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AN EMPIRICAL APPROACH TO IMMUNIZATION IN SOUTH AFRICA

By AJ Maitland

ABSTRACT
This paper presents an empirical approach to immunizing South African nominal liabilities in the presence of non-parallel yield-curve shifts. The results are compared with immunization strategies based on Fisher-Weil duration and illustrate the value in immunizing against non-parallel shifts.

KEYWORDS
Immunization; South Africa; principal components; nominal liabilities; arbitrage

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1. INTRODUCTION

1.1 In his seminal paper, Redington (1952) developed a theory of immunization under which the cash flow stream generated by an investor's nominal liabilities is protected against small changes in the valuation rate of interest. The immunization strategy is achieved by selecting a portfolio of assets with present value and Macaulay duration equal to those of the liabilities and convexity greater than that of the liabilities. Fisher & Weil (1971) adapted the measure of a bond's price elasticity developed by Macaulay (1938) to incorporate a term structure that is not flat and developed an immunization strategy based on this measure of duration. It is well known that these immunization strategies are valid only if term-structure shifts are parallel.

1.2 Consider representing the yields at d points along the yield curve as a single point in d dimensions, $\boldsymbol{c} = (c_1, c_2, \ldots, c_d)$, each dimension corresponding to the maturity of one of the chosen points. A parallel shift in yields results in a new point in d dimensions equal to the original point plus a constant addition to each coordinate. Clearly, the points resulting from any parallel shift in yields will lie along the straight line passing through these two points, $\boldsymbol{c} + k\mathbf{1}$, where $\mathbf{1}$ is the unit vector and $k \in \Re$. Hence, parallel shifts are restricted to a shift in the direction of the unit vector.

1.3 A more general and empirically plausible model allows the term structure to shift in multiple directions. Using factor analysis, Litterman & Scheinkman (1991) provide empirical evidence that three factors are required to explain the term structure of US interest rates. Similar results are found by Sherris (1994) using Australian yield-curve data, D'Ecclesia & Zenios (1994) using Italian bond-market data, Bühler & Zimmermann (1996) using Swiss and German interest rates and Feldman *et al* (1998) using real and nominal UK forward rates. A principal components analysis (PCA) on the covariance matrix of monthly changes in par yields for the JSE-Actuaries Yield Curve indicates that the first principal component explains 92,8% of the total variability and the first two principal components together explain 97,3%, while three principal components are required to explain 98,4% of the total variability, see Maitland (2000). More detail is given in Section 2 below.

1.4 Reitano (1991, 1992) developed a general framework for hedging interest-rate uncertainty that immunizes against term-structure shifts in multiple directions. Vectors whose elements correspond to rate changes at different maturity dates describe the shift in each direction. Each additional shift direction specified for immunization imposes at least one extra constraint to the portfolio selection problem. Barber & Copper (1996) use PCA to estimate the minimum number of fundamental directions in which to anticipate spot-rate changes. Unlike the decomposition of yield-curve shifts into parallel shifts, stylized slope changes and stylized curvature changes, or into key rate durations (see Ho, 1992), PCA provides the minimum number of components to explain any desired proportion of the total variability. Further, each subsequent principal component (PC) provides the direction of maximum variability orthogonal to the previous set (also referred to hereafter as a *fundamental* direction). Hence, the largest shifts are immunized as completely as possible and the effect of non-infinitesimal movements in any fundamental direction can be considered independently of non-infinitesimal movements in any other fundamental direction.

1.5 In this paper, the principal components analysis in Maitland (2000) is updated to include data to May 2000 and the work of Barber & Copper (1996) is extended by optimizing the immunized portfolio subject to principal component shifts. Hence, the suggested approach does not rule out the possibility of arbitrage. This is discussed further in Section 4. Hedging strategies immunized to an increasing number of principal component shifts are compared with hedging strategies based on Fisher-Weil duration and the risk of ignoring shape changes is illustrated using the yield curves of 30 September, 31 October and 30 November 1998. Finally, the optimization models introduced in Section 4 are used to identify "conditional" arbitrage opportunities in the September 1998 yield curve. The availability of conditional arbitrage opportunities then suggests a method for choosing the number of principal-component constraints required for immunization. This is discussed further in Section 6.

2. PRINCIPAL COMPONENTS OF THE JSE-ACTUARIES YIELD CURVE

2.1 Let $\mathbf{z}$ be a random d-vector with mean μ and covariance matrix Σ, and let $\mathbf{T} = (\mathbf{t}_1,\mathbf{t}_2,\ldots,\mathbf{t}_d)$ be an orthogonal matrix (i.e. $\mathbf{TT}'=\mathbf{T}'\mathbf{T}=\mathbf{I}$) such that $\mathbf{T}'\Sigma\mathbf{T} = \mathrm{diag}(\lambda_1,\lambda_2,\ldots,\lambda_d)$, where $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_d$ are the eigenvalues of Σ. If $\mathbf{y} = \mathbf{T}'(\mathbf{z}-\mu)$, then $y_j = \mathbf{t}_j'(\mathbf{z}-\mu)$ is called the *j*th principal component score of **z** and is the orthogonal projection of $\mathbf{z}-\mu$ in the direction $\mathbf{t}_j$, the *j*th principal component (Kendall, Stewart & Ord, 1983; 43.4). Hence, the scores at time *t* are that linear combination of the principal components required to reconstruct the yield curve at that time. Unpacking the matrix notation and letting $\mathbf{z}_k$ represent the *k*th realization of the random vector **z**, we can see that

$$\mathbf{z}_k = \mu + y_{1,k}\mathbf{t}_1 + \ldots + y_{d,k}\,\mathbf{t}_d,$$

from which it becomes clear that the variance of $\mathbf{z}_k$ is

$$V(\mathbf{z}_k) = V(y_{1,k})\,\mathbf{t}_1\mathbf{t}_1^{\mathrm{T}} + \ldots + V(y_{d,k})\,\mathbf{t}_d\mathbf{t}_d^{\mathrm{T}},$$

