

# Linear predictor of the discounted renewal aggregate claims with dependent inter-occurrence times

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## ABSTRACT

In this paper we derive the first two moments and a linear predictor of the compound discounted renewal aggregate claims when taking into account dependence within the inter-occurrence times. Using specific mixtures of exponential distributions to define the dependence structure between the inter-occurrence times, we compare the accuracy of the proposed linear predictor to the simulated value of that sum.

## KEYWORDS

Discounted compound renewal aggregate sums; moments; Archimedean copula; random interest rate; linear predictor

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## 1. INTRODUCTION

1.1 The model proposed and studied in the present paper was first presented in Section 3 of Albrecher et al. (2011), where the authors used a simple mixing idea to establish a number of explicit formulae for ruin probabilities in renewal risk models with dependence among claim sizes and among claim inter-occurrence times. In that model, the authors relax the assumption of independence between the inter-occurrence times through an arbitrary copula structure, such as the Archimedean copula.

1.2 All the previous papers that assume dependence between the subsequent inter-arrival times have restrictive applications and generalisations to dependent scenarios are called for. For instance, in car insurance, if there has been a long waiting time before a claim, the next inter-arrival time can be long as well, because the policyholders are potentially “good drivers”. Alternatively, when some policyholders only start to use their cars a long time after purchasing them, claims would arrive more frequently after a long claim-free period.

1.3 Another context in finance serves as a motivation for this study. Consider a portfolio of homogeneous credit risks according to S&P’s ratings. For risks with credit rating of single B, the probability of default is 0.049 and the Pearson’s correlation coefficient between the occurrences of two risks is 0.00156. Is this correlation negligible? Can we assume that the risks are independent? If we ignore it, does it have an impact on the riskiness of the portfolio? To answer these questions, we can use the concept of a sequence of exchangeable random variables (mixture of distributions). To take into account dependency, one can consider an extension of the classical discrete-time individual risk model, with exchangeability, which involves two important contributions by De Finetti (1957), i.e. the Representation Theorem for a sequence of exchangeable random variables and the classical discrete-time individual risk model.

1.4 Two types of competing risk models can be considered in actuarial science, namely, individual and collective risk models. The individual risk model is more suited to life insurance but the calculations involved can become very laborious, especially when it comes to finding the distribution function of the aggregate risks. To facilitate the calculation of the distribution of the total amount of portfolio claims, the individual risk model is often approximated by a collective risk model such as the Compound Poisson model. The collective risk model is better suited to risk modelling in automobile insurance or damage insurance.

1.5 In finance, the individual and collective risk models are known respectively as bottom-up and top-down models.

1.6 In the bottom-up approach, aggregate loss dynamics are inferred from the individual specification of portfolio entity dynamics while in the top-down approach, loss dynamics are directly specified. The individual dynamics of portfolio entity deficiencies can be inferred

by random thinning techniques. The precursors of these techniques in the area of credit risk modelling are shown in Giesecke & Goldberg (2006).

1.7 Bottom-up models have certain limitations when it comes to valuing exotic products such as Collateralised Debt Obligation (CDO) tranche options for which the dynamics of aggregate loss could be complex. Static bottom-up models are often used for large portfolios when it is required to take the heterogeneity of credit spreads into account. For a meticulous comparison of these two competing approaches, the reader is referred to the article by Giesecke (2008).

1.8 Our goal is to extend the results of Léveillé & Adékambi (2011) under Albrecher et al.'s (2011) models. We use simpler models for which explicit formulae exist and afterwards mix over involved parameters. That is, the mixing of parameters can be carried over to the mixing of the moments under study.

1.9 The first moment of the discounted aggregate claims or its mathematical expectation intuitively represents the central tendency of that random variable, as well as the average of its distribution. The justification for the popularity of the notion of mathematical expectation comes from the Law of Large Numbers which essentially says that the average of the successive realisations of a random variable tends towards the expectation of this random variable when the number of realisations tends to infinity. This result gives an almost experimental status to the mathematical notion of mathematical expectation.

1.10 The mathematical expectation plays an important role in determining the pure premium. The expectation of a random variable gives information on the central tendency of the distribution, but no information on the dispersion of values around their average value. A natural idea to quantify this dispersion would be to measure how far from the mean a realisation of that random variable falls. We could thus consider the expectation of the square of the distance from its mean, which is the second central moment.

1.11 The paper is organised as follows: in Section 2, we present the theoretical background of the study; in Section 3, we present the ordinary renewal process with dependence. The first two moments and joint moment of the aggregate discounted claims are derived in Sections 4, 5 and 6. In Section 7, we apply the results of the preceding section when the subsequent inter-arrival times have a conditional exponential distribution, with a parameter  $\theta$  that follows an Erlang (2,2) distribution to find a linear predictor of the aggregate discounted claims for a constant instantaneous interest rate. In Section 8, the conclusion follows.

## 2. THEORETICAL BACKGROUND

2.1 Collective risk theory originated in the doctoral thesis of Filip Lundberg (1903) and it was subsequently developed by a small group, of mainly Scandinavian actuaries. Lundberg's

insight was to model an insurance company as a repository of continuous flows of premiums, and as a source of flows of claim payments.

2.2 Lundberg's ideas were subsequently further developed by Cramer (1930, 1955). Andersen (1957) generalised the basic risk process to the compound ordinary renewal risk process and Thorin (1975) extended to the more general, delayed renewal risk process.

2.3 As the theory of the basic risk process and related problems matured, researchers increasingly turned their attention to the problem of finding the distribution of the aggregated discounted claims under the classical risk model and to more general problems that incorporated macroeconomic variables or that relaxed some of the assumptions of the classical risk process.

2.4 Much has been written on the problem of the distribution of aggregated discounted claims. The literature review in this paper is restricted to those papers that are relevant to this study. In a sequence of papers, Léveillé & Garrido (2001a, 2001b) tackled the problem of the distribution of aggregated compound discounted claims. They were interested in computing the moments of the distribution. In the first paper, they derived asymptotic expressions of the first two moments using renewal theory arguments under inflationary conditions. In the second, they obtained recursive formulae for the moments. Further improvements to these results were made by Leveille et al. (2010), who computed the asymptotic and finite time moment generating functions of the discounted aggregate claims process.

2.5 Taylor (1979) considered the classical ruin problem studied by Lundberg (1903) and Cramer (1955) and studied the effect of inflationary conditions. He proved that the probability of ruin is always increased when the (constant) inflation rate is increased. Perhaps this is not surprising because, intuitively the presence of inflation means that if the effects of inflation are not compensated for through the adjustment of premiums, say, it is the insurer who will be prejudiced when claims are eventually presented.

2.6 Delbaen & Haezendonck (1987) studied the influence of both the inflation force and the interest force. They found that the incorporation of interest and inflation forces greatly improved the estimation of bankruptcy probabilities both for the finite and infinite time horizons. Other noteworthy works are Yuen et al. (2006), who considered the effect of stochastic interest rates in a renewal risk process, and Leveille & Adékambi (2011), who considered the effect of stochastic interest rates in the problem of the moments of the distribution of discounted compound renewal sums for an ordinary or a delayed renewal process.

