SOUTH AFRICAN ACTUARIAL JOURNAL
SAAJ **15** (2015) 109–29
http://dx.doi.org/10.4314/saaj.v15i1.5

Reserves in the multi-state health insurance model with stochastic interest of diffusion type

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Submission date 12 June 2015
Acceptance date 16 October 2015

ABSTRACT
In this paper, we consider the Markovian model for the actuarial modelling of health insurance policies modified by the inclusion of durational effects (the time elapsed since entering a given state) on the aggregate payment streams, where the force of interest is a diffusion process. We derive differential equations for the first moment of the present value of the aggregate amount of benefits. We also give two examples to illustrate our results.

KEYWORDS
Multi-state life insurance; semi-Markov model; counting process; first conditional moment; partial differential equations; Markov chain

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1. INTRODUCTION

1.1 Health insurance plays a vital and increasing role worldwide. It provides financial protection in case of unfortunate events such as injury from accidents, sickness, costs of healthcare or loss of earnings (Christiansen, 2012). Low cost health insurance may contribute towards improving the quality of healthcare among lower- and middle-income groups and in particular individuals who cannot afford healthcare.

1.2 The computation of the expected present values of a health policy involves substantial uncertainty. The source of this uncertainty includes estimates of the probability of a policyholder becoming sick, the number of claims received during a life insurance contract, and other variables such as the age and the health status of the policyholder. The claims received during a life contract are variables whose probability distribution is crucial for actuarial estimations. The moments of the probability distribution give insight into the likelihood distribution of claims, premiums and reserves and the riskiness of the insurance (Norberg, 1995).

1.3 This paper shows the possibility of presenting a more elegant modelling of the actuarial reserves when interest rates take the form of a diffusion process. We then calculate the premium by using the equivalence principle where the reserve at time zero is zero in the active state.

1.4 The future cost of healthcare is often dependent on economic behaviour. The probability distribution of random pattern states such as being active or disabled, for example, may fluctuate considerably during the term of the insurance contract. This variation and the behaviour of interest rates may have an effect on the estimation of premiums and benefits (Christiansen, op. cit.).

1.5 A semi-Markov chain is a stochastic process that occurs in continuous time. Under the properties of semi-Markov chains, the probability of achieving a particular situation depends on the probabilities of transitions and the time spent in each state. Therefore, this paper focuses on the calculations of the first conditional moment of present values and premiums associated with the time elapsed since entering the current state and the transition from a health state.

1.6 Considerable research has been undertaken in the area of the mathematics of health insurance. Pitacco (1995) reviewed a multi-state model for pricing disability benefits using the stochastic process framework. He presented a mathematical model for permanent disability and lump-sum benefits and illustrated a multi-state model for dread disease insurance and long-term care annuities. He outlined the need to find a unified multi-state model for pricing disability. He concluded that implementing such a multi-state model tends to be more complicated than the classical two-states model, because it requires data on populations covered and premiums charged which are very limited. As a result, it can be difficult to implement a unified model for disability insurance.

1.7 In this regard, Norberg (1995) proposed a time-continuous Markov chain model for health insurance. He obtained non-central and central higher conditional moments of present values of a standard life insurance contract with different policies and interest rates for different values of net premium by deriving an appropriate set of differential equations.

1.8 Christiansen (op. cit.) calculated the premiums and reserves, erring on the safe side. He suggested that other models can be implemented by considering the dependence on the age of the policyholder at the time of issuing of the policy and the dependence on the time spent in the respective policyholder states.

1.9 Limited research has been carried out on the estimation of standard insurance products whose payments depend on the time elapsed since entering the health status. The question is whether a method exists to obtain moments of present values by taking into consideration the time elapsed in the respective policyholder states.

1.10 This work aims to address these challenges by deriving differential equations for the first conditional moment of the present values and annuity payment for health contracts with a stochastic interest rate.

1.11 Analysing the durational effects based on the time elapsed in each state using semi-Markov properties is possible, because it considers the transition and individual state probabilities. This study uses this framework as it is one of the most appropriate approximating mathematical frameworks.

1.12 The stochastic process has been expanded in insurance applications; Stenberg, Manca and Silvestrov (2006), for example, developed a model for calculating expectations and higher order moments for accumulated rewards for disability insurance contracts using semi-Markov processes.

1.13 The differential equations for the first-order moment are the well-known Thiele equations. While Thiele introduced his equations for the Markovian framework, Hoem (1972) and Helwich (2008) generalised Thiele's equations to the semi-Markovian framework. In the Markovian framework, differential equations are also available for all higher-order moments: the variance was obtained as a double integral in the multi-state Markov model by Hoem (1969), (see also Amsler (1968) and Norberg (1991)). Norberg (1992) used martingale techniques to express the variance as a single integral. Higher order conditional moments of present values of payments related to a life insurance policy are presented in Norberg (1995). In the semi-Markovian case, Helwich (op. cit.) presented integral equations for loss variances.

1.14 The paper is organised as follows. Section 2 gives an overview of commonly used multi-state models for health insurance policies. In section 3, we give differential equations for the first moment of present value when the force of interest is of the diffusion type and provide examples to illustrate our results. Concluding comments are provided in section 4.

2. MULTI-STATE MODEL FOR HEALTH STATUS

2.1 Description of the Model

2.1.1 Throughout this section we follow the presentation and notation of Christiansen (op. cit.).

2.1.2 Let the random pattern of states of an individual policyholder be given by a pure jump process $\left(\Omega, \mathfrak{I}^X, P\left(X_t\right)_{t\geq 0}\right)$ with finite state space S and right continuous paths with left-hand limits, representing the state of the policy at time $t\geq 0$. We further define the transition space

$$J := \{(i, j) \in S \times S \mid i \neq j\},$$

the counting processes

$$N_{jk}(t) := \#\{\tau \in (0,t] \| X_\tau = k, X_{\tau-} = j\}, (j,k) \in J,$$

the time of the next jump after t

$$T(t) := \min\{\tau > t | X_\tau \neq X_{\tau-}\},$$

the series of the jump times

$$S_0 := 0,\ S_n := T(S_{n-1}),\ n \in \mathbb{N} - \{1\},$$

and a process that gives for each time the time elapsed since entering the current state,

$$U_t := \max\ \{\tau \in [0, t] \mid X_u = X_t \ \textit{for all}\ u \in [t - \tau, t]\},$$

also called the duration process. Instead of using a jump process $(X_t)_{t\geq 0}$, some authors describe the random pattern of states by a chain of jumps. The two concepts are equivalent.