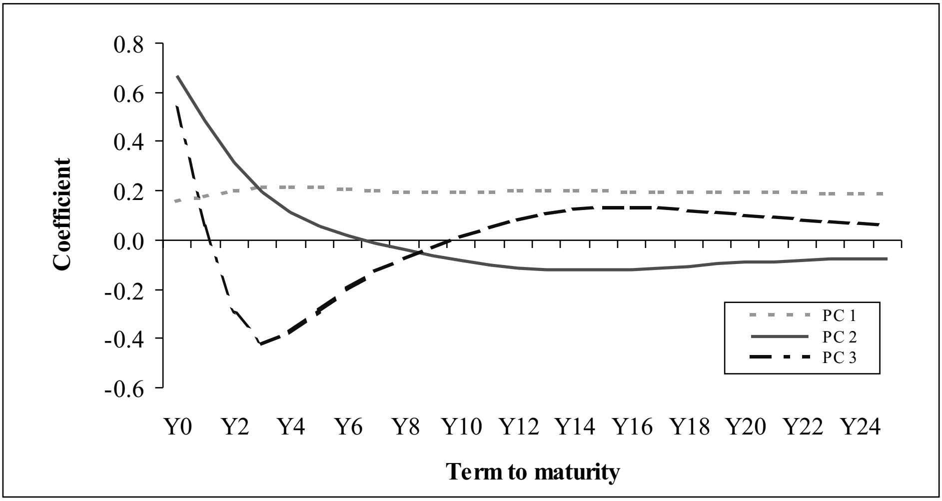
since the eigenvectors are orthogonal. Since $V(\mathbf{y}) = \mathbf{T}'\Sigma\mathbf{T} = \mathrm{diag}(\lambda_1,\lambda_2,\ldots,\lambda_d)$, the scores are uncorrelated. Truncating this series to include only the first *n* terms gives an approximation to $\mathbf{z}_k$, the accuracy of which depends on the cumulative variability explained by those terms. The decision whether or not to include one more term is based on the incremental variability explained by the additional term.

2.2 Table 1 indicates the additional and cumulative proportions explained by the first ten principal components from the covariance matrix of monthly changes in par yields for the JSE-Actuaries Yield Curve for the period February 1986 to May 2000. It is clear that the first two principal components describe most of the variability of term-structure shifts but that immunization against higher-order shifts may be desired in order to further reduce risk. The last two columns of Table 1 give the months in which the minimum and maximum scores occurred for each of the first ten principal components and indicate months in which extreme exposure to the various risk factors could give cause for concern.

TABLE 1. Variability explained by the first ten principal components

PC No.	Eigenvalues	Incremental variability (%)	Cumulative variability (%)	Minimum score	Maximum Score
1	9,928712	92,415	92,415	Sep 98	Aug 98
2	0,519766	4,838	97,253	Sep 86	Jun 98
3	0,128539	1,196	98,450	Jan 88	Jun 86
4	0,088378	0,823	99,272	Jul 90	Oct 91
5	0,057733	0,537	99,810	Aug 86	Apr 86
6	0,016355	0,152	99,962	Jul 98	Jun 98
7	0,002045	0,019	99,981	Jan 90	Oct 91
8	0,001200	0,011	99,992	Feb 86	May 00
9	0,000346	0,003	99,995	Feb 86	Dec 90
10	0,000141	0,001	99,997	Feb 86	Jul 90

FIGURE 1. Coefficients for the first three principal components



2.3 Figure 1 illustrates the coefficients of the first three principal components by term to maturity. The first principal component affects all maturities by similar amounts and in the same direction. It can be interpreted as a level shift factor but not as a parallel shift factor since the coefficients are unequal. The second principal component has an opposite effect on short and long yields and can be viewed as a slope change factor or twist. The third principal component has a negative effect on medium yields and a positive effect on short- and long-term yields and hence can be interpreted as a curvature factor or butterfly. Figures 2 to 4 illustrate the principal component scores for the first three principal components of yield-curve changes for the period February 1986 to May 2000.

FIGURE 2. Principal component 1 scores for yield changes (Feb 1986–May 2000)

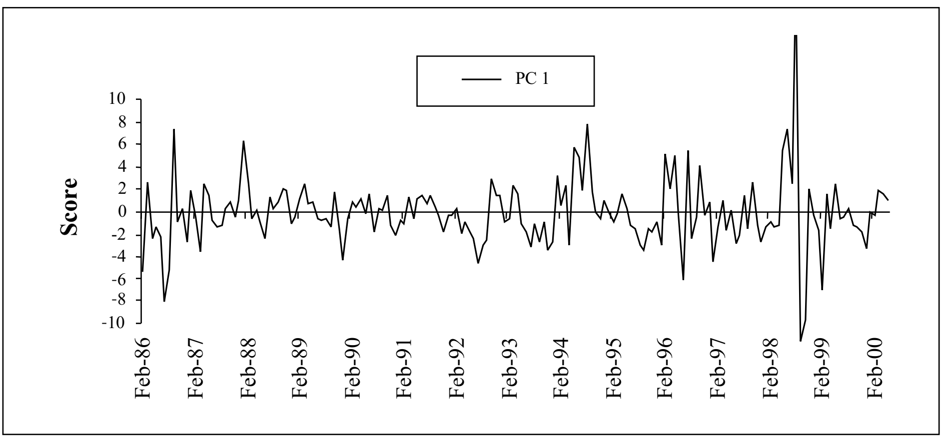


FIGURE 3. Principal component 2 scores for yield changes (Feb 1986–May 2000)

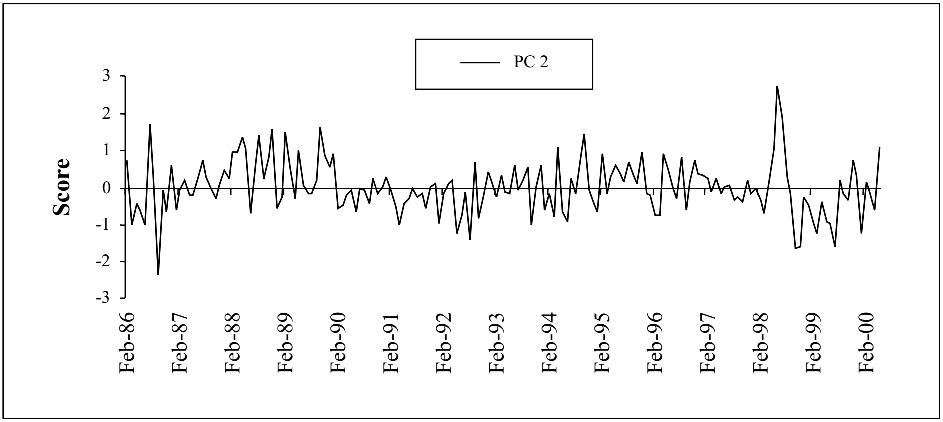
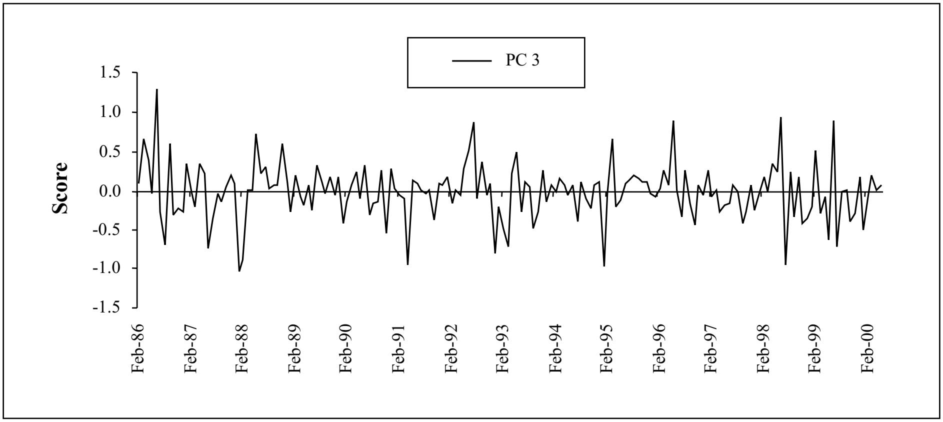


