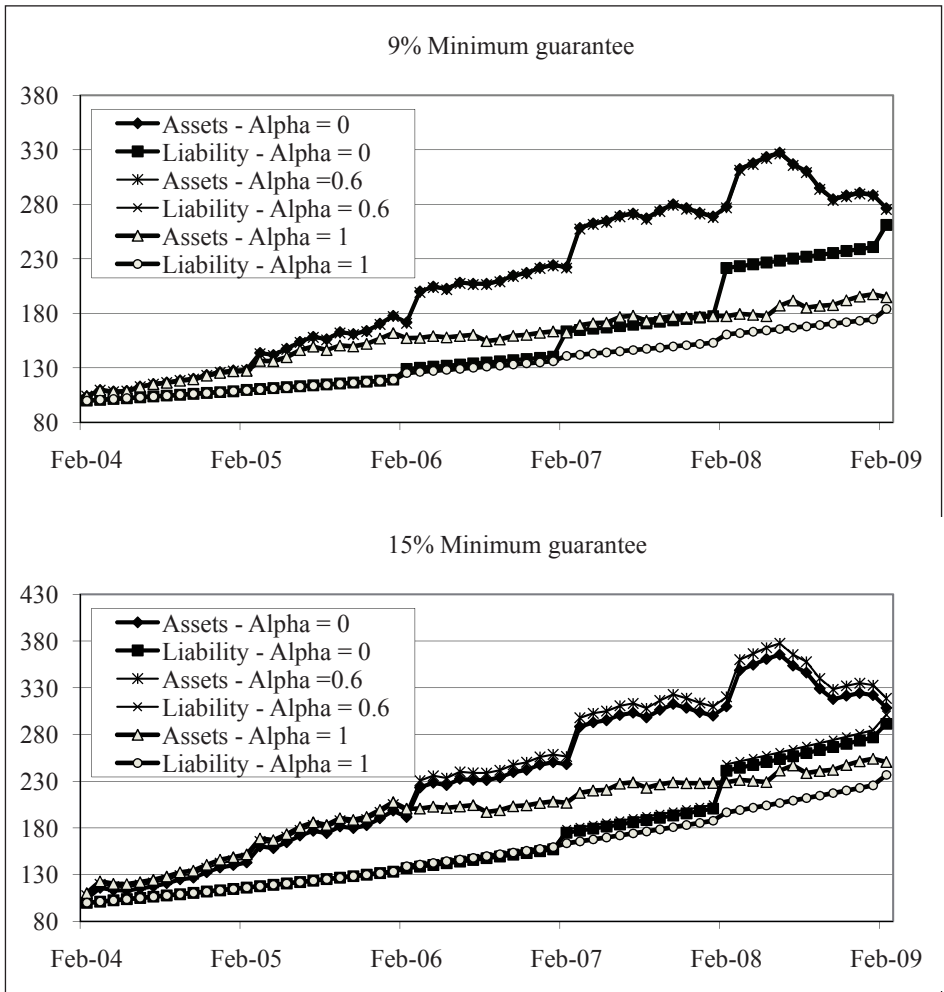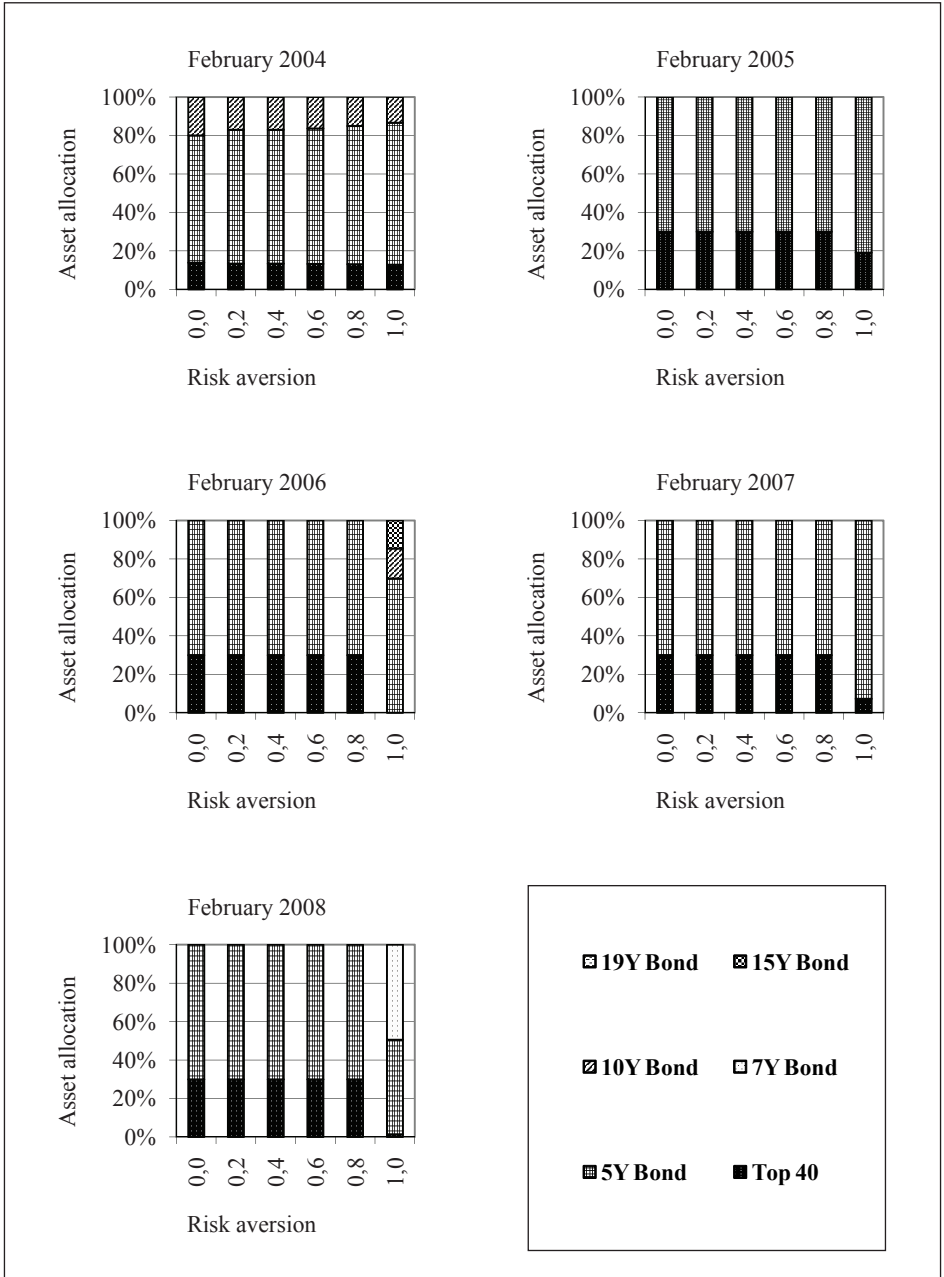Figure 9: Asset and liability accounts at different levels of risk-aversion