2.7 Albrecher & Boxma (2004) employed a threshold approach and showed that if a claim exceeded a certain threshold, then the parameters of the distribution for the next claim inter-arrival time would be modified. In contrast, Boudreault et al. (2006) assumed that if a

claim in an inter-arrival time is greater than a certain threshold then the parameters of the distribution of the next claim amount is modified. Albrecher & Teugels (2006) considered modelling dependence with the use of an arbitrary copula. Kim & Kim (2007) and Ren (2008) modelled dependence through a two-state Markovian environment in which claim rates and sizes fluctuated according to the state of risk of the business. Barges et al. (2011) adapted the copula approach earlier introduced by Albrecher & Teugels (2006) to compute the moments of the distribution of aggregate compound renewal sums when the force of interest is constant. Adékambi & Dziwa (2016) and Adékambi (2017) derived the first two moments of the compound discounted renewal cashflow when taking into account dependence between the cashflow and its occurrence time. The dependence structure between the two random variables is defined by a Farlie–Gumbel–Morgenstern copula.

2.8 Sarabia et al. (2017) obtain analytic expressions for the probability density function (PDF) and the cumulative distribution function (CDF) of aggregated risks, modelled according to a mixture of exponential distributions. Cossette et al. (2018) investigate dependent risk models in which the dependence structure is defined by an Archimedean copula.

### 3. THE MODEL

3.1 In this section, we introduce the ordinary renewal case with dependence.

3.1.1 The claims counting process  $\{N(t), t \geq 0\}$  forms an ordinary renewal process and, for  $k \in \mathbb{N} = \{1, 2, 3, \dots\}$ :

- the positive claim occurrence times are given by  $\{T_k, k \in \mathbb{N}\}$ ; and
- the positive claim inter-arrival times are given by  $W_k = T_k - T_{k-1}, k \in \mathbb{N}, T_0 = 0$ .

3.1.2 The corresponding claim severities  $\{X_k, k \in \mathbb{N}\}$  are such that:

- $\{X_k, k \in \mathbb{N}\}$  are independent and identically distributed (i.i.d),

- $\{X_k, W_k; k \in \mathbb{N}\}$  are mutually independent; and

- the first two moments of  $X_1$  exist.

3.1.3 The aggregate discounted value at time 0 of the claims recorded over the period  $[0, t]$  is given by:

$$Z(t) = \sum_{i=1}^{N(t)} e^{-\delta T_i} X_i = \sum_{i=1}^{N(t)} D(T_i) X_i,$$

with  $Z(t) = 0$  if  $N(t) = 0$ , and  $D(T_k) = \exp\left\{-\int_0^{T_k} \delta(v) dv\right\}$ .

3.2 In the usual ordinary renewal risk process, the sequences  $\{W_j\}_{j=1}^\infty$  are assumed to be mutually independent. In this paper, we assume that  $W_1, W_2, \dots$  are dependent, with dependence given by Archimedean copulas.

3.2.1 Let  $\Theta$  be a random variable with PDF  $f_\Theta(\theta)$  and suppose that the Laplace transform of  $\Theta$  given by  $f_\Theta^*(s) = \int_0^\infty e^{-s\theta} f_\Theta(\theta) d\theta$  exists over a subset  $K \subset \mathbb{R}$  including a neighbourhood of the origin.

3.3 For a general set-up, the formula above by using an exponential distribution for the conditional distribution of the time between successive claims can be extended to other conditionally independent distributions.

3.4 For example, the conditional distribution of the inter-claims time can be written in the power form  $\Pr(W_i \geq x_i | \Theta = \theta) = (\bar{H}(x_i))^\theta$  for some distribution function  $H(x)$  and

$$\Pr(W_1 \geq x_1, W_2 \geq x_2, \dots, W_n \geq x_n | \Theta = \theta) = \prod_{k=1}^n (\bar{H}(x_k))^\theta. \tag{1}$$

For all  $n$ , i.e.  $\Theta$  is the common mixing parameter, then

$$\begin{aligned} \bar{F}_{W_1, \dots, W_n}(x_1, \dots, x_n) &= \int_0^\infty \Pr(W_1 \geq x_1, \dots, W_n \geq x_n | \Theta = \theta) f_\Theta(\theta) d\theta, \\ &= \bar{F}_{W_1, \dots, W_n}(x_1, \dots, x_n) = \int_0^\infty (\bar{H}(x_1))^\theta \dots (\bar{H}(x_n))^\theta f_\Theta(\theta) d\theta, \\ &= f_\Theta^*(-\log \bar{H}(x_1) - \dots - \bar{H}(x_n)), \\ &= f_\Theta^*(f_\Theta^{*-1}(\bar{F}_{W_1}(x_1)) + \dots + f_\Theta^{*-1}(\bar{F}_{W_n}(x_n))). \end{aligned} \tag{2}$$

The form of dependence structure is again Archimedean with generator  $\mathcal{O}(t) = f_\Theta^{*-1}(t)$ , where  $\bar{F}_{W_i}(x_i) = f_\Theta^*(-\log \bar{H}(x_i))$ .

**3.5 Remark: Specific mixture of exponential distributions**

3.5.1 ORDINARY RENEWAL CASE

Suppose that random variables  $W_1, W_2, \dots, W_n$  are  $n$  dependent, positive and continuous random variables and  $\Theta = \theta$ , then the random variables  $W_1, W_2, \dots, W_n$  are conditionally independent and distributed as  $Exp(\theta)$ , i.e.

$$\begin{aligned} \Pr(W_1 \geq x_1, W_2 \geq x_2, \dots, W_n \geq x_n | \Theta = \theta) &= \Pr(W_1 \geq x_1 | \Theta = \theta) \dots \Pr(W_n \geq x_n | \Theta = \theta) \\ &= e^{-\theta x_1} \dots e^{-\theta x_n}. \end{aligned} \tag{3}$$

It follows that

$$\bar{F}_{W_1}(x_1) = \Pr(W_1 \geq x_1) = \int_0^\infty e^{-\theta x_1} f_\Theta(\theta) d\theta = f_\Theta^*(x_1) = \int_0^\infty \Pr(W_1 \geq x_1 | \Theta = \theta) f_\Theta(\theta) d\theta. \tag{4}$$

The joint distribution of the tail of  $W_1, W_2, \dots, W_n$  can be written as

$$\begin{aligned} \overline{F}_{W_1, \dots, W_n}(x_1, \dots, x_n) &= \int_0^\infty \Pr(W_1 \geq x_1, \dots, W_n \geq x_n | \Theta = \theta) f_\Theta(\theta) d\theta \\ &= \int_0^\infty e^{-\theta \sum_{i=1}^n x_i} f_\Theta(\theta) d\theta, \\ &= f_\Theta^* \left( \sum_{i=1}^n x_i \right) = f_\Theta^* \left( f_\Theta^{*-1}(\overline{F}_{W_1}(x_1)) + \dots + f_\Theta^{*-1}(\overline{F}_{W_n}(x_n)) \right). \end{aligned} \tag{5}$$