Figure 4. Principal component 3 scores for yield changes (Feb 1986–May 2000)



2.4 Estimates of variances, covariances and correlations can be very sensitive to outliers and so we can expect principal components to have the same sensitivity. The extreme scores for the first principal component between August and October 1998 shown in Figure 2 and the corresponding large changes in the level of the yield curve suggest the need for a PCA for sub-periods of the data. A number of alternative sub-periods have been considered and the results of the full period appear to be relatively robust to the choice of sub-period. In particular, the principal components are robust to outliers from August to November 1998, indicating that the shocks experienced over this period were of the same nature as previous shocks, despite their increased magnitude. The incremental proportions of the total variability explained by each of the first three principal components are also almost identical to those based on data to December 1998 and discussed in Maitland (2000).

3. IMMUNIZATION USING PRINCIPAL COMPONENTS

3.1 Given the par yield curve at time t, $\mathbf{x}_t$, and assuming that $\mathbf{z}_t=\mathbf{x}_t-\mathbf{x}_{t-1}$, the par yield curve one month forward is given by

$$\mathbf{x}_{t+1} = \mathbf{x}_t + \sum_{j=1}^{d} y_{j,t}\mathbf{t}_j + \mu \tag{1}$$

3.2 In general, d principal components are required to reproduce all possible term-structure movements but the first $n < d$ principal components may explain a sufficient proportion of shifts. In this case,

$$\mathbf{x}_{t+1} \approx \mathbf{x}_t + \sum_{j=1}^{n} y_{j,t}\mathbf{t}_j + \mu \tag{2}$$

3.3 Barber & Copper (1996) immunize against spot-curve changes but it is equally possible to immunize against changes in the par-yield curve. It may be theoretically more transparent to analyse spot rates than par yields but the majority of domestic bonds currently in issue have coupons in excess of 10%, so par bonds are more representative of the market than zero-coupon bonds. Further, local market practitioners are often more familiar with par yields than spot rates and the results from a PCA of par yields are more intuitive than those from a PCA of spot rates. More importantly, a PCA analysis of the bootstrapped spot curves indicates that the first n components consistently explain a smaller proportion of the total variability than the corresponding number of components of the par-yield curve. This indicates that a more parsimonious linear model is possible using par yields than spot rates, even though both contain the same information. This is due to the non-linear relationship between par yields and spot rates.

3.4 In the subsequent analysis, it is assumed that par bonds are available at any time for annual maturities between 0 and 25 years and that both liability and coupon cash flows occur at annual intervals. In practice, a liquid market in par bonds at annual maturities along the curve does not exist. The spread of bonds in issue is lumpy (over 50% of the market capitalization being concentrated in just two bonds) so practitioners would probably wish to immunize using corporate debt as well as government bonds. However, for the purposes of this paper, credit and liquidity considerations are ignored. Hence, the empirical nature of the analysis refers to the use of empirical shifts in the yield curve and not to the use of actual bonds available in the market at the time. As a caveat to the subsequent analysis, it should be noted that the JSE-Actuaries Yield Curve is an artificial construct that may poorly reflect the yields of actual bonds traded, which may either disguise true arbitrage opportunities or create their illusion.

3.5 Since the PCA is based on monthly changes in yields, it is necessary to value bonds a month later following the change in yields. To simplify calculations, it is assumed that the clean price of a bond of s years and eleven months is equal to the clean price of an (s+1)-year bond with the same coupon. Hence, capital returns from the roll-down effect over the month are ignored and only those from fundamental shifts in the par yield curve are considered. The effect of this assumption is minor given that bonds are priced at par at the start of the month and that the yield on a bond of s years and eleven months is almost identical to the yield on an (s+1)-year par bond. Apart from fundamental shifts, it is assumed that coupon income represents the only income generated over the month.

3.6 Let $B_{s,t}$ represent the present value (PV) at time t of a par bond maturing s years hence. If bonds are available at maturities from 0 to 25 years, then

$$V_{A,t} = \sum_{s=0}^{25} \alpha_{s,t} B_{s,t} \tag{3}$$

where $V_{A,t}$ represents the present value of the assets at time t and $\alpha_{s,t}$ represents the holding in bond $B_{s,t}$. Suppose $L_{s,t}$ represents the PV at time t of a liability cash flow payable s years hence. Then the PV of the liabilities at time t is

$$V_{L,t} = \sum_{s=0}^{25} L_{s,t} \tag{4}$$

3.7 At time t, the assets are immunized with respect to the liabilities relative to shifts described by the first n principal components if (see Barber & Copper, 1996)

$$\frac{\partial V_{A,t}}{\partial y_j} = \frac{\partial V_{L,t}}{\partial y_j} \quad \text{for } j = 1, 2, \ldots, n. \tag{5}$$

3.8 In addition, Barber & Copper impose the wealth constraint:

$$\mathrm{V}_{A,t} = V_{L,t}. \tag{6}$$

4. OPTIMAL IMMUNIZATION

4.1 Since the number of constraints is usually much less than the number of bonds, there exist a variety of portfolios from which to choose. The absolute-match portfolio will always satisfy constraints (5) and (6) but may require short positions in certain par bonds. It may be desirable to impose the non-negativity constraints $\alpha_{s,t} \geq 0 \;\forall\; s \in S$, $S = \{0,1,2,\ldots,25\}$, although a solution under such constraints may not exist. For the moment, we will assume the existence of a feasible solution such that $\alpha_{s,t} \geq 0 \;\forall\; s \in S$.