Figure 10: Asset allocation at different levels of risk-aversion
for a minimum guarantee of 9% a year

3.3.13   Figure 10 shows the first-stage optimal asset allocation at rebalancing times for different levels of risk-aversion for a minimum guarantee of 9% a year. As the level of risk-aversion increases the portfolio is more conservative and allocates a lower proportion of asset share to the risky asset; this is also apparent from the above discussion. In the most risk-averse situation, where only the shortfall is minimised (i.e. $\alpha = 1$), a much lower proportion of asset share is invested in the risky asset. Furthermore, the asset allocation does not change dramatically from one year to the next. The asset allocation for a minimum guarantee of 15% a year does not differ much from the asset allocation for a minimum guarantee of 9% a year, except that more asset share is invested in longer-term bonds in the first stage (see Figure 6).

4.   CONCLUSION

4.1   The model presented above is a multi-stage dynamic stochastic-programming model for the integrated asset and liability management of insurance products with guarantees that minimise the down-side risk of these products. Regular bonus payments have been included and the optimisation problem has been kept linear, so as to facilitate the modelling of the rebalancing of the portfolio at future decision times. Also, by keeping the optimisation problem linear, the model is flexible enough to take into account portfolio constraints such as the prohibition of short-selling, transaction costs and coupon payments. It has also been shown that the bonus assumption mimics the more realistic assumptions proposed by Consiglio *et al.* (2006).