From Sklar’s theorem,

$$\Pr(W_1 \geq x_1, \dots, W_n \geq x_n) = \tilde{C}(\overline{F}_{W_1}(x_1) \dots \overline{F}_{W_n}(x_n)). \tag{6}$$

It follows that  $\tilde{C}(u_1, \dots, u_n)$  is an Archimedean copula with

$$\tilde{C}(u_1, \dots, u_n) = f_\Theta^* (f_\Theta^{*-1}(u_1) + \dots + f_\Theta^{*-1}(u_n)) = \varnothing^{-1}(\varnothing(u_1) + \dots + \varnothing(u_n)), \tag{7}$$

where  $\varnothing(t) = f_\Theta^{*-1}(t)$  is the generator of the Archimedean copula  $\tilde{C}$ .

#### 4. FIRST MOMENT

From the first moment of L evell e & Ad ekambi (2011), we afterwards mix over the involved parameter  $\Theta$ .

##### Lemma 4.1

Consider a renewal counting process, such as defined in Section 2. Then the conditional density probability functions of  $T_k | N(t) = n, \Theta = \theta$  are given, for  $0 < s \leq t$  and  $k \leq n$ , by:

$$f_{T_k | N(t)=n, \Theta=\theta}(s | n, \theta) = \frac{P(N(t-s) = n-k | \Theta = \theta) f_{T_k | \Theta=\theta}(s | \theta)}{P(N(t) = n | \Theta = \theta)}. \tag{8}$$

##### Proof

For  $n \in \mathbb{N} \cup \{0\}$ :

$$\begin{aligned} P(T_k \leq s | N(t) = n, \Theta = \theta) &= \frac{P(N(t) = n, \Theta = \theta, T_k \leq s)}{P(N(t) = n, \Theta = \theta)} \\ &= \frac{P(N(t) = n | T_k \leq s, \Theta = \theta) P(T_k \leq s, \Theta = \theta)}{P(N(t) = n, \Theta = \theta)} \\ &= \int_0^s P(N(t) = n | T_k \leq s, T_k = v, \Theta = \theta) f_{T_k | T_k \leq s, \Theta = \theta}(v) dv \\ &= \frac{\times P(T_k \leq s, \Theta = \theta)}{P(N(t) = n | \Theta = \theta) f_\Theta(\theta)} \end{aligned}$$

$$\begin{aligned}
 P(T_k \leq s \mid N(t) = n, \Theta = \theta) &= \frac{\int_0^s P(N(t) - N(v) = n - k \mid T_k = v, \Theta = \theta) f_{T_k}(v) dv}{P(N(t) = n \mid \Theta = \theta) f_{\Theta}(\theta)} \\
 &= \frac{\int_0^s P(N(t) - N(v) = n - k \mid T_k = v, \Theta = \theta) f_{T_k | \Theta = \theta}(v | \theta) f_{\Theta}(\theta) dv}{P(N(t) = n \mid \Theta = \theta) f_{\Theta}(\theta)} \\
 &= \frac{\int_0^s P(N(t - v) = n - k \mid \Theta = \theta) f_{T_k | \Theta = \theta}(v | \theta) dv}{P(N(t) = n \mid \Theta = \theta)}
 \end{aligned}$$

which establishes equation (8), with  $P(N(t) - N(v) = n - k) = P(N(t - v) = n - k)$ .

**Theorem 4.1**

Given the assumptions of Section 3, the first moment of the discounted aggregate claims is given, for  $t > 0$ , by:

$$E[Z(t)] = E[X_1] \int_0^{\infty} \int_0^t E[D(v)] dm(v | \theta) f_{\Theta}(\theta) d\theta,$$

where  $m(v | \theta) = \sum_{k=1}^{\infty} F_{W_k}^{*k} (v | \theta) = E[N(v) | \Theta = \theta]$ .

**Proof**

Conditioning on  $N(t)$  and  $\Theta$ , then using independence between the number and the severity of claims yields:

$$E[Z(t) | N(t) = n, \Theta = \theta] = E[X_1] \sum_{k=1}^n E[D(T_k) | N(t) = n, \Theta = \theta].$$

From equation (8) of Lemma 4.1, we have:

$$\begin{aligned}
 E[Z(t) | \delta(x), x \in [0, t]] &= E[E[Z(t) | N(t) = n, \Theta = \theta, \delta(x), x \in [0, t]]] \\
 &= E[X_1] \sum_{k=1}^n \int_0^t D(v) f_{T_k | N(t)=n, \Theta=\theta}(v) dv \\
 &= E[X_1] \sum_{k=1}^n \int_0^t D(v) \frac{P(N(t - v) = n - k \mid \Theta = \theta) f_{T_k | \Theta = \theta}(v | \theta) f_{\Theta}(\theta)}{P(N(t) = n \mid \Theta = \theta) f_{\Theta}(\theta)} \\
 &= E[X_1] \sum_{k=1}^n \int_0^t D(v) \frac{P(N(t - v) = n - k \mid \Theta = \theta) f_{T_k | \Theta = \theta}(v | \theta) f_{\Theta}(\theta)}{P(N(t) = n, \Theta = \theta)}.
 \end{aligned}$$



Then,

$$\begin{aligned}
 E[Z(t) | \delta(x), x \in [0, t]] &= E[E[Z(t) | N(t) = n, \Theta = \theta, \delta(x), x \in [0, t]]], \\
 &= E[X_1 \sum_{n=0}^{\infty} \sum_{k=1}^n \int_0^t D(v) P(N(t-v) = n-k | \Theta = \theta) \\
 &\quad \times f_{T_k | \Theta = \theta}(v | \theta) f_{\Theta}(\theta) dv] \\
 &= E[X_1 \sum_{k=1}^{\infty} \int_0^t D(v) f_{T_k | \Theta = \theta}(v | \theta) \sum_{n=k}^{\infty} P(N(t-v) = n-k | \Theta = \theta) \\
 &\quad \times f_{\Theta}(\theta) dv] \\
 &= E[X_1 \sum_{k=1}^{\infty} \int_0^t D(v) f_{T_k | \Theta = \theta}(v | \theta) f_{\Theta}(\theta) dv d\theta] \\
 &= E[X_1 \int_0^t \int_0^{\infty} D(v) d \sum_{k=1}^{\infty} F_{W_k | \Theta = \theta}^{*k}(v | \theta) f_{\Theta}(\theta) dv d\theta] \\
 &= E[X_1 \int_0^t \int_0^{\infty} D(v) dm(v | \theta) f_{\Theta}(\theta) d\theta]
 \end{aligned}$$

where  $m(v | \theta) = \sum_{k=1}^{\infty} F_{W_k | \Theta = \theta}^{*k}(v | \theta) = E[N(v) | \Theta = \theta]$ .