2.1.3 We assume that the random pattern of states $(X_t)_{t\geq 0}$ is semi-Markovian, i.e. the bivariate process $(X_t, U_t)_{t\geq 0}$ is a Markovian process, which means that for all $i \in S$, $u \geq 0$ and $t \geq t_n \geq \dots t_1 \geq 0$ we have

$$P\left((X_t, U_t) = (i, u) \middle| X_{t_n}, U_{t_n}, \dots, X_{t_1}, U_{t_1}\right) = P\left((X_t, U_t) = (i, u) \middle| X_{t_n}, U_{t_n}\right).$$

2.1.4 We now assume that the initial state (X_0, U_0) is deterministic. (Note that $U_0 = 0$ by definition). In practice that means that we know the state of the policyholder when signing the contract. With this assumption and the Markov property of $(X_t, U_t)_{t\geq 0}$, the probability distribution of $(X_t, U_t)_{t\geq 0}$ is already uniquely defined by the transition probability matrix

$$p(s,t,u,v) = \left(P\left(X_t = k, U_t \leq v \middle| X_s = j, U_s = u\right)\right)_{(j,k)\in S^2},$$

$$0 \leq u \leq s \leq t < \infty,\ v \geq 0.$$

2.1.5 We can uniquely define the probability distribution of $(X_t, U_t)_{t\geq 0}$ by specifying the probabilities

$$\overline{p}(s,t,u)=\left(\overline{p}_{jk}(s,t,u)\right)_{(j,k)\in S^2},$$

$$\overline{p}_{jk}(s,t,u):=P\left(T(s)\le t,\ X_{T(s)}=k\ \middle|\ X_s=j, U_s=u\right),\ j\ne k,$$

$$\overline{p}_{jj}(s,t,u):=-P\left(T(s)\le t\ \middle|\ X_s=j, U_s=u\right).$$

2.1.6 A third way to define uniquely the probability distribution of $(X_t, U_t)_{t\ge 0}$ is to specify the cumulative transition intensity matrix

$$q(s,t)=\left(q_{jk}(s,t)\right)_{(j,k)\in S^2},$$

$$q_{jk}(s,t):=\int_{(s,t]}\frac{\overline{p}_{jk}(s,d\boldsymbol{\tau},0)}{1-\overline{p}_{jj}(s,\boldsymbol{\tau}-,0)},\ 0\le s\le t<\infty.$$

2.1.7 If $q(s,t)$ is differentiable with respect to *t*, we can also define the transition intensity matrix

$$\boldsymbol{\mu}_{jk}(t,t-s):=\frac{d}{dt}q(s,t)=\left(\frac{\frac{d}{dt}\overline{p}_{jk}(s,t,0)}{1-\overline{p}_{jj}(s,t,0)}\right)_{(j,k)\in S\times S},$$

which is expressed in the form of a multi-state hazard rate. The quantity $\boldsymbol{\mu}_{jk}(t,t-s)$ gives the rate of transitions from state *j* to state *k* given that the current duration of stay in *j* is *t*−*s*.

2.2 Application to the Health Insurance Contract

2.2.1 Payments between an insurer and policyholder are of two types:

- The amount payable is $b_{jk}(t,u)$ if the policy jumps from state *j* to state *k* at time *t* and the duration of stay in state *j* was *u*. In the Markovian approach the parameter *u* plays no role, and we write $b_{jk}(t,u)=b_{jk}(t)$. In order to distinguish between payments from insurer to insured and vice versa, benefit payments are given a positive sign and premium payments are given a negative sign.
- Annuity payments fall due during sojourns in a state and are defined by deterministic functions $B_j(s,t),\ j\in S$. Given that the last transition occurred at time *s*, $B_j(s,t)$ is the total amount paid in $[s,t]$ during a sojourn in state *j*. We assume that elements of the series $B_j(s,.)$ are right continuous and of bounded variation on compacts. We count the amount paid in state active as negative since this amount is the premium paid by the insured.

2.2.2 We assume that all contractual payments happen only on the time interval $[0,n]$. In insurance practice, *n* might be, for example the maximum age of a life table.

3. INTEREST RATE MODELS

3.1 First Moment of Present Value

3.1.1 For the insurer to honour its future liabilities towards the insured with certainty, it must hold adequate reserves at all times. The interest rate operating on the reserve is represented by $\delta(t)$. We then define a discounting function,

$$v(s,t) := e^{-\int_s^t \delta(\alpha)d\alpha} = e^{-(\Delta(t)-\Delta(s))}$$

where $\Delta(u)$ is the log accumulation function defined by $\Delta(u) = \int_0^u \delta(\tau)d\tau$.

3.1.2 We can interpret $v(s,t)$ as the value at time s of a unit payable at time $t \geq s$.

3.2 Log Accumulation Function of Diffusion Type

3.2.1 Here we let $\Delta(u)$ be a stochastic process. Many possibilities exist, but we assume that

$$d\Delta(t) = \delta(t)dt + \sigma(t)dW(t), \tag{1}$$

where δ and σ are deterministic positive functions and W is a standard Brownian motion. The interpretation of this model is given in Norberg & Moller (1996). The interest yield per unit of savings in the time interval $[t, t+dt]$ deviates from its mean $\delta(t)dt$ by a white noise term with variance $\sigma^2(t)dt$. The force of interest does not exist in the present model, since Δ is not of bounded variation.

3.2.2 For the model in equation (1), $\Delta(t)$ has independent and normally distributed increments,

$$\Delta(t) - \Delta(s) \sim N\left(\int_s^t \delta(\alpha)d\alpha, \int_s^t \sigma^2(\alpha)d\alpha\right),$$

which are independent of $\Im_t = \{I_j(t) = 1, U_t = u\}$. Thus, using the formula for the moment generating function of a normal variate, the function $\phi(t,\tau) = E[v(t,\tau)|\Im_t]$ is now

$$\phi(t,\tau) = \exp\left(-\int_t^\tau \delta^*(s)ds\right),$$

with

$$\delta^*(t) = \delta(t) - \frac{1}{2}\sigma^2(t).$$

3.2.3 Using Itô's lemma, $dv(t)$ can be expressed as

$$dv(t) = -v(t)\left(\delta(t)dt + \sigma(t)dW(t) - \frac{1}{2}\sigma^2(t)dt\right). \tag{2}$$

3.2.4 The discounted aggregate sum of all future benefits and premium payments is given by

$$A(t) := \sum_{j\in S}\sum_{l=0}^{\infty}\int_{(t,n]} v(t,\boldsymbol{\tau})1_{\{S_l\le\boldsymbol{\tau}<S_{l+1}\}}B_j(S_l,\,d\boldsymbol{\tau}) + \sum_{(j,\,k)\in J}\int_{(t,n]} v(t,\boldsymbol{\tau})b_{jk}(\boldsymbol{\tau},\,U_{\boldsymbol{\tau}})dN_{jk}(\boldsymbol{\tau}). \tag{3}$$

3.2.5 The quantity $A(t)$ is the amount that an insurer needs at time t in order to meet all future obligations in respect of the contract. Since we have assumed that there are no payments after time n, we have $A(t)=0$ for $t>n$.

3.2.6 Linking our development to Norberg (1995), we may alternatively write

$$A(t) = \int_{(t,n]} v(t,\tau)dB(\tau),$$

where

$$dB(\boldsymbol{\tau}) := \sum_{j\in S}\sum_{l=0}^{\infty} v(t,\boldsymbol{\tau})1_{\{S_l\le\boldsymbol{\tau}<S_{l+1}\}}B_j(S_l,\,d\boldsymbol{\tau}) + \sum_{(j,\,k)\in J} v(t,\boldsymbol{\tau})b_{jk}(\boldsymbol{\tau},\,U_{\boldsymbol{\tau}})dN_{jk}(\boldsymbol{\tau}).$$

3.2.7 $B(\tau)$ is the random total amount paid in the time interval $[0,\,t]$.