4.2   Furthermore, the features of the model at different levels of minimum guarantee and different levels of risk-aversion have been shown. The back-tested results show that the proposed stochastic-optimisation framework successfully deals with the risks created by the guarantee and declaration of bonus payments. As Consiglio *et al.* (2006) have shown, the model can also be used for analysing the investment decision made by the insurer.

4.3   The with-profit guarantee funds discussed in this paper are operated on a 90–10 basis: the policyholders benefit in 90% of the growth in asset share and the shareholders in 10%. These products do, however, occur to a lesser extent in the South African insurance market. One of the main goals of this paper was to extend the work of Dempster *et al.* (2006) and Consiglio *et al.* (2006). There remains wide scope, however, for further research of these products in the South African insurance market.

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APPENDIX A

THE LINEAR PROGRAMMING MODEL

A.1 Given a set of scenarios the stochastic programming model results in a linear programme.

A.2 All variables are indexed by a time index t which takes values over the times $t = 0, \frac{1}{12}, \frac{2}{12}, \ldots, T$, and states index s from the set $\Sigma_t = \left\{ s_t^v \mid v = 1, 2, \ldots, S_t \right\}$.

The time sets are:
- all times $T^{total} = \left\{ 0, \frac{1}{12}, \frac{2}{12}, \ldots, T \right\}$;

- decision times $T^d = \left\{ 0, 1, 2, \ldots, T-1 \right\}$;

- intermediate times $T^i = T^{total} \setminus T^d$; and

- coupon payment times between decision times $T^c = \left\{ \frac{1}{2}, \frac{3}{2}, \ldots, T - \frac{1}{2} \right\}$.

Other index sets are: stock indices SI; government bonds with maturity τ $B=\{B_\tau\}$; all instruments $I=SI \cup B$.

A.3 The parameters in the linear programme are: δ_{B_τ}, F_{B_τ}, $r_{t,\tau}^s$, $g, \rho, r_{t,b}^s, \gamma, \beta, P_{t,i}^{a,s}, P_{t,i}^{b,s}$, f_a, f_b and p_t^s.

A.4 The resulting linear programme is:

$$\max_{\left\{ x_{t,i}^s ; y_{t,i}^s ; z_{t,i}^s ; A_t^s ; L_t^s ; E_t^s ; c_t^s ; SF_t^s ; RB_t^s ; TB_t^s \right\}} \left\{ (1-\alpha) \sum_{s\in\Sigma_T} p^s (1-\gamma)\left(\left(A_T^s - L_T^s \right) - E_T^s \right) - \alpha \sum_{t\in T^{total}} \sum_{s\in\Sigma_t} p^s \frac{SF_t^s}{\left| T^{total} \right|} \right\}.$$

subject to:

$$\sum_{i\in I} P_{0,i}^{a,s} x_{0,i}^s \left(1 + f_a \right) = A_0^s , \text{ for } t\in\{0\} \text{ and } s\in\Sigma_t;$$

$$\sum_{i\in I} P_{t,i}^{b,s} y_{t,i}^s \left(1 - f_b \right) + \sum_{i\in I\setminus\{SI\}} \tfrac{1}{2}\delta_i F_i y_{t,i}^s + c_t^s = \sum_{i\in I} P_{t,i}^{a,s} x_{t,i}^s \left(1 + f_a \right) \text{ for } t\in T^d\setminus\{0\} \text{ and}$$

$$s\in\Sigma_t;$$

$$x_{t,i}^s \geq 0, \text{ for all } i\in I, t\in T^{total}\setminus\{T\} \text{ and } s\in\Sigma_t;$$

$$y_{t,i}^s \geq 0, \text{ for all } i\in I, t\in T^{total}\setminus\{0\} \text{ and } s\in\Sigma_t;$$

$$z_{t,i}^s \geq 0, \text{ for all } i\in I, t\in T^{total}\setminus\{T\} \text{ and } s\in\Sigma_t;$$

$z_{0,i}^s = x_{0,i}^s$, for $t \in \{0\}$ and $s \in \Sigma_t$;

$z_{t,i}^s = z_{t,i}^{s-} + x_{t,i}^s - y_{t,i}^s$, for $i \in I$, $t \in T^{total} \setminus \{0\}$ and $s \in \Sigma_t$;