Since the last integral is a random variable, we use a well-known theorem of stochastic processes theory (see Karatzas & Shreve, 1991: 3) to finally obtain:

$$\begin{aligned}
 E[Z(t)] &= E[E[Z(t) | \delta(x), x \in [0, t]]] \\
 &= E[X_1 E\left[\int_0^t D(v) dm(v | \theta) f_{\Theta}(\theta) d\theta\right]] \\
 &= E[X_1 \int_0^t \int_0^{\infty} E[D(v)] dm(v | \theta) f_{\Theta}(\theta) d\theta].
 \end{aligned}$$

**Example 4.1**

Let  $\{\delta(t), t \geq 0\}$  be an Itô process satisfying the stochastic differential equation of Ho–Lee–Merton:

$$d\delta(t) = rdt + \sigma dB(t),$$

with constant drift  $r$ , constant diffusion coefficient  $\sigma$ , and where  $B(t)$  is a standard Brownian motion (see Cairns, 2004: 87).

From the Itô theory (see Karatza & Shreve, 1991; Oksendal, 1992), it can be shown that:

$$\int_0^t \delta(x) dx \sim N\left(\delta(0)t + r\frac{t^2}{2}, \sigma^2\frac{t^3}{3}\right).$$

We assume that:

$$W_1 | \Theta = \theta \sim \text{Exp}(\theta), E[X_1 = 1], \delta(0) = 0.03, r = 0.002 \text{ and } \sigma = 0.001.$$

We set:  $\Theta \sim \Gamma(2, 2)$ , be a Gamma distribution of parameters (2,2).

Hence, Theorem 4.1 yields:

$$E[Z(t)] = E[X]E[\Theta] \int_0^t \exp\left\{-\delta(0)v - r\frac{v^2}{2} + \sigma^2\frac{v^3}{6}\right\} dv.$$

Then with the help of software Matlab, we have the table below.

TABLE 1. First moment of  $Z(t)$  Ho–Lee–Merton case

$E[Z(1)]$	$E[Z(5)]$	$E[Z(10)]$	$E[Z(15)]$	$E[Z(20)]$
0.9848	4.6061	8.3807	11.3241	13.5086
$E[Z(30)]$	$E[Z(40)]$	$E[Z(50)]$	$E[Z(60)]$	$E[Z(70)]$
16.0895	17.1590	17.5241	17.6270	17.6509

The above results are similar to the one obtained in Léveillé & Adékambi (2011) because in our example  $E[\Theta] = 1$ .

### 5. SECOND MOMENT

From the results of Léveillé & Adékambi (2011), we mix over the involved parameter  $\Theta$ .

#### Lemma 5.1

Consider a renewal counting process, such as defined in Section 3. The conditional joint density probability functions of  $T_i, T_j | N(t) = n$  are given for  $0 < x < y < t$  and  $1 \leq i < j \leq n$  by:

$$f_{T_i, T_j | N(t)=n, \Theta=\theta}(x, y | n, \theta) = \frac{P(N(t-y) = n-j | \Theta = \theta) f_{T_{j-i} | \Theta=\theta}(y-x | \theta) f_{T_i | \Theta=\theta}(x | \theta)}{P(N(t) = n | \theta)}. \quad (9)$$

**Proof**

As in Lemma 4.1, we get for  $n \in \mathbb{N} - \{0\}$ :

$$\begin{aligned} P(T_i \leq x, T_j \leq y \mid N(t) = n, \Theta = \theta) &= \frac{P(N(t) = n, T_i \leq x, T_j \leq y, \Theta = \theta)}{P(N(t) = n, \Theta = \theta)} \\ &= \frac{P(N(t) = n \mid T_i \leq x, T_j \leq y, \Theta = \theta)}{P(N(t) = n, \Theta = \theta)} \\ &\quad \times P(T_i \leq x, T_j \leq y, \Theta = \theta). \end{aligned}$$

The last equation can be written as:

$$\begin{aligned} &P(T_i \leq x, T_j \leq y \mid N(t) = n, \Theta = \theta) \\ &= \int_0^x \int_0^y P(N(t) = n \mid T_i = u, T_j = v, \Theta = \theta) f_{T_i, T_j \mid T_i \leq x, T_j \leq y, \Theta = \theta}(u, v) dv du \\ &\quad \times \frac{P(T_i \leq x, T_j \leq y, \Theta = \theta)}{P(N(t) = n, \Theta = \theta)} \\ &= \frac{\int_0^x \int_0^y P(N(t-y) = n-j \mid \Theta = \theta) f_{T_i, T_j}(u, v) dv du}{P(N(t) = n, \Theta = \theta)} \\ &= \frac{\int_0^x \int_0^y P(N(t-y) = n-j \mid \Theta = \theta) f_{T_i, T_j \mid \Theta = \theta}(u, v \mid \theta) dv du}{P(N(t) = n \mid \Theta = \theta)} \end{aligned}$$

with

$$\begin{aligned} P(T_i \leq u, T_j \leq v \mid \Theta = \theta) &= P(T_j \leq v \mid T_i \leq u, \Theta = \theta) P(T_i \leq u \mid \Theta = \theta) \\ &= \int_0^u P(T_j \leq v \mid T_i = z, \Theta = \theta) f_{T_i \mid T_i \leq u, \Theta = \theta}(z) P(T_i \leq u \mid \Theta = \theta) dz \\ &= \int_0^u P(W_{i+1} + \dots + W_j \leq v - z \mid \Theta = \theta) f_{T_i \mid \Theta = \theta}(z \mid \theta) dz. \end{aligned}$$

We have

$$f_{T_i, T_j \mid \Theta = \theta}(u, v \mid \theta) = f_{T_{j-i}}(v-u \mid \theta) f_{T_i \mid \theta}(u \mid \theta).$$

Finally,

$$P(T_i \leq x, T_j \leq y \mid N(t) = n, \Theta = \theta) = \frac{\int_0^x \int_0^y P(N(t-y) = n-j \mid \Theta = \theta) f_{T_{j-1} \mid \Theta = \theta}(v-u \mid \theta) f_{T_i \mid \Theta = \theta}(u \mid \theta) dv du}{P(N(t) = n \mid \theta)},$$

which gives equation (9).