4.2 One alternative for optimizing the immunized portfolio is to drop the wealth constraint and minimize the capital required. It may be possible to hold a portfolio of assets that actually costs less than the absolute-match portfolio (i.e. $V_{A,t} < V_{L,t}$) and which is also immunized with the respect to the liabilities. Ignoring capital returns from the roll-down effect, the only difference in monthly income generated from the immunized portfolio versus the absolute-match portfolio results from the difference in coupon income. To incorporate the coupon income earned on assets into the selection process, it is necessary to impose the additional constraint that this coupon income equals that from the absolute match. Since the monthly coupon income is proportionate to the par yield, this defines the following linear programming problem:

$$\textit{Minimize} \quad V_{A,t} = \sum_{s=0}^{25} \alpha_{s,t} B_{s,t} \quad w.r.t. \{\alpha_{s,t}\}_{s=0}^{25}$$

$$\textit{subject to} \quad \frac{\partial V_{A,t}}{\partial y_j} = \frac{\partial V_{L,t}}{\partial y_j} \quad \forall\, j = 1,2,\ldots,n$$

$$\alpha_t{}' \mathbf{x}_t = \varpi_t{}' \mathbf{x}_t$$

$$\alpha_t \geq \mathbf{0} \qquad (7)$$

where α_t represents the column vector with elements $\alpha_{s,t}$ and ω_t represents the mix of bonds in the absolute-match portfolio, $\omega_t \geq \mathbf{0}$.

4.3 The optimum asset mix α_t^*, satisfies $1'\alpha_t \leq 1'\omega_t$ since, for $\omega_t \geq \mathbf{0}$, ω_t is a feasible solution. If $\alpha_t^* = \omega_t$, there is no arbitrage. If $\alpha_t^* \leq \omega_t$, $100{*}1'\alpha_t^*$(=A^*, say) represents the reduced capital required to immunize the liabilities. For a system subject only to infinitesimal shifts of the form $\mathbf{t}_1,\mathbf{t}_2, \mathbf{t}_3,\ldots,\mathbf{t}_n$, the asset mix α_t^* is riskless in relation to the liabilities. In other words, for such shifts, the change in the nominal value of the absolute match will equal that in A^* and the coupon income from both portfolios will be the same. This follows from the first two constraints in equation (7). Hence, following such shifts, the *free capital*, defined as the difference between the present value of the liabilities and the assets (i.e. $100*(1'\omega_t - 1'\alpha_t^*) = V_{L,t} - A^* = k$, say), will remain unchanged. (It is important to note that at the end of the month an amount of capital equal to k is still required for there to be sufficient funds to meet the liabilities. However, there are no constraints as to how this temporarily freed-up capital should be invested and the capital could simply be held in bank notes if desired.)

4.4 It is tempting to invest the free capital in the risk-free asset to generate additional funds with certainty (i.e. arbitrage profits conditional on the absence of shifts of the form $\mathbf{t}_{n+1},\ldots,\mathbf{t}_d$) and so improve the funding ratio, but this strategy is not optimal (see Model (8)). Also, since the portfolio is immunized only with respect to shifts of the form $\mathbf{t}_1,\mathbf{t}_2, \mathbf{t}_3,\ldots,\mathbf{t}_n$, the apparent arbitrage profits represented by these additional funds are not generated with certainty, however unlikely such shifts may appear historically. "Conditional arbitrage" is discussed further in Section 6.

4.5 Another alternative for optimizing the immunized portfolio is to reintroduce the wealth constraint and maximize the coupon income. The immunization model then becomes:

$$Maximize\ Coupon\ Income = \alpha_t{}'\mathbf{x}_t \qquad w.r.t.\left\{\alpha_{s,t}\right\}_{s=0}^{25}$$

$$subject\ to \quad \frac{\partial V_{A,t}}{\partial y_j} = \frac{\partial V_{L,t}}{\partial y_j} \quad \forall\ j = 1,2,\ldots,n$$

$$V_{A,t} = V_{L,t}$$

$$\alpha_t \geq \mathbf{0} \tag{8}$$

4.6 If we assume that the value of the assets at time t is equal to the cost of the absolute-match portfolio and that $\alpha_{s,t} \geq 0\ \forall\ s \in S$, the optimum portfolio from Model (8) will generally give a higher return than a composite portfolio comprising the optimum portfolio from (7) together with an amount of capital equal to the free capital invested in the risk-free asset. The reason for this is that the composite portfolio satisfies the constraints of model (8) but forces certain funds into the risk-free asset in contrast to model (8). Again, no arbitrage exists if $\alpha_t^* = \omega_t$.

4.7 In the absence of arbitrage opportunities, it is always possible to solve a system of linear equations to determine an appropriate portfolio. For the models described above, this system of equations is simply the set of constraints specified in the respective linear programming problems. If the number of constraints is less than the number of bonds available, the immunized portfolio will not be unique. Since arbitrage opportunities may arise from time to time, optimization is required to determine the optimum portfolio. For a discussion on the number of constraints to consider, see Section 6.

4.8 The yield to maturity (YTM) might be considered to be an alternative objective function to the coupon income suggested in Model (8). The YTM on a single par bond is equal to its coupon, but the YTM of a portfolio of par bonds is a non-linear function of the YTM of each bond. Although it is possible to obtain a first-order approximation to the portfolio YTM as a linear function of α, this objective function is inappropriate. Since immunization is with respect to monthly changes in yields, bonds must be rebalanced monthly and are unlikely to be held to maturity. Further, unless the optimum portfolio is also the absolute-match portfolio, the immunized portfolio's constraints are violated with the passage of time, even in the absence of yield-curve shifts.

4.9 Model (8) is optimal only if $V_{A,t} = V_{L,t}$. For example, if a surplus is also to be invested in bonds without taking a position on interest-rate movements, the wealth constraint can be replaced with the more general wealth constraint $V_{A,t} = F \cdot V_{L,t}$, where F is the funding ratio. There is then no risk that the funding ratio will change, except to the extent that additional income is received.

4.10 The last constraint, $\alpha_t \geq \mathbf{0}$, in Model (7) and Model (8) is necessary to avoid short positions in certain bonds. So far, it has been assumed that no short positions are required in the absolute-match portfolio, i.e. $\omega_t \geq \mathbf{0}$. If the absolute-match portfolio contains short positions but these are not permitted in the immunized portfolio, a feasible solution to models (7) or (8) may not exist. If short positions are permitted in the immunized portfolio, they may still be limited to the extent required by the absolute-match portfolio. This suggests two possible modifications to the last constraint in models (7) and (8):

(i) $\alpha_t \geq \min(\omega_t, \mathbf{0})$

(ii) $[\min(\alpha_t, \mathbf{0})]'\mathbf{1} \geq [\min(\omega_t, \mathbf{0})]'\mathbf{1}$ (9)

4.11 Under (i), the short position in any bond may not exceed the corresponding short position in the absolute-match portfolio. Since ω_t is known *a priori*, models (7) and (8) remain linear programming problems. Under (ii), the total funds generated from short positions in the immunized portfolio may not exceed those generated from short positions in the absolute-match portfolio. Since optimization is with respect to α and constraint (ii) is non-linear in α, with (ii) replacing $\alpha \geq \mathbf{0}$, models (7) and (8) are no longer standard linear programming problems. However, they can still be solved with minor modifications to the simplex algorithm.