$x_{t,i}^s = y_{t,i}^s = 0$, for $i \in I$, $t \in T^i \setminus T^c$ and $s \in \Sigma_t$;

$x_{t,i}^s = \dfrac{\frac{1}{2} \delta_i F_i z_{t,i}^{s-}}{P_{t,i}^{a,s} (1 + f_a)}$, for $i \in I \setminus \{SI\}$, $t \in T^c$ and $s \in \Sigma_t$;

$y_{t,i}^s = 0$, for $i \in I \setminus \{SI\}$, for $t \in T^c$ and $s \in \Sigma_t$;

$x_{t,SI}^s = 0, y_{t,SI}^s = 0$, for $t \in T^c$ and $s \in \Sigma_t$;

$A_0^s = L_0 + E_0^s$, for $t \in \{0\}$ and $s \in \Sigma_t$;

$A_t^s = \sum_{i \in I} P_{t,i}^{a,s} z_{t,i}^s (1 + f_a)$, for $t \in T^{total} \setminus \{T\}$ and $s \in \Sigma_t$;

$A_T^s = \sum_{i \in I} P_{T,i}^{b,s} z_{T-\frac{1}{12},i}^{s-} (1 - f_b) + \sum_{i \in I \setminus \{SI\}} \frac{1}{2} \delta_i F_i z_{T-\frac{1}{12},i}^{s-}$, for $s \in \Sigma_t$;

$L_0^s = L_0$, for $t \in \{0\}$ and $s \in \Sigma_t$;

$L_t^s = L_{t-\frac{1}{12}}^{s-} e^{\frac{1}{12}g} + RB_t^s$, for $t \in T^{total} \setminus \{T\}$ and $s \in \Sigma_t$;

$E_0^s = c_0^s$, for $t \in \{0\}$ and $s \in \Sigma_t$;

$E_t^s = E_{t-\frac{1}{12}}^{s-} e^{r_{t-\frac{1}{12},t}^{s-}} + c_t^s$, for $t \in T^{total} \setminus \{T\}$ and $s \in \Sigma_t$;

$RB_t^s >= \dfrac{\gamma(1-\beta) A_t^{b,s} e^{r_{t,b}^s(T-t)} - (\beta + \gamma(1-\beta)) L_{t-\frac{1}{12}}^{s-} e^{g(T-t+\frac{1}{12})}}{(\beta + \gamma(1-\beta))\left(\dfrac{e^{g(T-t)}-1}{e^g - 1} + e^{g(T-t)}\right)}$, for $t \in \left(T^d \setminus \{0\}\right) \cup \{T\}$, and

$s \in \Sigma_t$;

$RB_t^s >= 0$ for $t \in \left(T^d \setminus \{0\}\right) \cup \{T\}$, and $s \in \Sigma_t$;

$RB_t^s = 0$ for $t \in \left(T^i \cup \{0\}\right) \setminus \{T\}$, and $s \in \Sigma_t$;

$SF_0^s + L_0 >= (1+\rho) L_0^s$, for $t \in \{0\}$ and $s \in \Sigma_t$;

$SF_t^s + \sum_{i \in I} P_{t,i}^{b,s} z_{t-\frac{1}{12},i}^{s-} (1 - f_b) + \sum_{i \in I \setminus \{SI\}} \frac{1}{2} \delta_i F_i z_{t-\frac{1}{12},i}^{s-} >= (1+\rho) L_{t-\frac{1}{12}}^s e^{g\frac{1}{12}}$, for

$t \in \left(T^{total} \setminus \{0\}\right)$, and $s \in \Sigma_t$;

$SF_t^s >= 0$ for $t \in T^{total}$, and $s \in \Sigma_t$;

$c_t^s = SF_t^s$ for $t \in \left(T^d \right) \cup \{T\}$ and $s \in \Sigma_t$

$c_t^s = 0$ for $t \in T^i \setminus \{T\}$, and $s \in \Sigma_t$; and

$$\frac{\sum_{i \in \tilde{I}} P_{t,i}^{a,s} z_{t,i}^s}{W_t^s} < \pi_{\tilde{I}} \text{ for } \tilde{I} \subset I.$$