3.2.8 Here, we use the result stating that the processes $M_{jk}(t)$ defined for $j\neq k$ by

$$dM_{jk}(t) = dN_{jk}(t) - I_j(t)\boldsymbol{\mu}_{jk}(t,\,U_t)dt \tag{4}$$

are martingale processes.

3.3 Differential Equations for the First Moment of Present Value

3.3.1 STATEMENT OF EQUATION

3.3.1.1 Our goal is to derive

$$\begin{aligned} V_j(t,\,u) &= E\left[\int_{(t,n]} v(t,\tau)dB(\tau)\Big|I_j(t)=1,\,U_t=u\right] \\ &= \int_{(t,n]} \phi(t,\tau)\left[dB(\tau)\Big|I_j(t)=1,\,U_t=u\right] \end{aligned} \tag{5}$$

the first conditional moments of the present value in (3), given the information available at time t, where for ,

$$\phi(t,\tau) = E\left[v(t,\tau)\Big|I_j(t)=1,\,U_t=u\right] = r(t)E\left[v(\tau)\Big|I_j(t)=1,\,U_t=u\right], \tag{6}$$

with $v(t)=r^{-1}(t)$.

3.3.1.2 Throughout this paper, we consider that the first conditional moment of the present value, $A(t)$ exists. In mathematical formulation, we have: $E\left[A(t)\middle|I_j(t)=1, U_t=u\right]<\infty$.

3.3.2 THEOREM

3.3.2.1 The functions $V_j(t,u)$ are determined by the differential equations:

$$\begin{aligned}&\frac{\partial}{\partial t}V_j(t,u)+\frac{\partial}{\partial r}V_j(t,u)=-b_j(t,u)+\left(\boldsymbol{\delta}(t)-\frac{1}{2}\boldsymbol{\sigma}^2(t)\right)V_j(t,u)\\&\quad-\sum_{k:k\neq j}\left(V_k^l(t,0)+b_j(t,u)-V_j(t,u)\right)\boldsymbol{\mu}_{jk}(t,r),\\&V_j(n,u)=0\end{aligned} \tag{7}$$

valid on $(0,\ n)/\wp$ and subject to the condition

$$V_j(t_-,u)=\Delta B_j(t,u)+V_j(t,u). \tag{8}$$

3.3.3 REMARK

Equation (7) implies the existence of the derivatives on the left. They exist, only at those $t\in(0,\ n)/\wp$ where the δ, b_j, b_{jk} and μ_{jk} are continuous. The interpretation of equation (7) is expressed when writing $dV_j(t,u)$ on the left and multiplying by dt on the right. The resulting differential equation can then be solved. Refer to ¶3.4.4.

3.3.4 PROOF OF THEOREM

3.3.4.1 It is easier to work with present values evaluated at time 0 and use the function

$$\tilde{V}_j(t,u)=v(t)V_j(t,u). \tag{9}$$

3.3.4.2 We also note that, from equations (2) and (9)

$$\begin{aligned}d\bar{\tilde{V}}_j(t,u)=-v(t)\sum_j I_j(t)&\left(\delta(t)dt+\sigma(t)dW(t)-\frac{1}{2}\sigma^2(t)dt\right)\\&V_j(t,u)dt+v(t)d\bar{V}_j(t,u)\end{aligned} \tag{10}$$

3.3.4.3 We denote by $\Im_t$ the information carried by the counting processes by time t (i.e, the sigma-field generated by $N_{jk}(\boldsymbol{\tau}),\ j\neq k, 0\leq\boldsymbol{\tau}\leq t$). We define the martingale M by

$$M(t)=E\left(\int_0^n v\,dB\middle|\Im_t\right). \tag{11}$$

3.3.4.4 Upon expanding the expression $\int_0^n v\,dB=\int_0^t v\,dB+\int_t^n v\,dB$ with

$$K(t) = \int_0^t v\, dB,$$

and from the Markov property, we have

$$E\left(\left(\int_t^n v\, dB\right) \| \Im_t\right) = \sum_j I_j(t)\tilde{V}_j(t,u) \tag{12}$$

giving

$$M(t) = \sum_j I_j(t)\left(K(t) + \tilde{V}_j(t,u)\right).$$

3.3.4.5 Note that M is right continuous, and $\tilde{V}_j(t,u)$ is also right-continuous since K and I_j are. Since our goal is to derive a differential equation for $\tilde{V}_j(t,u)$, we need the following differential form for M:

$$dM(t) = \sum_j d\left(I_j(t)\left(K_j(t) + \tilde{V}_j(t,u)\right)\right). \tag{13}$$

3.3.4.6 We apply the change of variable formula to the terms on the right hand side and use the fact that the continuous part of a function X is denoted by $\overline{X}$:

$$\sum_j d\left(I_j(t)\left\{K_j(t) + \tilde{V}_j(t,u)\right\}\right)$$

$$= \sum_j I_j(t)\left\{d\overline{K}_j(t) + d\overline{\tilde{V}}_j(t,u)\right\} \quad (14)$$

$$+ \sum_j \left\{K_j(t) + \tilde{V}_j(t,u)\right\} d\overline{I}_j(t) \ (15)$$

$$+\left[\sum_j I_j(t)\left\{K_j(t) + \tilde{V}_j(t,u)\right\} - I_j(t_-)\left\{K_j(t_-) + \tilde{V}_j(t_-,u)\right\}\right] (16)$$

3.3.4.7 We express the terms on the right as follows:

– In expression (14) put $d\overline{K}(t) = v(t)\sum_k I_k(t)b_k(t,u)\,dt$ and use $I_k(t)I_j(t) = I_j(t)\boldsymbol{\delta}_{jk}$.

– In expression (15) the first term disappears since I_j is constant.

3.3.4.8 The jump term in (16) can be of two types. Firstly, a possible transition between states causes a jump of size

$$\sum_{k \neq j}\left(K(t_-) + v(t)b_{jk}(t,u) + \tilde{V}_k(t,0)\right) - \left(K(t_-) + \tilde{V}_j(t_-,u)\right)dN_{jk}(t).$$

3.3.4.9 Secondly, a possible lump sum annuity payment causes a jump of size

$$\sum_{k \neq j}\left(K(t_-) + v(t)\Delta B_j(t,u) + \tilde{V}_j(t,u)\right) - \left(K(t_-) + \tilde{V}_j(t_-,u)\right),$$

which can differ from 0 only at time $t \in \wp$. Here, we replace $I_j(t)$ by $I_j(t_-)$ since these coincide with probability 1 on $\wp$. Using the same argument, jumps of contractual annuity functions and transitions between states are null with probability 1 and can be disregarded. Since $\int_t^n v\,dB$ is continuous at each fixed $t \notin \wp$, we do not have any further jump terms from the functions $\tilde{V}_j(t, u)$ at such t.

