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Jump tests for semimartingales

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ABSTRACT
This paper aims to introduce jump tests to the actuarial community. In actuarial science, semimartingales are extensively used in the models for interest rates, options, variable annuities and equity-linked annuities. Those models usually assume without justification that the underlying asset process follows a continuous stochastic process such as a geometric Brownian motion, for the market data sometimes tell a different story. Choosing between a continuous model and a model with jumps is not only important for pricing of insurance products but also crucial for implementing other post-sales risk management measures such as dynamic liability hedging. A test for jumps allows actuaries to rigorously test whether the underlying asset process has jumps, which is the first critical step in model selection. The ability to conduct the test should thus belong to the repertoire of every expert and practitioner working in this field. In this paper, we review several major tests for jumps, describe their advantages and disadvantages, and offer suggestions for their implementation. We also implement several tests using real data, enabling practitioners to apply these tests in their work.

KEYWORDS
Asset price; Black–Scholes; equity-linked annuity; variable annuity

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1. INTRODUCTION

1.1 The set of semimartingales form a very rich grouping that includes the class of jump diffusions, the class of pure jump processes, the class of local martingales, and (under fairly mild conditions) the class of Lévy processes. Hence, it has been widely used in financial modelling. Recently, the semimartingale model has gained popularity among actuaries for modelling stochastic behaviour of asset prices such as asset returns, stock prices, and interest rates. Two popular choices are (1) the continuous semimartingale model (especially the diffusion model)—see, for example, Boyle & Tian (2009), Giovanni (2010), Bernard, Maj & Vanduffel (2011), Costabile, Massabò & Russo (2011) and Piscopo & Haberman (2011); and (2) the general semimartingale model—see, for example, Biffis, Denuit & Devolder (2010), Lin & Li (2011) and Gerber, Shiu & Yang (2013). The diffusion model possesses many desirable properties: it makes good integrators that lead to closed-form formulas in many cases. This allows one to derive the First Fundamental Theorem of Asset Pricing; see, for example, Shreve (2005). However, the diffusion model is a continuous model; this rules out the possibility of modelling market noises such as asset price jumps. In contrast, the general semimartingale model is able to serve this purpose, because a semimartingale might be considered as the sum of a martingale (intuitively, a ‘trendless noise’) and a process of finite variation (intuitively, a ‘drift’). Moreover, the general semimartingale model still has many useful properties. For example, it obeys a generalised Itô formula; it has an integral representation; and it is stable under stopping, stochastic integration, C^2-transformation and the change of measures.

1.2 A close scrutiny of the above papers reveals that all of the authors assumed that the underlying asset price follows either a continuous semimartingale model or a general semimartingale model. However, none of them justified the choice based on a rigorous statistical test. In practice, actuaries first need to decide whether jumps are allowed in their models before starting the analysis described in these papers. In other words, actuaries should first make a decision between a purely continuous model and a model with jumps. This decision is of paramount importance because (1) model selection has a direct impact on pricing of insurance products, and (2) the effectiveness of post-sales risk management activities such as dynamic liability hedging also depends on model selection; see, for instance, Hardy (2003). To our best knowledge, no work in the existing actuarial literature discusses a rigorous test for such a decision. This paper aims to fill this gap by introducing tests for jumps to the experts and practitioners in this area. The major contributions of the paper are (1) it reviews several major tests for jumps—to the best of our knowledge no such a review paper even exists in the statistical literature; (2) it describes the advantages and disadvantages of these tests; and (3) it points out some pitfalls and offers guidelines for practitioners to implement these tests.

1.3 Aït-Sahalia (2002) first introduced a jump test for diffusion models. Since then much work has been done to test whether the underlying stochastic process has jumps. We do not intend to give an exhaustive list of all the available tests. Instead, we focus on those that are relatively easy for practitioners to implement. The theory of semimartingales and the associated statistical theory are two highly technical subjects; readers who are interested in technical details may consult Dellacherie & Meyer (1978, 1982), He, Wang & Yan (1992), Jacod & Protter (2012) and Jacod & Shiryaev (2003). While we give precise statements, we also provide intuitive explanations for each test, hoping that this approach will serve both experts and non-experts. In particular, we hope that a number of actuaries working in the field would pick up some of the tests reviewed here and use them in their work.

1.4 The remainder of the paper is organised as follows. Section 2 reviews four jump tests. Section 3 provides three examples to demonstrate the applications of these jump tests in actuarial science. Section 4 concludes the paper with a summary.

2. TESTS FOR JUMPS

2.1 The Carr–Wu Test

Aït-Sahali (op. cit.) proposed the first jump test based on the sample paths of the underlying asset processes. Shortly after the appearance of the Aït-Sahali test, Carr & Wu (2003) proposed a test for the presence of jumps which does not examine the sample paths of the underlying asset process.