**Theorem 5.1**

Given assumptions of Section 3, the second moment of the discounted aggregate claims is given, for  $t > 0$ , by:

$$E[Z^2(t)] = E[X_1^2] \int_0^t \int_0^t E[D^2(v)] dm(v \mid \theta) f_\Theta(\theta) d\theta + 2E^2[X_1] \int_0^t \int_0^{t-v} \int_0^t E[D(u+v)D(v)] dm(u \mid \theta) dm(v \mid \theta) f_\Theta(\theta) d\theta .$$

**Proof**

Conditioning on  $N(t)$  and  $\Theta$  and using independence between the number and the severity of claims yields:

$$E[Z^2(t) \mid N(t) = n, \Theta = \theta] = E\left[\sum_{k=1}^n D^2(T_k) X_k^2 \mid N(t) = n, \Theta = \theta\right] + 2 \sum_{i=1}^{n-1} \sum_{j=i+1}^n E\left[D(T_i)D(T_j)X_iX_j \mid N(t) = n, \Theta = \theta\right], = E[X_1^2] \sum_{k=1}^n E\left[D^2(T_k) \mid N(t) = n, \Theta = \theta\right] + 2E^2[X_1] \sum_{i=1}^{n-1} \sum_{j=i+1}^n E\left[D(T_i)D(T_j) \mid N(t) = n, \Theta = \theta\right].$$

From equation (8) of Lemma 4.1 and of equation (9) of Lemma 5.1, we have:

$$E[Z^2(t) \mid \delta(x), x \in [0, t]] = E\left[E[Z^2(t) \mid N(t) = n, \delta(x), x \in [0, t]]\right], = E[X_1^2] \sum_{n=0}^\infty \sum_{k=1}^n \int_0^\infty \int_0^t D^2(v) f_{T_k \mid \Theta = \theta}(v \mid \theta) P(N(t-v) = n-k \mid \Theta = \theta) dv d\theta + 2E^2[X_1] \sum_{n=0}^\infty \sum_{i=1}^{n-1} \sum_{j=i+1}^n \int_0^\infty \int_0^t \int_0^t D(u)D(v) P(N(t-u) = n-j \mid \Theta = \theta) \times f_{T_{j-1} \mid \Theta = \theta}(u-v \mid \theta) f_{T_i \mid \Theta = \theta}(v \mid \theta) du dv d\theta.$$

The permutation of the sums gives:

$$\begin{aligned}
 & E \left[ Z^2(t) \mid \delta(x), x \in [0, t] \right] \\
 &= E \left[ X_1^2 \right] \sum_{k=1}^{\infty} \sum_{n=k}^{\infty} \int_0^t D^2(v) f_{T_k | \Theta = \theta}(v | \theta) P(N(t-v) = n-k | \Theta = \theta) dv d\theta \\
 &+ 2E^2 \left[ X_1 \right] \sum_{i=1}^{\infty} \sum_{j=i+1}^{\infty} \sum_{n=j}^{\infty} \int_0^t \int_0^{t-v} D(u) D(v) P(N(t-u) = n-j | \Theta = \theta) \\
 &\quad \times f_{T_{j-i} | \Theta = \theta}(u-v | \theta) f_{T_i | \Theta = \theta}(v | \theta) du dv d\theta \\
 &= E \left[ X_1^2 \right] \sum_{k=1}^{\infty} \int_0^t \int_0^{t-v} D^2(v) dm(v | \theta) f_{\Theta}(\theta) d\theta \\
 &+ 2E^2 \left[ X_1 \right] \sum_{i=1}^{\infty} \sum_{j=i+1}^{\infty} \int_0^t \int_0^{t-v} D(u+v) D(v) P(N(t-u) = n-j | \Theta = \theta) \\
 &\quad \times f_{T_{j-i} | \Theta = \theta}(u-v | \theta) f_{T_i | \Theta = \theta}(v | \theta) du dv d\theta.
 \end{aligned}$$

Hence, following the same reasoning used in Theorem 4.1, we have:

$$\begin{aligned}
 E \left[ Z^2(t) \right] &= E \left[ E \left[ Z^2(t) \mid \delta(x), x \in [0, t] \right] \right] \\
 &= E \left[ X_1^2 \right] \int_0^t \int_0^{t-v} E \left[ D^2(v) \right] dm(v | \theta) f_{\Theta}(\theta) d\theta \\
 &\quad + 2E^2 \left[ X_1 \right] E \left[ \int_0^t \int_0^{t-v} \int_0^{t-u-v} D(u+v) D(v) f_{T_i | \Theta = \theta}(v | \theta) \right. \\
 &\quad \left. d \left\{ \sum_{j=i+1}^{\infty} F_{\tau_j | \Theta = \theta}^{*(j-i)}(u | \theta) \right\} f_{\Theta}(\theta) dv d\theta \right] \\
 &= E \left[ X_1^2 \right] E \left[ \int_0^t \int_0^{t-v} D^2(v) dm(v | \theta) f_{\Theta}(\theta) d\theta \right] \\
 &\quad + 2 E^2 \left[ X_1 \right] E \left[ \int_0^t \int_0^{t-v} \int_0^{t-u-v} D(u+v) D(v) dm(u | \theta) dF_{\tau_i}^{*i}(v | \theta) f_{\Theta}(\theta) d\theta \right] \\
 &= E \left[ X_1^2 \right] \int_0^t \int_0^{t-v} E \left[ D^2(v) \right] dm(v | \theta) f_{\Theta}(\theta) d\theta \\
 &\quad + 2 E^2 \left[ X_1 \right] \int_0^t \int_0^{t-v} \int_0^{t-u-v} E \left[ D(u+v) D(v) \right] dm(u | \theta) dm(v | \theta) f_{\Theta}(\theta) d\theta. \quad \square
 \end{aligned}$$

Let  $\{\delta(t), t \geq 0\}$  be an Itô process satisfying the stochastic differential equation of Ho–Lee–Merton. Hence, we have:

$$E\left[D(v)^2\right] = \exp\left\{-2\delta(0)v - rv^2 + \frac{2}{3}\sigma^2v^3\right\},$$

$$E\left[D(u)D(u+v)\right] = \exp\left\{-[\delta(0)(u+2v)] - \frac{r}{2}[u^2 + 2uv + 2v^2] + \frac{\sigma^2}{2}\left[\frac{(u+2v)^3 + u^3}{6}\right]\right\}.$$

We assume that:

$$W_1 | \Theta = \theta \sim \text{Exp}(\theta), E[X_1] = 1, E[X_1^2] = 2, \delta(0) = 0.03, r = 0.002 \text{ and } \sigma = 0.001.$$

We set:  $\Theta \sim \Gamma(2, 2)$ , be a Gamma distribution with parameters (2,2). Then with the help of software Matlab, we have the table below for the second moment.

TABLE 2. Second moment of  $Z(t)$  Ho–Lee–Merton case

$E\left[Z^2(1)\right]$	$E\left[Z^2(5)\right]$	$E\left[Z^2(10)\right]$	$E\left[Z^2(15)\right]$	$E\left[Z^2(20)\right]$
2.9098	29.7246	84.4707	145.9729	202.1786
$E\left[Z^2(30)\right]$	$E\left[Z^2(40)\right]$	$E\left[Z^2(50)\right]$	$E\left[Z^2(60)\right]$	$E\left[Z^2(70)\right]$
280.0771	315.9861	328.7406	332.3814	333.2318

The above results are similar to the one obtained in Léveillé & Adékambi (2011) because in our example  $E[\Theta] = 1$ .