3.3.4.10 We define the functions $\tilde{b}_j, \tilde{b}_{jk}, \Delta\tilde{B}_j$ by:

$$\tilde{b}_j(t, u) = v(t) b_j(t, u),\ \tilde{b}_{jk}(t, u) = v(t) b_{jk}(t, u),\ \Delta\tilde{B}_j(t, u) = v(t)\Delta B_j(t, u),$$

where

$$\Delta B_j(s,u) := B_j(s,u) - B_j(s,u-)$$

and we gather:

$$\begin{aligned} d\left(\sum_j I_j(t)\left\{K(t)+\tilde{V}_j(t, u)\right\}\right) &= \sum_j I_j(t)\left\{\tilde{b}_j(t, u)\,dt + d\tilde{V}_j(t, u)\right\} \\ &+\sum_{j\neq k}\left\{\left(K(t_-)+\tilde{b}_j(t, u)+\tilde{V}_k(t, 0)\right)-\left(K(t_-)+\tilde{V}_j(t_-, u)\right)\right\} dN_{jk}(t) \\ &+\sum_j I_j(t_-)\left\{\left(K(t_-)+\Delta\tilde{B}_j(t, u)+\tilde{V}_j(t, u)\right)-\left(K(t_-)+\tilde{V}_j(t_-, u)\right)\right\}. \end{aligned}$$

3.3.4.11 Inserting this into equation (13) and then using equation (4) and, noting the general rule $X(t_-)dt = X(t)dt$, we obtain:

$$\begin{aligned} &dM(t) - \sum_{j\neq k}\left\{\left(K(t_-)+\tilde{b}_j(t, u)+\tilde{V}_k(t, 0)\right)-\left(K(t_-)+\tilde{V}_j(t_-, u)\right)\right\} dM_{jk}(t) \\ &= \sum_j I_j(t)\left(\tilde{b}_j(t, u)\,dt + d\tilde{V}_j(t, u)\right) \\ &+\sum_j I_j(t)\sum_{k:k\neq j}\left\{\left(K(t_-)+\tilde{b}_{jk}(t, u)+\tilde{V}_k(t, 0)\right)-\left(K(t_-)+\tilde{V}_j(t_-, u)\right)\right\} \\ &\qquad \boldsymbol{\mu}_{jk}(t, u)\,dt \\ &+\sum_j I_j(t_-)\left\{\left(K(t_-)+\Delta\tilde{B}_j(t, u)+\tilde{V}_j(t, u)\right)-\left(K(t_-)+\tilde{V}_j(t_-, u)\right)\right\} \end{aligned} \qquad (17)$$

3.3.4.12 From equation (10), the term $d\tilde{V}_j(t, u)$ appearing inside the parentheses multiplied by $I_j(t)$ has the value

$$\begin{aligned} d\tilde{V}_j(t, u) &= v(t)\left(\left(-\boldsymbol{\delta}(t)+\frac{1}{2}\boldsymbol{\sigma}^2(t)\right)V_j(t, u)\,dt - \boldsymbol{\sigma}(t)V_j(t, u)\,dW(t)\right). \\ &+v(t)\,dV_j(t, u) \end{aligned} \qquad (18)$$

3.3.4.13 But $dM'(t) = -\boldsymbol{\sigma}(t)V_j(t, u)\,dW = 0$ since

$M'(t) = -\int_0^t \boldsymbol{\sigma}(\tau) V_j(\tau, u)\, dW(\tau)$ is a martingale, continuous and of bounded variation, it must be constant (see Chung & Williams, 1990: 87).

3.3.4.14 Equation (18) becomes

$$d\bar{\bar{V}}_j(t, u) = -v(t)\left(\boldsymbol{\delta}(t) - \frac{1}{2}\boldsymbol{\sigma}^2(t)\right) V_j(t, u)\, dt + v(t)\, d\bar{V}_j(t, u). \tag{18a}$$

3.3.4.15 We then take equation (18) as a notation.

3.3.4.16 Now, on the left of equation (17) is the differential of a sum of martingales (note that the coefficients of $dM_{jk}(t)$ are predictable). Thus, the expression on the right must also be the differential of a martingale. Since this martingale is predictable and of bounded variation, it must be constant due to the Doob-Meyer decomposition (refer to chapter 2, Andersen et al., 1993). We conclude that both the continuous part and the discrete part must have increments that are identically 0, and that this is true for all outcomes of the indicator processes if and only if

$$\tilde{b}_j(t, u)\, dt + d\bar{\bar{V}}_j(t, u) + \sum_{k:k \neq j} \left\{ \tilde{b}_{jk}(t, u) + \tilde{V}_k(t, 0) - \tilde{V}_j(t, u) \right\} \boldsymbol{\mu}_{jk}(t, u)\, dt = 0 \tag{19}$$

for all j and all $t \in (0, n) \setminus \wp$.

3.3.4.17 Finally, replacing $\tilde{b}_j(t, u) = v(t) b_j(t, u)$, $\tilde{b}_{jk}(t, u) = v(t) b_{jk}(t, u)$ and equation (17) in (19) we have:

$$\begin{aligned}
0 &= v(t) b_j(t, u)\, dt - v(t)\left(\boldsymbol{\delta}(t) - \frac{1}{2}\boldsymbol{\sigma}^2(t)\right) V_j(t, u)\, dt + v(u)\, dV_j(t, u) \\
&\qquad + \sum_{k:k \neq j} \left\{ \left(v(t) b_{jk}(t, u) + v(t) V_k(t, 0) \right) - v(t) V_j(t, u) \right\} \boldsymbol{\mu}_{jk}(t, u)\, dt \\
&= b_j(t, u) - \left(\boldsymbol{\delta}(t) - \frac{1}{2}\boldsymbol{\sigma}^2(t)\right) V_j(t, u) + dV_j(t, u) \\
&\qquad + \sum_{k:k \neq j} \left\{ \left(b_{jk}(t, u) + V_k(t, 0) \right) - V_j(t, u) \right\} \boldsymbol{\mu}_{jk}(t, u) \\
&= b_j(t, u) - \left(\boldsymbol{\delta}(t) - \frac{1}{2}\boldsymbol{\sigma}^2(t)\right) V_j(t, u) + \frac{\partial V_j(t, u)}{\partial t} + \frac{\partial V_j(t, u)}{\partial u} \\
&\qquad + \sum_{k:k \neq j} \left\{ \left(b_{jk}(t, u) + V_k(t, 0) \right) - V_j(t, u) \right\} \boldsymbol{\mu}_{jk}(t, u).
\end{aligned} \tag{20}$$

3.3.5 REMARK

$$\text{From } V_j(t, u) = E\left[\int_{(t,n]} v(t,\tau)dB(\tau)\middle| I_j(t)=1, U_t = u\right]$$

$$= \int_{(t,n]} \phi(t,\tau)\left[dB(\tau)\middle| I_j(t)=1, U_t = u\right]$$

and since $v(t,\tau)$ is lognormal, $\phi(t,\tau) = E\left[v(t,\tau)\right] = e^{-\left(\delta(\tau-t)-\frac{1}{2}\sigma^2(\tau-t)\right)}$. The solutions of equations (18) when the force of interest is constant denoted by $\delta(t)$ are equivalent to the case where the log accumulation is following equation (1). We just need to rewrite the accumulation function as $\Delta(u) = \int_0^u \boldsymbol{\delta}^*(\boldsymbol{\tau})d\boldsymbol{\tau}$ where $\delta^*(t) = \delta(t) - \frac{1}{2}\sigma^2(t)$.