4.12 Another alternative when short sales are required in the absolute-match portfolio is to minimize the total funds generated from short positions (FGSP):

$$\textit{Minimize } FGSP = [\min(\alpha_t, \mathbf{0})]'\mathbf{1} \qquad w.r.t. \left\{\alpha_{s,t}\right\}_{s=0}^{25}$$

$$\textit{subject to} \quad \frac{\partial V_{A,t}}{\partial y_j} = \frac{\partial V_{L,t}}{\partial y_j} \quad \forall\ j = 1,2,\ldots,n$$

$$V_{A,t} = V_{L,t}$$

$$\alpha_t{}'\mathbf{x}_t = \omega_t{}'\mathbf{x}_t \qquad (10)$$

4.13 In addition to the constraints suggested in models (7), (8) and (10), we may also wish to impose the second-order constraints:

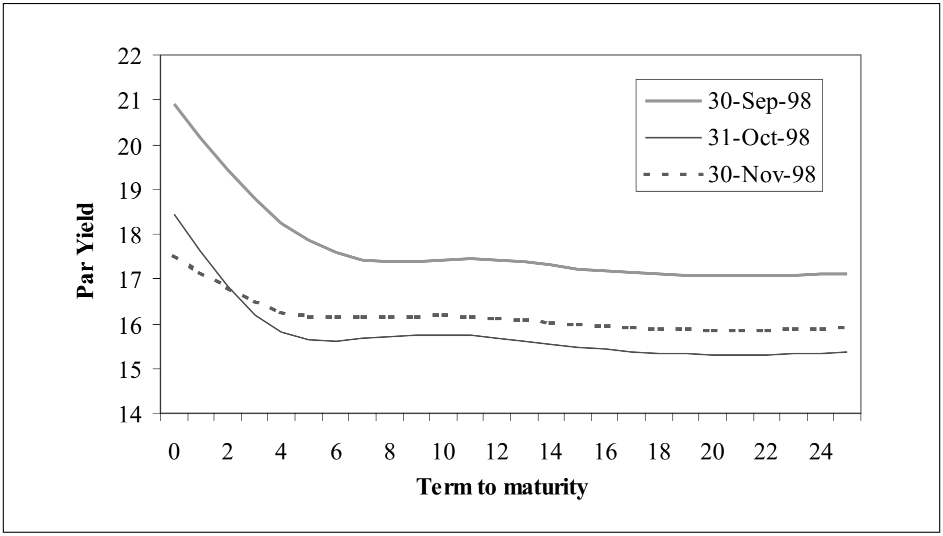
$$\frac{\partial^2 V_{A,t}}{\partial y_j{}^2} \geq \frac{\partial^2 V_{L,t}}{\partial y_j{}^2} \quad \forall\ j = 1,2,\ldots,n \qquad (11)$$

4.14 These constraints ensure that the value of the assets is greater than or equal to that of the liabilities for non-infinitesimal shifts of the form $\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3, \ldots, \mathbf{t}_n$. This represents an additional source of arbitrage in the qualified sense that it is known with certainty that only the immunized fundamental shifts will occur. "Conditional arbitrage" is discussed further in Section 6.

5. EMPIRICAL ANALYSIS

5.1 In this section, the immunization strategy described in (8) is used to construct portfolios for a level stream of liability cash flows of R100 payable annually in arrear for five years. The term structures of 30 September, 31 October and 30 November 1998 shown in Figure 3 are used to illustrate these strategies.

Figure 5. JSE-Actuaries Yield Curve



5.2 The perfect-match portfolios as at 30 September and 31 October 1998 are shown in Table 2, together with the optimum immunized portfolios subject to one, two and three principal-component, partial-derivative constraints. The principal-component constraints use *ex-ante* estimates of the principal components so that the results provide an *ex-post* test of the data. The optimum immunized portfolios subject only to parallel shifts and built using standard duration-matching techniques, namely the Fisher-Weil duration, are also shown in Table 2. The holdings represent the rand investment in each par bond and maturities with zero holdings are omitted for brevity.

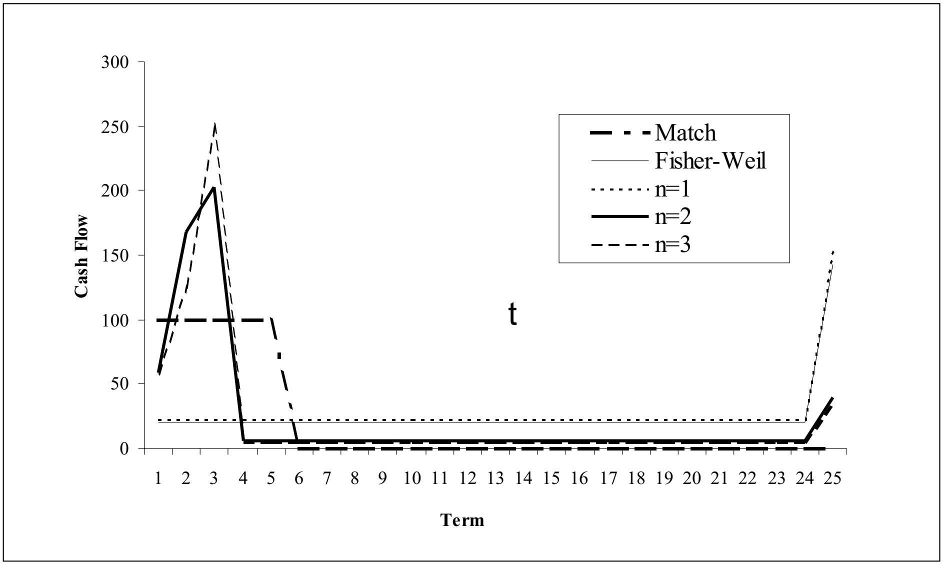
5.3 The no-arbitrage values of the liabilities at 30 September and 31 October 1998 are R309-71 and R327-12 respectively. In both cases it is assumed that the liability cash flows occur at annual intervals from the date of immunization. That is, for the portfolio identified on 30 September 1998, liability cash flows occur at the end of September in subsequent years, while for the portfolio identified on 31 October 1998, liability cash flows occur at the end of October in subsequent years. The future cash flows generated by each of the portfolios constructed for 30 September 1998 are shown in Figure 4. (Cash flows for portfolios constructed in October are not shown since they are almost identical.)