## 6. JOINT MOMENT

As in Section 4, we mix over the involved parameter  $\Theta$ .

### Theorem 6.1

According to the assumptions of Section 3, the joint moments of order 2 of the discounted aggregate claims are given for  $t > 0$  and  $h > 0$  by:

$$E\left[Z(t)Z(t+h)\right] = E\left[Z^2(t)\right] + E^2[X_1] \int_0^t \int_{t-v}^{t+h-v} \int_0^\infty E\left[D(u+v)D(v)\right] dm(u|\Theta = \theta) dm(v|\Theta = \theta) f_\Theta(\theta) d\theta. \tag{10}$$

**Proof**

We have:

$$\begin{aligned} E\left[E\left[Z(t)Z(t+h)\right]\right] &= E\left[\sum_{i=1}^{N(t)} D(T_i)X_i \sum_{j=1}^{N(t+h)} D(T_j)X_j\right] \\ &= E\left[Z(t)^2\right] + E\left[\sum_{i=1}^{N(t)} D(T_i)X_i \sum_{j=N(t)+1}^{N(t+h)} D(T_j)X_j\right]. \end{aligned}$$

Conditioning on  $N(t)$ ,  $N(t+h)$  and  $\Theta$ , we obtain for the second term:

$$\begin{aligned} &E\left[\sum_{i=1}^{N(t)} D(T_i)X_i \sum_{j=N(t)+1}^{N(t+h)} D(T_j)X_j \mid \delta(x), x \in [0, t+h]\right] \\ &= E\left[E\left[\sum_{i=1}^{N(t)} D(T_i)X_i \sum_{j=N(t)+1}^{N(t+h)} D(T_j)X_j \mid N(t), N(t+h), \Theta, \delta(x), x \in [0, t+h]\right]\right] \\ &= E\left[X_1^2\right] E\left[\sum_{i=1}^{N(t)} D(T_i) \sum_{j=N(t)+1}^{N(t+h)} D(T_j) \mid \delta(x), x \in [0, t+h]\right]. \end{aligned}$$

Now,

$$\begin{aligned} &E\left[\sum_{i=1}^{N(t)} D(T_i) \sum_{j=N(t)+1}^{N(t+h)} D(T_j) \mid \delta(x), x \in [0, t+h]\right] \\ &= E\left[E\left[\sum_{i=1}^{N(t)} D(T_i) \sum_{j=N(t)+1}^{N(t+h)} D(T_j) \mid N(t), N(t+h), \Theta, \delta(x), x \in [0, t+h]\right]\right] \\ &= \int_0^\infty \sum_{n=1}^\infty \sum_{m=n+1}^\infty \sum_{i=1}^n \sum_{j=1}^m E\left[D(T_i)D(T_j) \mid N(t), N(t+h), \Theta\right] \\ &\quad \times P(N(t) = n, N(t+h) = m, \Theta = \theta) d\theta, \\ &= \sum_{i=1}^\infty \sum_{j=i+1}^\infty \sum_{n=j}^\infty \sum_{m=n}^\infty \int_0^t \int_0^{t+h} \int_0^t \int_0^{t+h} D(u)D(v) f_{T_i, T_j, N(t), N(t+h), \Theta}(v, u, n, m, \theta) du dv d\theta, \\ &= \sum_{i=1}^\infty \sum_{j=i+1}^\infty \sum_{m=j}^\infty \int_0^t \int_0^{t+h} \int_0^t \int_0^{t+h} D(u)D(v) f_{T_i, T_j, N(t+h), \Theta}(v, u, m, \theta) du dv d\theta, \\ &= \sum_{i=1}^\infty \sum_{j=i+1}^\infty \sum_{m=j}^\infty \int_0^t \int_0^{t+h} \int_0^t \int_0^{t+h} D(u)D(v) f_{T_i, T_j \mid N(t+h)=m, \Theta=\theta}(v, u) P(N(t+h) = m \mid \Theta = \theta) \\ &\quad \times f_\Theta(\theta) dv du d\theta. \end{aligned}$$

Then from equation (9) of Lemma 5.1, we have:

$$\begin{aligned}
 & E \left[ \sum_{i=1}^{N(t)} D(T_i) \sum_{j=N(t)+1}^{N(t+h)} D(T_j) \middle| x \in [0, t+h] \right] \\
 &= \sum_{i=1}^{\infty} \sum_{j=i+1}^{\infty} \int_0^t \int_0^{t+h} D(u) D(v) \left[ \sum_{m=j}^{\infty} P[N(t+h-u) = m-j | \Theta = \theta] \right] \\
 &\quad \times f_{T_{j-i} | \Theta = \theta}(u-v | \theta) f_{T_i | \Theta = \theta}(u | \theta) f_{\Theta}(\theta) du dv d\theta \\
 &= \sum_{i=1}^{\infty} \sum_{j=i+1}^{\infty} \int_0^t \int_0^{t+h} D(u) D(v) f_{T_{j-i} | \Theta = \theta}(u-v | \theta) f_{T_i | \Theta = \theta}(u | \theta) f_{\Theta}(\theta) du dv d\theta \\
 &= \sum_{i=1}^{\infty} \int_0^t \int_0^{t+h} D(u) D(v) f_{T_i | \Theta = \theta}(v) d \sum_{j=i+1}^{\infty} F_{T_i | \Theta = \theta}^{*(j-i)}(u-v) f_{\Theta}(\theta) dv d\theta, \\
 &= \int_0^t \int_0^{t+h} D(u) D(v) dm(u-v | \Theta = \theta) \sum_{i=1}^{\infty} F_{T_i | \Theta = \theta}^{*i}(v) f_{\Theta}(\theta) d\theta, \\
 &= \int_0^t \int_{t-v}^{t+h-v} D(u+v) D(v) dm(u | \Theta = \theta) dm(v | \Theta = \theta) f_{\Theta}(\theta) d\theta.
 \end{aligned}$$

Finally, as theorems 4.1 and 5.1, we have:

$$\begin{aligned}
 & E[Z(t)Z(t+h)] \\
 &= E \left[ E[Z(t)Z(t+h) | \delta(x), x \in [0, t+h]] \right], \\
 &= E[Z^2(t)] \\
 &\quad + E^2[X_1] E \left[ \int_0^t \int_{t-v}^{t+h-v} E[D(u+v)D(v)] dm(u | \Theta = \theta) dm(v | \Theta = \theta) f_{\Theta}(\theta) d\theta \right], \\
 &= E[Z^2(t)] \\
 &\quad + E^2[X_1] \int_0^t \int_{t-v}^{t+h-v} E[D(u+v)D(v)] dm(u | \Theta = \theta) dm(v | \Theta = \theta) f_{\Theta}(\theta) d\theta.
 \end{aligned}$$

## 7. LINEAR PREDICTOR

7.1 Having all the information on our risk process to time  $t$ , we are able to predict the behaviour of this risk process to better re-evaluate the premium. Since it is usually difficult to obtain the distribution explicitly, one must rely on methods of estimation, regression or simulation.