3.4 Examples

3.4.1 As analytical solutions for the differential equation (19) are generally out of reach, we discuss numerical methods here.

3.4.2 Under the assumption that the derivatives $b_j(t,t-s) := \frac{\partial}{\partial t} B_j(s,t)$ exist, we now discuss a numerical solution method for the ordinary differential equations.

3.4.3 THE ACTIVE-DEAD MODEL

3.4.3.1 We take the following example from Buchardt, Møller & Schmidt (2014) where they have only calculated the first moment numerically. Our results allow us to calculate all the higher condition moments of their two-states survival model.

3.4.3.2 Buchardt, Møller & Schmidt (op. cit.) consider a two-state Markov model with states 0, alive, and 1, dead. They consider an insured male of age 40 with pension age 65, and two products:

– a life annuity, starting at age 65, and
– a 10-year annuity upon death, if death occurs before age 65.

3.4.3.3 They also specify the payment functions as follows:

– $b_a(t,u) = 37,404 \times 1_{\{t \geq 25\}}$, a life annuity, and
– $b_d(t,u) = 18,702 \times 1_{\{t-u < 25\}} \times 1_{\{u < 10\}}$, a death annuity upon death.

3.4.3.4 We then have:

– Log accumulation process, $d\Delta(t) = 0.015dt + 0.25dW(t)$, and
– $\boldsymbol{\mu}_x = 0.0005 + 0.000075858 \times 1.09144^x$

3.4.3.5 The annuity upon death is dependent on the duration *u*, since it is only for the 10 first years after death that there is a payment.

3.4.3.6 We are then in practice in the semi-Markov set-up, even though the transition rates are not duration dependent, but $(X_t)_{t\geq 0}$ is in fact a Markov process.

3.4.3.7 If the state space consists only of the states a = active and d = dead, explicit solutions for the first moment can be derived from the differential equations in (19) by using the method of variation of constants.

3.4.3.8 Since the transition intensity from dead to active is zero, the system of ordinary partial differential equations according to equation (19) has for the first moment the form

$$\frac{d\tilde{V}_d(t,u)}{dt} = -\tilde{b}_d(t,u)$$

for the state d = dead and

$$\frac{d\tilde{V}_d(t,u)}{dt} = \boldsymbol{\mu}_{ad}(t,u)\tilde{V}_d(t,u) + \tilde{F}_1(t,u)$$

for the state a = active, where

$$F_1(t,s) = -b_a(t,u) - \boldsymbol{\mu}_{ad}(t,u)\big(b_{ad}(t,u) + V_d(t,u)\big) \text{ and } \tilde{F}_1(t,u) = e^{-\delta^* t}F_1(t,u).$$

3.4.3.9 The ordinary differential equations can be successively solved by applying the method of variation of constants. For the first-order moment conditional on state dead, the solution of the homogeneous problem is

$$V_d(t,u) = \int_t^n e^{-(\tau-t)\delta^*} b_d(\tau,u)d\tau \tag{21}$$

3.4.3.10 This trivial structure of the solution in state dead is due to the fact that there is no randomness once the policyholder is in state dead. Now we continue with the ordinary differential equations for state active. By applying the method of variation of constants for the first-order moment, we obtain

$$\tilde{V}_a(t,s) = \int_t^n \Big(\tilde{b}_a(\tau,u) + \boldsymbol{\mu}_{ad}(\tau,u)\big(\tilde{b}_{ad}(\tau,u) + \tilde{V}_d(\tau,u)\big)\Big) e^{-\int_t^\tau \mu_{ad}(u,\, u-s)du} d\tau,$$

where we may replace $\tilde{V}_d(\tau,u)$ with the explicit formula (21). Similar and successive calculations lead to

$$\tilde{V}_a(t,s) = \int_t^n \tilde{L}_1(\tau,u) e^{-\int_t^\tau \left(\delta^* + \mu_{ad}(\alpha,u)\right)d\alpha} d\tau$$

with

$$L_1(\tau,\, u) = F_1(\tau,u) - b_a(\tau,u),\ \tilde{L}_1(\tau,\, u) = e^{-\delta^* t}L_1(\tau,\, u).$$

3.4.3.11 Note that $L_1(\tau,\, s)$ depends only on expressions that have already been calculated.

3.4.3.12 We may also apply the above formulas to a two-state disability model, where the disability payment has a durational effect.

3.4.4 THE DISABILITY MODEL

3.4.4.1 The example described here is similar to the example set out in Norberg & Moller (op. cit.). We change the values of the benefits between transitions and the resulting accumulation factor follows a diffusion process. The policy terms are set out below.

3.4.4.2 The insurance period is n years and the premium is paid continuously with a constant rate of P for a period of at most m years, $m<n$, as long as the insured is in state 1. In state 2, an annuity is paid continuously to the insured with a constant rate b and a qualifying period of one year before receiving benefits as disabled. This gives that $b_2(t,r)=bI(U_t\geq 1)$. A benefit amount S, the sum insured, is paid immediately upon death within time n. Furthermore, the reserve $V_1(t,r)$ is paid to the insured if a transition from state 1 occurs at time t. Thus, $b_{12}(t,r)=V_1(t,r)$, $b_{13}(t,r)=S+V_1(t,r)$, $b_{23}(t,r)=S$ and $b_{21}(t,r)=0$. We consider the same interest model as in the active-dead model.

3.4.4.3 Since $V_3(t,r)=0$ it remains to find the equivalence premium and the state reserves $V_j(n,r)$, $j=1, 2$, with initial condition $V_j(n,r)=0$. Now

$$\frac{d\tilde{V}_j^1(t,r)}{dt}=\tilde{V}_j^1(t,r)\boldsymbol{\mu}_{j\bullet}(t,r)-\tilde{b}_j(t,r)-\sum_{k:k\neq j}\tilde{V}_k^1(t,0)\boldsymbol{\mu}_{jk}(t,r)-\sum_{k:k\neq j}\tilde{b}_{jk}(t,r)\boldsymbol{\mu}_{jk}(t,r)$$

3.4.4.4 We have the following system of differential equations which are similar to the corresponding system given in Norberg & Moller (op. cit.):

$$\begin{cases}\dfrac{d\tilde{V}_a^1(t)}{dt}=-\boldsymbol{\mu}_{ai}(t)\tilde{V}_i^1(t,0)+v^t\left[P-S\boldsymbol{\mu}_{ad}(t)\right]\\[2ex] \dfrac{d\tilde{V}_i^1(t,0)}{dt}=-\boldsymbol{\mu}_{ia}(t)\tilde{V}_a^1(t)+\tilde{V}_i^1(t,0)\boldsymbol{\mu}_{i\bullet}(t)-v^t\left[S\boldsymbol{\mu}_{id}(t)+bI(U_t\geq 1)\right]\end{cases}$$

or the following matrix representation

$$\begin{pmatrix}\dfrac{d\tilde{V}_a^1(t)}{dt}\\[2ex] \dfrac{d\tilde{V}_i^1(t,0)}{dt}\end{pmatrix}=\begin{pmatrix}0 & -\boldsymbol{\mu}_{ai}(t)\\ -\boldsymbol{\mu}_{ia}(t) & \boldsymbol{\mu}_{i\cdot}(t)\end{pmatrix}\begin{pmatrix}\tilde{V}_a^1(t)\\ \tilde{V}_i^1(t,0)\end{pmatrix}+\begin{pmatrix}v^t\left[P-S\boldsymbol{\mu}_{ad}(t)\right]\\ -v^t\left[S\boldsymbol{\mu}_{id}(t)+bI(U_t\geq 1)\right]\end{pmatrix} \quad (22)$$

and

$$\frac{d\tilde{V}_i^1(t,r)}{dt}=\tilde{V}_i^1(t,r)\boldsymbol{\mu}_{i\cdot}(t)-\tilde{V}_a^1(t)\boldsymbol{\mu}_{ia}(t)-v^t\left[S\boldsymbol{\mu}_{id}(t)+bI(U_t\geq 1)\right].$$