TABLE 2. Optimum immunized portfolios using parallel and principal component shifts

30 Sep 98					
Maturity	Match	Fisher-Weil	n=1	n=2	n=3
0	0,000	187,436	178,753	0,000	5,827
1	42,094	0,000	0,000	0,000	0,000
2	50,584	0,000	0,000	110,263	68,206
3	60,417	0,000	0,000	165,965	207,524
4	71,764	0,000	0,000	0,000	0,000
5	84,854	0,000	0,000	0,000	0,000
25	0,000	122,277	130,960	33,484	28,156
Monthly Coupon Income					
	4,826	5,009	4,982	4,861	4,856
31 Oct 98					
Maturity	Match	Fisher-Weil	n=1	n=2	n=3
0	0,000	205,764	195,080	0,000	0,000
1	46,755	0,000	0,000	0,000	0,000
2	54,993	0,000	0,000	133,811	108,799
3	64,249	0,000	0,000	158,578	180,885
4	74,657	0,000	0,000	0,000	0,000
5	86,468	0,000	0,000	0,000	0,000
25	0,000	121,358	132,042	34,733	17,457
Monthly Coupon income					
	4,437	4,717	4,689	4,462	4,453

5.4 The Fisher-Weil immunization strategy and the strategy immunized against changes in only the first principal component produce almost equivalent barbell strategies, since the first principal component is almost a parallel shift. In contrast, immunization against changes in the first two and the first three principal components produces strategies with cash flows that more closely match the liability cash flows. Clearly, as the number of constraints increases, the maximum coupon income decreases. The perfect-match portfolio is immunized against all changes to the par-yield curve but has the lowest yield.

FIGURE 6. Cash flows of assets and liabilities using Fisher-Weil and n=1,2 & 3 principal component immunization (30 Sep 98)



5.5 The advantage of the principal-component immunization approach is that it allows the portfolio manager to hedge against different types of risk while quantifying the sacrifice in yield. By dropping certain principal-component, partial-derivative constraints, the manager can take active risk in those yield-curve dynamics while hedging against others. For example, with the downward-sloping yield curves of September and October 1998, it may be reasonable to expect a downward shift in yields. Under this expectation, a manager might structure a barbell portfolio similar to the Fisher-Weil and single-principal-component portfolios described in Table 2.

5.6 The results are somewhat unexpected. For the portfolios of September 1998 when yields were relatively high, the drop in yields to 31 October appears to result from a roughly parallel shift (see Figure 3). This might suggest positive results from the barbell strategies. However, the post-shift present values of R327-51 and R327-57 for the hedges with two and three principal-component, partial-derivative constraints respectively exceed the present values of R323-30 for the Fisher-Weil and R324-27 for those with one. The PV of the liabilities a month later is R327-12, indicating substantial risks in the barbell strategies compared with the more fully immunized strategies. This paradox is resolved by noting that the score for the second principal component is relatively large and negative, indicating a flattening of the curve.

5.7 The benefits of the barbell portfolio one month hence for the October 1998 portfolios are again questionable. Under parallel shifts, the convexity effects of the barbell strategy will produce excess funds; however, it is well known that parallel shifts are more the exception than the rule. Portfolio managers often use rules of thumb when assessing basis risk. One such rule is that when yields rise, shorter rates tend to rise faster than longer rates. Under such circumstances, a barbell portfolio immunized using the traditional Fisher-Weil duration will outperform the benchmark because of convexity effects, and because shorter rates may rise faster than longer rates. However, for the October 1998 portfolios, the post-shift present values of R323-11 for the Fisher-Weil and R322-75 for the single-principal-component hedges indicate poor immunity to the November shift since the present value of the liabilities at the end of November is R324-55. In contrast, the post-shift present values of R325-07 and R325-09 for the hedges with two and three principal-component, partial-derivative constraints respectively are roughly equivalent to the present value of the liabilities a month later. The reason for the poor results from the Fisher-Weil and the single-principal-component hedges is that the score for the second principal component is again relatively large and negative, indicating a further flattening of the curve.

5.8 Figures 5 to 8 show the price movements in response to parallel and fundamental yield-curve shifts for the absolute-match portfolio and each of the four immunized portfolios constructed for 30 September 1998. Considering each type of shift in isolation, price movements for the four portfolios relative to price movements in the absolute-match portfolio illustrate how well each portfolio is immunized against that type of shift.