7.2 Since we have obtained the simple and joint moments of the discounted renewal claims, we can construct linear predictors based on the minimisation of the linear distance to predict the value of  $Z(t+h)$  if we know that of  $Z(t)$ . These predictors could help to assess our risk better, to estimate its variability over time or to readjust the premium, as and when information is received by the insurer.

7.3 Consider a constant instantaneous interest rate  $\delta > 0$  and a conditional counting Poisson process with parameter  $\Theta \sim \Gamma(2, 2)$ . Then Equation (10) yields:

$$\begin{aligned} & E[Z(t)Z(t+h)|\Theta] \\ &= E[X_1^2] \left( \frac{1-e^{-2\delta t}}{2\delta} \right) \Theta + E^2[X_1] \left\{ \left( \frac{1-e^{-\delta t}}{\delta} \right)^2 + e^{-\delta t} \frac{(1-e^{-\delta t})(1-e^{-\delta h})}{\delta^2} \right\} \Theta^2, \quad (11) \\ &= E[X_1^2] \left( \frac{1-e^{-2\delta t}}{2\delta} \right) \Theta + \frac{E^2[X_1]}{\delta^2} (1-e^{-\delta t})(1-e^{-\delta(t+h)}) \Theta^2, \end{aligned}$$

and,

$$E[Z(t)Z(t+h)] = E[X_1^2] \left( \frac{1-e^{-2\delta t}}{2\delta} \right) E[\Theta] + \frac{E^2[X_1]}{\delta^2} (1-e^{-\delta t})(1-e^{-\delta(t+h)}) E[\Theta^2].$$

It follows that,

$$\begin{aligned} Cov[Z(t)Z(t+h)|\Theta] &= E[X_1^2] \left( \frac{1-e^{-2\delta t}}{2\delta} \right) \Theta + \frac{1}{\delta^2} E^2[X_1] (1-e^{-\delta t})(1-e^{-\delta(t+h)}) \Theta^2 \\ &\quad - \frac{1}{\delta^2} E^2[X_1] (1-e^{-\delta t})(1-e^{-\delta(t+h)}) \Theta^2 \\ &= E[X_1^2] \left( \frac{1-e^{-2\delta t}}{2\delta} \right) \Theta, \end{aligned}$$

and,

$$Cov[Z(t)Z(t+h)] = E[X_1^2] \left( \frac{1-e^{-2\delta t}}{2\delta} \right) E[\Theta],$$

which is independent of  $h$  and then equal to  $V[Z(t)]$ , and is almost constant for large  $t$ .

If  $\rho(t, h)$  is the correlation coefficient between  $Z(t)$  and  $Z(t+h)$ , then

$$\begin{aligned} \rho(t, h) &= \frac{E[X_1^2] \left( \frac{1 - e^{-2\delta t}}{2\delta} \right) E[\Theta]}{\left[ E[X_1^2] \left( \frac{1 - e^{-2\delta t}}{2\delta} \right) E[\Theta] \right]^{1/2} \left[ E[X_1^2] \left( \frac{1 - e^{-2\delta(t+h)}}{2\delta} \right) E[\Theta] \right]^{1/2}} \\ &= \left[ \frac{1 - e^{-2\delta t}}{1 - e^{-2\delta(t+h)}} \right]^{1/2}. \end{aligned}$$

So  $\rho(t, h) \rightarrow [1 - e^{-2\delta t}]^{1/2}$  when  $h \rightarrow \infty$ , and  $\rho(t, h)$  is close to 0 for a small  $t$  and large  $h$  as expected. This strong result just shows us that the correlation coefficient  $\rho(t, h)$  doesn't depend on a specific mixture of exponential distributions for the inter-occurrence times. We consider a linear predictor  $L(t, h) = a_{t,h} + b_{t,h} Z(t)$ , where  $a_{t,h}$  and  $b_{t,h}$  eventually depend on  $t$  and  $h$ , by minimising the function  $A_{t,h}$  defined by:

$$A_{t,h}(a_{t,h}, b_{t,h}) = E\left[ [Z(t+h) - a_{t,h} - b_{t,h} Z(t)]^2 \right].$$

The partial derivative of  $A_{t,h}$  with respect to  $a_{t,h}$  and  $b_{t,h}$ , that we set equal to 0, gives

$$\begin{aligned} \frac{\partial A_{t,h}(a_{t,h}, b_{t,h})}{\partial a_{t,h}} &= -2E[Z(t+h) - a_{t,h} - b_{t,h} Z(t)] = 0, \\ \frac{\partial A_{t,h}(a_{t,h}, b_{t,h})}{\partial b_{t,h}} &= -2E[Z(t)[Z(t+h) - a_{t,h} - b_{t,h} Z(t)]] = 0. \end{aligned}$$

This results in the following system of linear equations

$$\begin{pmatrix} 1 & E[Z(t)] \\ E[Z(t)] & E[Z^2(t)] \end{pmatrix} \begin{pmatrix} a_{t,h} \\ b_{t,h} \end{pmatrix} = \begin{pmatrix} E[Z(t+h)] \\ E[Z(t)Z(t+h)] \end{pmatrix},$$

with solutions,

$$\begin{aligned} b_{t,h} &= \frac{Cov(Z(t), Z(t+h))}{Var[Z(t)]}, \\ a_{t,h} &= \frac{E[Z^2(t)]E[Z(t+h)] - E[Z(t)]E[Z(t+h)Z(t)]}{Var[Z(t)]}. \end{aligned}$$

Assume that the correlation is sufficiently strong on the period  $[t, t+h]$  and that the equation of the linear predictor of  $Z(t+h)$ , having the value of  $Z(t)$ , is given by

$$L(t, h) = E[Z(t+h)] + \rho(t, h) \left[ \frac{V[Z(t+h)]}{V[Z(t)]} \right]^{1/2} [Z(t) - E[Z(t)]] .$$

We now consider the case where the claims amount is 1 and the number of claims follow a conditional Poisson distribution with parameter  $\Theta \sim \Gamma(2, 2)$  and  $\delta = 0.03$ . Then from results of Section 3 and 4, we have

$$E[Z(t)] = E[\Theta] \left( \frac{1 - e^{-\delta t}}{\delta} \right), \text{Var}[Z(t)] = \left( \frac{1 - e^{-2\delta t}}{2\delta} \right) E[\Theta], \text{ and } \rho(t, h) = \left[ \frac{1 - e^{-2\delta t}}{1 - e^{-2\delta(t+h)}} \right]^{1/2} .$$

We can compare the simulated value of  $Z(t+h)$  to the value of  $L(t, h)$  in the table below for different values of  $t$  and  $h$ .