3.4.4.5 The premium P is calculated such that $\tilde{V}_a^1(0)=0$.

3.4.4.6 To solve equation (22) we use the procedure described in Moller (1993) to obtain the following differential system for $\left(\tilde{V}_a^1(t), \tilde{V}_i^1(t,0)\right)$

$$\begin{pmatrix} \dfrac{d\tilde{V}_a^1(t)}{dt} \\ \dfrac{d\tilde{V}_i^1(t,0)}{dt} \end{pmatrix} = \begin{pmatrix} 0 & -\boldsymbol{\mu}_{ai}(t) \\ -\boldsymbol{\mu}_{ia}(t) & \boldsymbol{\mu}_{i\bullet}(t) \end{pmatrix} \begin{pmatrix} \tilde{V}_a^1(t) \\ \tilde{V}_i^1(t,0) \end{pmatrix} + \begin{pmatrix} v^t\left[P - S\boldsymbol{\mu}_{ad}(t)\right] \\ -v^t\left[S\boldsymbol{\mu}_{id}(t) + b_2(t)\right] \end{pmatrix} \quad (23)$$

where $b_2(t) = be^{-\int_t^{t+1}\{\delta + \mu_{i\bullet}(u)\}du}$.

3.4.4.7 Define the following vectors:

$$\tilde{V}(t) = \left(\tilde{V}_a^1(t), \tilde{V}_i^1(t,0)\right)'$$

$$H(t) = \left(H_1(t), H_2(t)\right)'$$

where system (23) can then be written as

$$\frac{d\tilde{V}(t)}{dt} = \Lambda(t)\tilde{V}(t) + H(t),\ t \in I,$$

and where

$$-\Lambda(t) = \begin{pmatrix} 0 & -\boldsymbol{\mu}_{ai}(t) \\ -\boldsymbol{\mu}_{ia}(t) & \boldsymbol{\mu}_{i\bullet}(t) \end{pmatrix} \text{ and } H(t) = \begin{pmatrix} v^t\left[P - S\boldsymbol{\mu}_{ad}(t)\right] \\ -v^t\left[S\boldsymbol{\mu}_{id}(t) + b_2(t)\right] \end{pmatrix}.$$

3.4.4.8 A system like (23) is solved by considering the corresponding homogeneous system obtained by putting $H(t) \equiv 0$. The unique solution $\tilde{V}^u(t)$ to the homogeneous system with initial condition $\tilde{V}^u(t_0) = X_0$ for some fixed $t_0 \in I$, is of the form $\tilde{V}^u(t) = \hat{\boldsymbol{\phi}}(t_0, t)X_0,\ t \in I$, where $\hat{\boldsymbol{\phi}}(t_0, t)$ is a 2×2 matrix called the fundamental matrix or the basic solution chosen such that $\hat{\boldsymbol{\phi}}(t_0, t_0)$ the unit matrix 1. The initial condition $\tilde{V}(w) = v^w a,\ w \le \infty$, is given by

$$\begin{aligned} \tilde{V}(t) &= v^w\hat{\boldsymbol{\phi}}(w,t)a - \hat{\boldsymbol{\phi}}(w,t)\int_t^w \hat{\boldsymbol{\phi}}^{-1}(w,\boldsymbol{\tau})H(\boldsymbol{\tau})d\boldsymbol{\tau} \\ &= v^w\hat{\boldsymbol{\phi}}^{-1}(w,t)a - \hat{\boldsymbol{\phi}}^{-1}(t,w)\int_t^w \hat{\boldsymbol{\phi}}(\boldsymbol{\tau},w)H(\boldsymbol{\tau})d\boldsymbol{\tau} \\ &= v^w\hat{\boldsymbol{\phi}}^{-1}(t,w)a - \int_t^w \hat{\boldsymbol{\phi}}^{-1}(t,\boldsymbol{\tau})H(\boldsymbol{\tau})d\boldsymbol{\tau}. \end{aligned}$$

3.4.4.9 Let $\hat{\boldsymbol{\phi}}$ be the basic solution to this system. Then with $\hat{\boldsymbol{\phi}}^{-1} = \left(\boldsymbol{\psi}_{ij}\right)_{1 \le i,j \le 3}$ we find the equivalence premium as the solution to $V_a^1(0) = 0$.

3.4.4.10 For the disability model example, we have:

$$V_a^1(t)=\int_t^n v^{\tau-t}\psi_{11}(t,\tau)S\mu_{ad}(\tau)d\tau-P\int_t^n v^{\tau-t}\psi_{11}(t,\tau)d\tau$$
$$+\int_t^n v^{\tau-t}\psi_{12}(t,\tau)\{S\mu_{id}(\tau)+b_2(\tau)\}d\tau$$

3.4.4.11 Using the equivalence principle, we calculate the premium P by setting $V_a^1(0)=0$. We have

$$P=\frac{\int_0^n v^{\tau}\psi_{11}(0,\tau)S\mu_{ad}(\tau)d\tau+\int_0^n v^{\tau}\psi_{12}(0,\tau)\{S\mu_{id}(\tau)+b_2(\tau)\}d\tau}{\int_0^n v^{\tau}\psi_{11}(0,\tau)d\tau}.$$

3.4.4.12 Once we have calculated the value P of the premium, using equation (23), we have the value $\tilde{V}_i^1(t,0)$ of the reserve at the invalid state

$$V_i^1(t,0)=\int_t^n v^{\tau-t}\left[\psi_{21}(t,\tau)S\mu_{12}(\tau)+\psi_{22}(t,\tau)(S\mu_{id}(\tau)+b_2(\tau))\right]d\tau$$
$$-P\int_t^n v^{\tau-t}\psi_{21}(t,\tau)d\tau$$

and finally

$$V_i^1(t,u)=\int_t^n e^{-\int_t^{\tau}\mu_{i\bullet}(s)ds}\left[\psi_{21}(t,\tau)\mu_{ia}(\tau)V_a^1(t)+v^{\tau}S\mu_{id}(\tau)\right]d\tau$$
$$+\int_{t+1-u}^n v^{\tau}e^{-\int_t^{\tau}\mu_{i\bullet}(s)ds}d\tau$$

TABLE 1. Values of the reserves and the premium when $S=4000$, $b=2000$, $i=3.5\%$

	$x=40$ / $n=25$	$x=50$ / $n=15$
P	79.7	131.4
$V_1=(5)$	92.3	239.1
$V_2=(5,0)$	13,590.2	7,398.16
$V_2=(5,1/5)$	13,067.6	7,799.6
$V_1=(10)$	398.2	479.31
$V_2=(10,0)$	10,102.1	3,815.9
$V_2=(10,2/5)$	11,016.5	4,345.5
$V_1=(15)$	441.5	–
$V_2=(15,0)$	7,930.68	–
$V_2=(15,3/5)$	8,063.66	–

3.4.4.13 We have empty cells in the table because for a 15-year term insurance, at year 15, the insured is no longer covered therefore the reserve is zero.