FIGURE 7. Price movements following parallel shifts in the yield curve

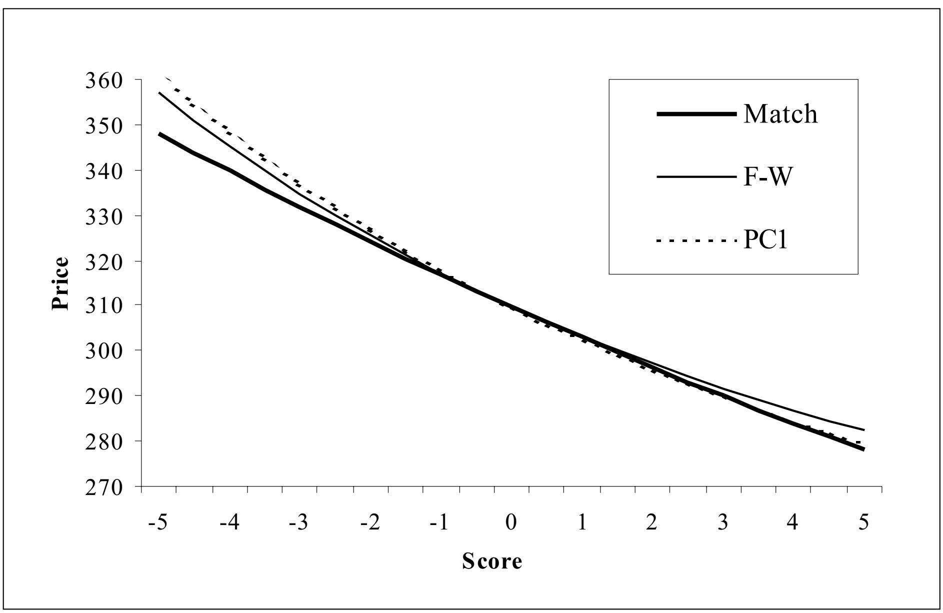
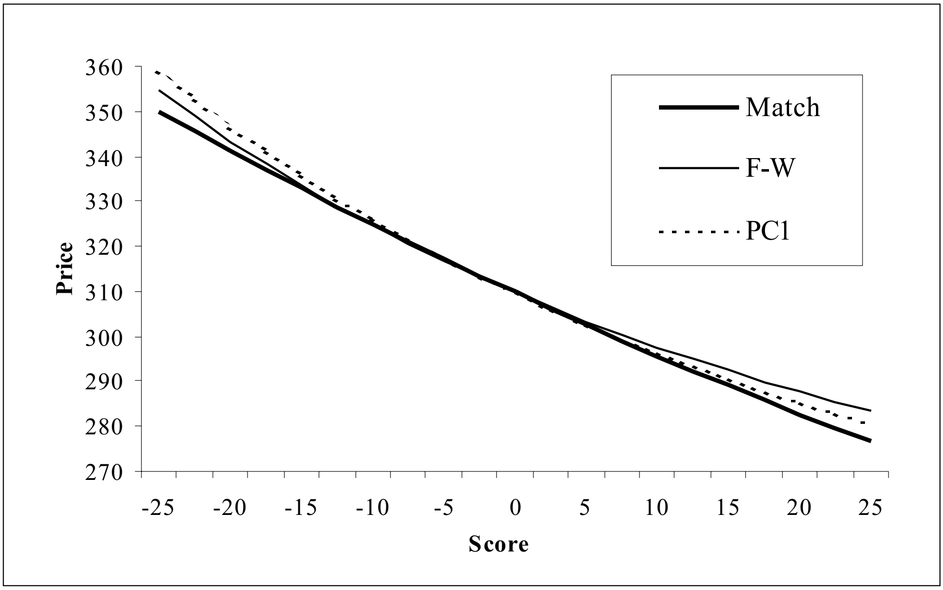


FIGURE 8. Price movements following PC1 shifts in the yield curve



5.9 The graphs of price movements for the two portfolios immunized against shifts in the first two and the first three principal components have been omitted from figures 5 and 6 for clarity. Following parallel or PC1-type shifts, prices for these two portfolios lie between the prices for the absolute-match portfolio and those for the portfolio immunized against a shift in only the first principal component.

5.10 While all portfolios are fairly well immunized against parallel and PC1-type shifts, the Fisher-Weil and single-principal-component hedges are poorly immunized against PC2 and PC3-type shifts. This is obvious to some extent since shifting the curve in a way that is not anticipated by the Fisher-Weil hedge, for example, will result in poor performance from that hedge. The extent to which the unanticipated shift affects the hedge portfolio depends on how exposed the hedge is to that type of shift. What is clear from Figures 9 and 10 is just how exposed the Fisher-Weil and single-principal-component hedges are to PC2 and PC3-type shifts. For September 1998, the tracking errors for the Fisher-Weil and single-principal-component hedges are 12‰ and 9‰ respectively, while for the hedges immunized against PC2 and PC3-type shifts the corresponding error is 1‰ in both cases.

5.11 Although the price graphs in Figures 5 to 8 are specific to the portfolios constructed for 30 September 1998, they give a good indication of price movements for 31 October 1998 portfolios since the corresponding portfolios are similar. The 30 September 1998 and 31 October 1998 portfolios immunized against PC2-type shifts are marginally

FIGURE 9. Price movements following PC2 shifts in the yield curve

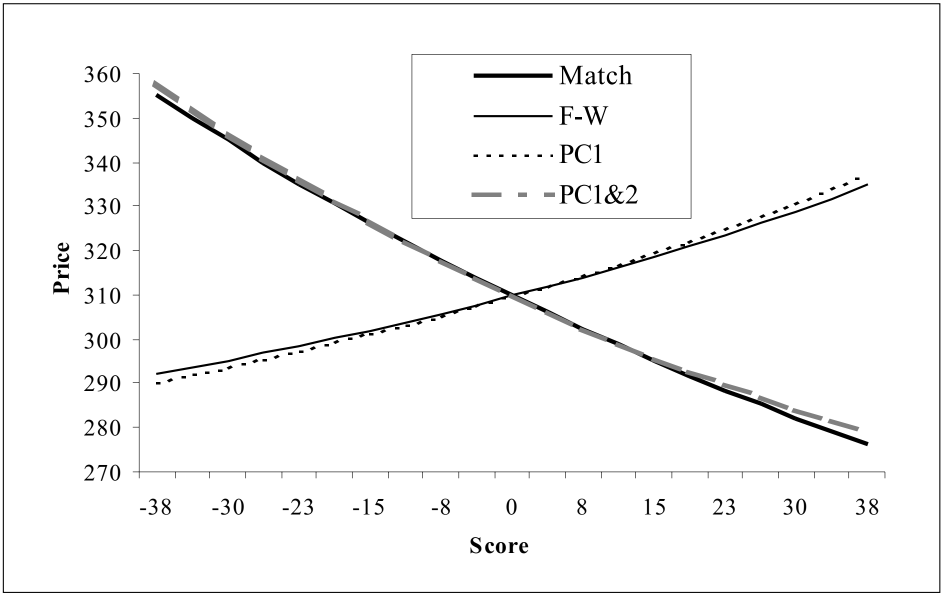
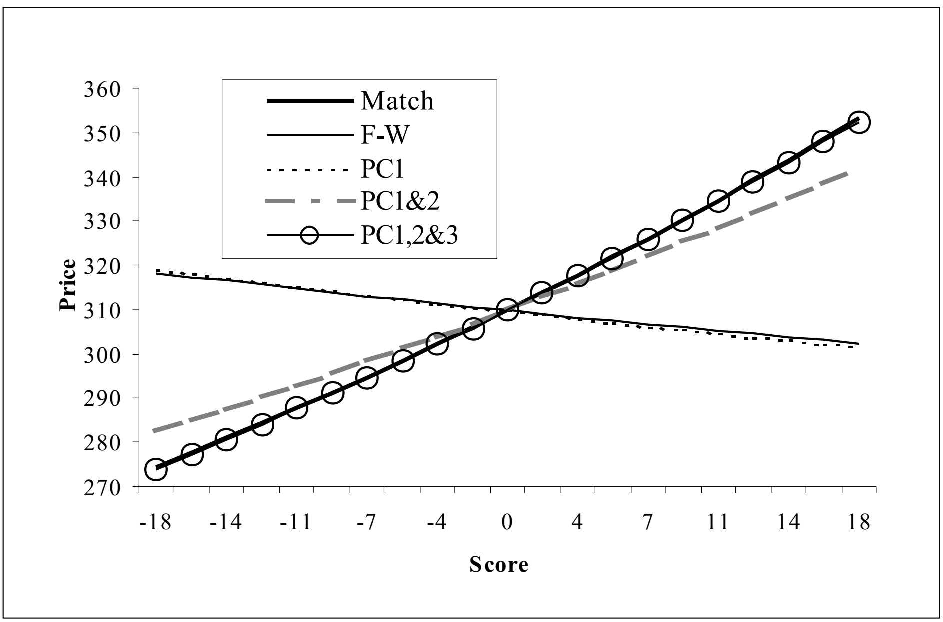