TABLE 3. Comparison between  $Z_{simul}(t+h) | Z(t)$  and  $L(t, h)$

$t$	$h$	$Z(t)$	$Z_{simul}(t+h)   Z(t)$	$L(t, h)$
1	0.01	0.967	0.998	0.977
1	1	0.967	1.925	1.923
1	10	0.967	9.476	9.351
10	0.01	9.985	9.996	9.992
10	1	9.985	10.895	10.715
10	10	9.985	16.523	16.385
100	0.01	100.1	100.153	100.011
100	1	100.1	100.087	100.059
100	10	100.1	100.624	100.440

7.4 We find that the difference between the estimated value and the simulated value is not very large for small  $h$  values relative to  $t$ . Similarly, when the value of  $h$  becomes large relative to  $t$ , the estimates are not very good. Obviously the price to pay with the estimated value is to calculate simple and joint moments, but with simulation we cannot perform sensitivity analysis on the discounted aggregate parameters. When the value of  $t$  is high, the discount factor cancels the effect of those moments.

## 8. CONCLUSION

8.1 We have constructed a linear predictor of the compound discounted renewal aggregate claims when taking into account dependence within the inter-occurrence times by giving explicit formulae for the first two moments of that sum. To evaluate the accuracy of the proposed linear predictor, we compare its value to the simulated value of the compound discounted renewal aggregate. The techniques used are an extension of Léveillé & Adékambi (2011, 2012).

8.2 Possible extensions to this research include the computation of the distribution of that sum with the same risk process.

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## APPENDIX

### A.1 Pareto inter-arrival claims and Clayton copula dependence

As in Albrecher et al. (2011), if  $\Theta \sim \Gamma(\alpha, \beta)$  with PDF  $f_{\Theta}(\theta) = \frac{\beta^{\alpha}}{\Gamma(\alpha)} \theta^{\alpha-1} e^{-\beta\theta}$ ,  $\theta > 0$ , it follows that  $W_i \sim \text{Pareto}(\alpha, \beta)$  with survival function  $\bar{F}_{W_1}(x) = \Pr(W_1 \geq x) = L_{\Theta}(x) = \left(1 + \frac{x}{\beta}\right)^{-\alpha}$ ,  $x > 0$  and  $f_{\Theta}^{*-1}(t) = \beta \left(t^{\frac{-1}{\alpha}} - 1\right)$ . The multivariate Pareto survival function of  $W_1, W_2, \dots, W_n$  can then be written as  $\bar{F}_{(W_1, W_2, \dots, W_n)}(x_1, x_2, \dots, x_n) = \left(1 + \sum_{i=1}^n \frac{x_i}{\beta}\right)^{-\alpha}$ ,  $x_i > 0, \forall i = 1, \dots, n$ , and  $\alpha, \beta > 0$ . The associated copula is the Clayton copula given by:

$$C_{\alpha}(u_1, \dots, u_n) = \left( u_1^{-1/\alpha} + \dots + u_n^{-1/\alpha} - n + 1 \right)^{-\alpha}.$$

### A.2 Dependent Gamma inter-arrival claims

The gamma distribution with shape parameter  $\alpha \in (0, 1]$  is completely monotone and can be accommodated in the general model introduced in Section 1. We have the following result (see Gleser, 1989; Albrecher & Kortschak, 2009).

Let  $W \sim \Gamma(\alpha, \lambda)$  be a gamma distribution with scale parameter  $\lambda$  and shape parameter  $\alpha \in (0, 1]$  and PDF

$$f_W(x) = \frac{\lambda^{\alpha}}{\Gamma(\alpha)} w^{\alpha-1} e^{-\lambda w}, \quad w > 0.$$

Then,

$$f_W(x) = \int_0^{\infty} e^{-\theta w} f_{\Theta}(\theta) d\theta,$$

where

$$f_{\Theta}(\theta) = \frac{(\theta - \lambda)^{-\alpha} \lambda^{\alpha}}{\theta \Gamma(1 - \alpha) \Gamma(\alpha)}, \quad \lambda \leq \theta < \infty, \quad (\text{A.1})$$

and  $f_{\Theta}(\theta) = 0$  otherwise.

#### Lemma A.1

The Laplace transform of the random variable  $\Theta$  with PDF (A.1) is

$$L_{\Theta}(s) = \frac{\Gamma(\alpha, \lambda s)}{\Gamma(\alpha)}, \quad s \geq 0,$$

where  $\Gamma(s, x) = \int_x^\infty t^{s-1} e^{-t} dt$  denotes the upper incomplete gamma function.

**Proof**

From the Laplace transform of  $\Theta$ ,  $f_\Theta^*(s) = \int_0^\infty e^{-s\theta} f_\Theta(\theta) d\theta$ , we take the derivative of  $f_\Theta^*(s)$  with respect to  $s$ , and obtain

$$\begin{aligned} \frac{df_\Theta^*(s)}{ds} &= \int_0^\infty -\theta e^{-s\theta} f_\Theta(\theta) d\theta, \\ &= -f_W(s), \\ &= -\frac{\lambda^\alpha}{\Gamma(\alpha)} s^{\alpha-1} e^{-\lambda s}. \end{aligned}$$

Thus,

$$f_\Theta^*(s) = \int_s^\infty \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x} dx.$$

With the change of variable  $\lambda x = u$ , we have  $f_\Theta^*(s) = \int_{\lambda s}^\infty \frac{u^{\alpha-1} e^{-u} du}{\Gamma(\alpha)} = \frac{\Gamma(\alpha, \lambda s)}{\Gamma(\alpha)}$ .

Using Lemma A.1, we get the generator function of the corresponding copula. This is given by

$$\varnothing(t) = f_\Theta^{*-1}(t) = Q_{G_\alpha}(1-t),$$

where  $Q_{G_\alpha}$  represents the quantile function of a gamma distribution with mean  $\alpha$  and unit scale parameter. From Lemma 2.1, the joint survival function is

$$\Pr(W_1 \geq x_1, W_2 \geq x_2, \dots, W_n \geq x_n) = \frac{\Gamma\left(\alpha, \lambda \sum_{j=1}^n x_j\right)}{\Gamma(\alpha)},$$

if  $x_1, \dots, x_n \geq 0$ , with marginal survival functions,  $\Pr(X_i > x) = \frac{\Gamma(\alpha, \lambda x)}{\Gamma(\alpha)}$ ,  $x \geq 0$ ,  $i = 1, 2, \dots, n$ .

The survival copula associated to the Exponential-Gamma dependent model is given by

$$\tilde{C}(u_1, \dots, u_n) = 1 - F_{G_\alpha}(G_{G_\alpha}(1-u_1) + \dots + G_{G_\alpha}(1-u_n)),$$

where  $F_{G_\alpha}$  and  $Q_{G_\alpha}$  represent the CDF and the quantile function, of a gamma distribution with shape parameter  $\alpha$ .