4. FORCE OF INTEREST OF DIFFUSION TYPE

4.1 Model Description

4.1.1 The model described in section 3.2 is an unsophisticated model of the interest rate. A more realistic model could be obtained by modelling the interest rate itself as a diffusion process given by a stochastic differential equation

$$d\boldsymbol{\delta}(t)=\boldsymbol{\kappa}(t,\boldsymbol{\delta}(t))\,dt+\boldsymbol{\sigma}(t,\boldsymbol{\delta}(t))\,dW(t),$$

where $\boldsymbol{\kappa}$ and $\boldsymbol{\sigma}$ are some appropriate functions and W is a standard Brownian motion. Since the process is Markovian, the conditionally expected discount rate, $E\left[v(t,\tau)\middle|I_j(t)=1, U_t=u\right]$ is now a function of $\boldsymbol{\delta}(t)$ (as well as t and τ) and we write $\boldsymbol{\phi}(t,\boldsymbol{\tau},\boldsymbol{\delta}(t))$.

4.1.2 We use the Vasicek (1977) model and the Ornstein–Uhlenbeck (OU) model which give explicit expressions for $\boldsymbol{\phi}(t,\boldsymbol{\tau},\boldsymbol{\delta}(t))$. In the Vasicek (op. cit.) model, the differential equation of $\boldsymbol{\delta}(t)$ is given by

$$d\boldsymbol{\delta}(t)=\boldsymbol{\kappa}\left(\bar{\boldsymbol{\delta}}-\boldsymbol{\delta}(t)\right)dt+\boldsymbol{\sigma}\,dW(t).$$

4.1.3 The key feature of this process is that it is mean reverting. If the process value, $\boldsymbol{\delta}(t)$, is below $\bar{\boldsymbol{\delta}}$ at time *t*, the drift coefficient $\boldsymbol{\kappa}\left(\bar{\boldsymbol{\delta}}-\boldsymbol{\delta}(t)\right)$ has a positive value, so that the process has a tendency to move upwards. If, on the other hand, the process value is above $\bar{\boldsymbol{\delta}}$, the drift coefficient $\boldsymbol{\kappa}\left(\bar{\boldsymbol{\delta}}-\boldsymbol{\delta}(t)\right)$ has a negative value, so that the process tends to move downwards. So the process is always drawn towards the constant value $\bar{\boldsymbol{\delta}}$. The constant parameter $\boldsymbol{\kappa}$ controls the rate of mean reversion, that is, how strongly the process is drawn back to the value $\bar{\boldsymbol{\delta}}$. A higher value of $\boldsymbol{\kappa}$ causes the process to pull more strongly towards the value $\bar{\boldsymbol{\delta}}$.

4.1.4 The parameter $\boldsymbol{\sigma}$ is a positive constant.

$$\Delta(\boldsymbol{\tau})-\Delta(t)\sim N\left(\boldsymbol{\mu}(t,\boldsymbol{\tau}),\boldsymbol{\sigma}^2(t,\boldsymbol{\tau})\right),$$

with

$$\boldsymbol{\mu}(t,\boldsymbol{\tau})=(u-t)\bar{\boldsymbol{\delta}}+\frac{1}{\boldsymbol{\kappa}}\left(1-e^{-\boldsymbol{\kappa}(u-t)}\right)\left(\boldsymbol{\delta}(t)-\bar{\boldsymbol{\delta}}\right),\text{ and}$$

$$\boldsymbol{\sigma}^2(t,\boldsymbol{\tau})=\frac{\boldsymbol{\sigma}^2}{\boldsymbol{\kappa}^3}\left((u-t)\bar{\boldsymbol{\delta}}+\frac{1}{2}\left(1-\left(2-e^{-\boldsymbol{\kappa}(u-t)}\right)^2\right)\right)$$

4.1.5 It follows that the function in equation (6) now is

$$\boldsymbol{\phi}(t,\boldsymbol{\tau},\boldsymbol{\delta}(t))=\exp\left(-\boldsymbol{\mu}(t,\boldsymbol{\tau})+\frac{1}{2}\boldsymbol{\sigma}^2(t,\boldsymbol{\tau})\right).$$

4.1.6 Cox, Ingersoll & Ross (1985) proposed to replace $\boldsymbol{\sigma}$ in the expression $d\boldsymbol{\delta}(t)=\boldsymbol{\kappa}\left(\overline{\boldsymbol{\delta}}-\boldsymbol{\delta}(t)\right)dt+\boldsymbol{\sigma}dW(t)$ with $\boldsymbol{\sigma}\sqrt{\boldsymbol{\delta}(t)}$. Their model is a reference model in financial economics and is represented by the acronym CIR. The CIR model also admits a closed formula for $\boldsymbol{\phi}(t,\boldsymbol{\tau},\boldsymbol{\delta}(t))$ but, in contrast to the OU model, it cannot take negative values. The fact that the CIR model can only take positive values makes it more suitable than the OU model to describe the behaviour of interest.

4.2 Differential Equations for the First Moment of Present Value

4.2.1 STATEMENT OF EQUATION

Our goal is to derive $V_j(t,r)=E\left[A(t)\middle|I_j(t)=1,U_t=r\right]$, the first conditional moment of the present value in equation (3), given the information available at time *t*.

4.2.2 THEOREM

The functions $V_j(t,r)$ are determined by the differential equations:

$$\begin{aligned}&\frac{\partial}{\partial t}V_j(t,u,\boldsymbol{\delta}(t))+\frac{\partial}{\partial u}V_j(t,u,\boldsymbol{\delta}(t))+b_j(t,u)-\boldsymbol{\delta}_j(t)V_j(t,u,\boldsymbol{\delta}(t))\\&+\frac{1}{2}\boldsymbol{\sigma}^2(t,\boldsymbol{\delta}(t))\frac{\partial^2}{\partial\boldsymbol{\delta}^2}V_j(t,u,\boldsymbol{\delta}(t))+\boldsymbol{\kappa}(t,\boldsymbol{\delta}(t))\frac{\partial}{\partial\boldsymbol{\delta}}V_j(t,u,\boldsymbol{\delta}(t))\\&+\sum_{k:k\neq j}\left(V_k^l(t,0,\boldsymbol{\delta}(t))+b_j(t,u)-V_j(t,u,\boldsymbol{\delta}(t))\right)\boldsymbol{\mu}_{jk}(t,u)=0,\end{aligned}\tag{24}$$