FIGURE 10. Price movements following PC3 shifts in the yield curve



exposed to PC3-type shifts. This accounts for most of the difference in PVs for the two- and three-principal-component hedges following the shocks of October and November 1998. However, since these errors are relatively small and since the error for the PC1&2 hedge shown in Figure 10 is relatively small, a fund manager hedging this liability stream might not be too concerned about immunizing against PC3-type shifts.

5.12 The first two principal components illustrated in Figure 1 are both roughly level beyond a maturity of ten years. This might suggest that for a liability cash flow with maturity greater than ten years, an immunized portfolio subject to both the first two principal-component constraints would be equivalent to an immunized portfolio subject to only the first. However, since an immunized portfolio may contain bonds of any maturity, including bonds with a maturity of less than ten years, these two immunized portfolios will not be equivalent in general. Hence, the five-year annuity considered in the above example is also illustrative of the risks faced by alternative nominal liabilities.

6. CONDITIONAL ARBITRAGE

6.1 For any nominal liability cash flow stream, the coupon income from the optimum portfolio given by Model (8) will be at least as great as that from the absolute-match portfolio since the latter also satisfies the constraints of Model (8). For a system subject only to infinitesimal shifts of the form $\mathbf{t}_1,\mathbf{t}_2,\mathbf{t}_3,\ldots,\mathbf{t}_n$, an immunized portfolio subject to the first n principal-component, partial-derivative constraints is risk-free in relation to the liabilities.

6.2 If short selling is permitted, the short sale of an s-year par bond creates a nominal liability, which can be immunized in the same manner. If the optimum portfolio consists of bonds other than the s-year par bond and if the coupon income from the optimum portfolio is greater than that from the s-year par bond, an arbitrage opportunity exists. By short selling the s-year par bond and using the proceeds to purchase the optimum portfolio, a monthly risk-free profit equal to the difference between the monthly coupon income from the optimum portfolio and that from the s-year par bond is created. It should now be clear why short sales must be limited: unlimited short sales give rise to infinite arbitrage profits and result in an unbounded objective function.

6.3 Table 3 illustrates the maximum arbitrage profits possible from the short sale of R100 nominal of each s-year par bond for optimum portfolios immunized against shifts in one, two and three principal components, while the last row gives the maximum arbitrage profits possible from the short sale of any par bond. For the short sale of any s-year par bond, as the number of principal-component partial-derivative constraints increases, the maximum possible arbitrage profit from the optimum immunized portfolio decreases.

6.4 Since the optimum portfolios generating the arbitrage profits illustrated in Table 3 are only immunized against at most the first three fundamental shifts, the apparent arbitrage profits represented by these additional funds are not generated with certainty. For fundamental shifts of the form $\mathbf{t}_4,\mathbf{t}_5,\ldots,\mathbf{t}_d$, such profits are not guaranteed. However unlikely such shifts may appear historically, there is always the possibility that they may occur in future. Hence, such arbitrage opportunities might be referred to as "conditional arbitrage".

TABLE 3. The excess monthly coupon income per R100 nominal short-sale of each *s*-year par bond produced by the optimum, immunized portfolio subject to 1, 2 and 3 principal-component, partial-derivative constraints (Sep 98).

s	n=1	n=2	n=3
1	0,019	0,000	0,000
2	0,035	0,000	0,000
3	0,049	0,000	0,000
4	0,060	0,000	0,000
5	0,067	0,000	0,000
6	0,071	0,025	0,000
7	0,068	0,043	0,000
8	0,059	0,048	0,000
9	0,045	0,042	0,000
10	0,030	0,029	0,000
11	0,016	0,014	0,000
12	0,008	0,004	0,000
13	0,005	0,000	0,000
14	0,006	0,000	0,000
15	0,007	0,000	0,000
16	0,007	0,000	0,000
17	0,009	0,000	0,000
18	0,010	0,003	0,000
19	0,010	0,006	0,000
20	0,010	0,008	0,000
21	0,009	0,008	0,000
22	0,008	0,007	0,000
23	0,006	0,006	0,000
24	0,003	0,003	0,000
25	0,000	0,000	0,000
Max	0,071	0,048	0,000

6.5 For 30 September 1998, this section shows that negligible arbitrage opportunities exist if one conditions on three principal-component constraints. Hence, for this period, it may be imprudent to condition on less than three PC constraints when selecting an immunized portfolio, since if the market did not think such shifts possible the arbitrage opportunities illustrated for n=1&2 in Table 3 should not exist. Although this logic is applicable to a stream of liability cash flows in general, certain portfolios may be only marginally exposed to PC3-type shifts so that the PC3-constraint can effectively be ignored, as discussed in Section 5.

6.6 It should be noted that any portfolio other than the absolute-match portfolio always contains risk since it is actually possible for the yield curve to move in any number of ways in reality. Since the scientific method is based on the presumption that the past will in some sense be like the future, the above analysis conditions on what was historically likely.

7. CONCLUSION

7.1 Principal components analysis provides a parsimonious description of historical South African yield-curve dynamics. In this paper, a variety of models have been introduced to immunize against such dynamics. These models are optimal in two senses: firstly, they minimize the number of constraints required to immunize against any desired proportion of the total variability and, secondly, they maximize the income in excess of that produced by the absolute-match portfolio. The optimization models can be used to identify optimal portfolios for immunizing any nominal cash flow stream, the short sale of any existing bond or a portfolio of bonds. Hence, these models can also be used for enhanced index tracking.

7.2 The immunization strategies presented in Section 5 accentuate the substantial risk of using the traditional Fisher-Weil duration as the only measure of risk. Clearly, the more fully immunized strategies bear less risk and illustrate the importance of immunizing against shifts other than parallel shifts. Further, it is likely that optimization will maximize exposure to non-immunized shifts. Hence, the increased income from optimum portfolios with fewer principal-component, partial-derivative constraints should always be weighed against the investor's risk tolerance to non-immunized shocks and the market's risk premium for these shocks as implied by the yield curve at that time.

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