$$V_j(n,u,\boldsymbol{\delta}(t))=0$$

valid on $(0,\ n)/\wp$ and subject to the condition

$$V_j(t_-,r)=\Delta B_j(t,r)+V_j(t,r)\tag{25}$$

4.2.3 PROOF

4.2.3.1 The functions V_j will now depend on not only *t* and *u* but also on $\delta(t)$.

4.2.3.2 From $dv(t)=-v(t)\delta(t)dt$, equation (18) becomes:

$$
\begin{aligned}
d\tilde{V}_j(t,u,\boldsymbol{\delta}(t)) &= -v(t)\boldsymbol{\delta}(t)V_j(t,u,\boldsymbol{\delta}(t))\,dt + v(t)\,dV_j(t,u,\boldsymbol{\delta}(t)) \\
&= -v(t)\boldsymbol{\delta}(t)V_j(t,u,\boldsymbol{\delta}(t))\,dt \\
&\quad + v(t)\left(\frac{\partial}{\partial t}V_j(t,u,\boldsymbol{\delta}(t))\,dt + \frac{\partial}{\partial u}V_j(t,u,\boldsymbol{\delta}(t))\,du\right) \\
&\quad + v(t)\frac{\partial}{\partial \boldsymbol{\delta}}V_j(t,u,\boldsymbol{\delta}(t))\left(\boldsymbol{\kappa}(t,\boldsymbol{\delta}(t))\,dt + \boldsymbol{\sigma}(t,\boldsymbol{\delta}(t))\,dW(t)\right) \\
&\quad + \frac{1}{2}v(t)\boldsymbol{\sigma}^2(t,\boldsymbol{\delta}(t))\frac{\partial^2}{\partial \boldsymbol{\delta}^2}V_j(t,u,\boldsymbol{\delta}(t))
\end{aligned}
\tag{26}
$$

but

$$
dM'(t) = \frac{\partial}{\partial \boldsymbol{\delta}}V_j(t,u,\boldsymbol{\delta}(t))\boldsymbol{\sigma}(t,\boldsymbol{\delta}(t))\,dW(t) = 0,
$$

since $M''(t) = -\int_0^t \frac{\partial}{\partial \boldsymbol{\delta}}V_j(\boldsymbol{\tau},u,\boldsymbol{\delta}(\boldsymbol{\tau}))\boldsymbol{\sigma}(\boldsymbol{\tau},\boldsymbol{\delta}(\boldsymbol{\tau}))\,dW(\boldsymbol{\tau})$ is a martingale, continuous and of bounded variation, it must be constant.

4.2.3.3 Then,

$$
\begin{aligned}
d\tilde{V}_j(t,u,\boldsymbol{\delta}(t)) &= -v(t)\boldsymbol{\delta}(t)V_j(t,u,\boldsymbol{\delta}(t))\,dt + v(t)\,dV_j(t,u,\boldsymbol{\delta}(t)) \\
&= -v(t)\boldsymbol{\delta}(t)V_j(t,u,\boldsymbol{\delta}(t))\,dt \\
&\quad + v(t)\left(\frac{\partial}{\partial t}V_j(t,u,\boldsymbol{\delta}(t))\,dt + \frac{\partial}{\partial u}V_j(t,u,\boldsymbol{\delta}(t))\,du\right) \\
&\quad + v(t)\frac{\partial}{\partial \boldsymbol{\delta}}V_j(t,u,\boldsymbol{\delta}(t))\boldsymbol{\kappa}(t,\boldsymbol{\delta}(t))\,dt \\
&\quad + \frac{1}{2}v(t)\boldsymbol{\sigma}^2(t,\boldsymbol{\delta}(t))\frac{\partial^2}{\partial \boldsymbol{\delta}^2}V_j(t,u,\boldsymbol{\delta}(t))\,dt
\end{aligned}
\tag{27}
$$

4.2.3.4 Finally, replacing $\tilde{b}_j(t,r) = v(t)b_j(t,r)$, $\tilde{b}_{jk}(t,r) = v(t)b_{jk}(t,r)$ and equation (27) in (19) we have:

$$
\begin{aligned}
&\frac{\partial}{\partial t}V_j(t,u,\boldsymbol{\delta}(t)) + \frac{\partial}{\partial u}V_j(t,u,\boldsymbol{\delta}(t)) + b_j(t,u) - \boldsymbol{\delta}_j(t)V_j(t,u,\boldsymbol{\delta}(t)) \\
&+ \sum_{k:k\neq j}\left(V_k^I(t,0,\boldsymbol{\delta}(t)) + b_j(t,u) - V_j(t,u,\boldsymbol{\delta}(t))\right)\boldsymbol{\mu}_{jk}(t,u) \\
&+ \frac{1}{2}\boldsymbol{\sigma}^2(t,\boldsymbol{\delta}(t))\frac{\partial^2}{\partial \boldsymbol{\delta}^2}V_j(t,u,\boldsymbol{\delta}(t)) + \boldsymbol{\kappa}(t,\boldsymbol{\delta}(t))\frac{\partial}{\partial \boldsymbol{\delta}}V_j(t,u,\boldsymbol{\delta}(t)) = 0,
\end{aligned}
\tag{28}
$$

$$
V_j(n,u,\boldsymbol{\delta}(t)) = 0.
$$

4.2.3.5 This differential equation differs from equation (20) in the last two terms.

4.3 Overcoming Practical Challenges

The differential equation (28) is in general difficult to solve. However, if we only have one state and the only payment is a single payment of 1 at time *n*, then dropping the top script, *j*, and setting all the transition intensities and payments to zero, equation (28) becomes

$$\frac{\partial}{\partial t}V(t,u,\boldsymbol{\delta}(t))+\frac{\partial}{\partial u}V(t,u,\boldsymbol{\delta}(t))-\boldsymbol{\delta}(t)V(t,u,\boldsymbol{\delta}(t))$$
$$+\frac{1}{2}\boldsymbol{\sigma}^2(t,\boldsymbol{\delta}(t))\frac{\partial^2}{\partial\boldsymbol{\delta}^2}V(t,u,\boldsymbol{\delta}(t))+\boldsymbol{\kappa}(t,\boldsymbol{\delta}(t))\frac{\partial}{\partial\boldsymbol{\delta}}V(t,u,\boldsymbol{\delta}(t))=0,$$
$$V(n,u,\boldsymbol{\delta}(t))=0$$

where $V(t,u,\boldsymbol{\delta}(t))$ is just the expected discounted factor $\boldsymbol{\phi}(t,\boldsymbol{\tau},\boldsymbol{\delta}(t))$. The usefulness of stochastic differential equations for reserves is discussed in more detail in Norberg & Moller (op. cit.).

5. CONCLUDING COMMENTS

5.1 The economic power of every insurance company is measured by two key indicators: solvency and reserves. Solvency indicates the adequacy of an insurance company's funding. The reserves, also called technical provisions, allow for complete and lasting long-term liabilities arising from insurance contracts. Reserves are therefore of utmost importance for all areas of insurance.

5.2 For policyholders to be protected, it is essential that reserves are not only calculated correctly, but also covered by free and unencumbered assets (called tied assets) throughout the contract term. Consequently, reserves determine the level of tied assets. They are used to meet claims arising from insurance contracts where an insurance company is found insolvent.

5.3 This paper gives a general framework for health insurance modelling by taking into account duration effect and a random interest rate. The particular application of this paper to the South African insurance industry is in disability insurance.

5.4 For future work, we intend to collect data for the South African market and estimate the different transition probabilities.

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