2.1.1 ASSUMPTIONS OF THE CARR–WU TEST

The Carr–Wu test makes the following assumptions:

(1) the market is frictionless without arbitrages: under this assumption, there exists a risk-neutral measure Q under which S_t is a solution to the following stochastic differential equation (SDE)

$$\frac{dS_t}{S_{t-}} = \left(r-q\right)dt + \sigma_t dB_t + \int_{R\backslash\{0\}} (e^x - 1)\left[\mu\left(dx,dt\right) - \nu_t\left(x\right)dxdt\right], t \in \left[0,T\right],$$

where
S_{t-} is the left-hand limit process of S_t,
r is the continuously compound risk-free interest rate,
σ_t is the volatility process,
q is the continually compound dividend yield rate,
B_t is a Q standard Brownian motion,
$\mu\left(dx,dt\right)$ is the random measure that counts the jumps at time t, and
v is the dual predictable projection or compensator of μ;

(2) both r and q are assumed to be constant;

(3) the asset price process S_t is non-negative and absorbing at the origin; intuitively, this means a firm has limited liability; and

(4) μ_t and v_t are bounded in some neighbourhood of $t=0$. Intuitively, this means neither the volatility of the firm's price nor the jump size can be too large.

2.1.2 INTUITION

The intuition behind the Carr–Wu test is that the market prices of all at-the-money (ATM) and out-of-money (OTM) options written on the asset converge to zero as the maturity T goes to zero. However, the speed of convergence implied by continuous processes, purely discontinuous processes and hybrid processes are different. Thus, by examining the speed with which an option approaches zero as the maturity T goes to zero, we can distinguish the nature of the underlying asset process.

2.1.3 THE TEST STATISTIC

2.1.3.1 The Carr–Wu test is based on the asymptotic behaviour of short-maturity options. The following table taken from Carr & Wu (op. cit.) summarises the key results.

TABLE 1. Asymptotic behaviour of short-maturity options

Process type	OTM options	ATM options
Purely continuous process	$O\left(e^{-c/T}\right), c>0$	$O\left(\sqrt{T}\right)$
Purely discontinuous process	$O(T)$	$O\left(T^p\right), p \in (0,1]$
Combinations	$O(T)$	$O\left(T^p\right), p \in (0,1/2]$

In Table 1, O is the Bachman-Landau O-symbol. Specifically, for two real-valued functions f and g defined on a set E, $f(x)=O(g(x)), x \in E$ means there exists a constant C such that $|f(x)| \leq C|g(x)|$ for all $x \in E$. Table 1 shows that the prices of OTM options converge to zero at an exponential rate if the underlying process is purely continuous. However, if the underlying process has jumps, OTM option prices converge to zero at a rate which is at most linear. For ATM option prices, the results in Table 1 can be interpreted similarly.

2.1.3.2 In practice, one can implement the Carr–Wu test using the steps that follow.

(1) First, plot $\ln(P/T)$ versus $\ln T$ of both ATM and OTM options, where P is the price of the ATM or OTM option and T is the option term. Following Carr & Wu (op. cit.), we call such a plot the term decay plot.

(2) As the option maturity approximates the valuation date, that is $T \to 0$, the term decay plot of ATM options will exhibit a flat line if the underlying asset price process is a finite variation pure jump process, a straight line with a downward slope if the underlying asset price process has a continuous martingale component or an infinite variation jump component.

(3) As $T \to 0$, the term decay plot of OTM options will exhibit a flat line when the underlying asset price process has jumps, a concave curve with an upward slope when the underlying asset price process is purely continuous, i.e. the underlying asset price process does not have any jumps.

(4) From the time decay plots of both ATM and OTM options, one can tell whether the underlying asset price process has jumps.

2.1.4 Advantages and Disadvantages

The Carr–Wu test is easy to implement if real data of short-maturity options prices are available. But the Carr–Wu test does not examine the asset prices directly, instead it looks at the asymptotic behaviour of short-maturity options. Therefore, to apply the Carr–Wu test, one must have data of short-term ATM and OTM option prices written on the asset. Since such data are usually not easy to obtain, this may restrict the implementation of the Carr–Wu test.

2.1.5 Suggestions for Implementation

Though the Carr–Wu test uses only option price data, we recommend that one obtain the corresponding asset price data too. According to assumption (4) above, we suggest that one first plot the corresponding asset price to make a visual inspection of the data. If any large jumps are present, one should remove them as outliers. If the data display large volatility, then one should be cautious about the results; in this case, we recommend that at least one more different jump test be implemented on the same data for cross-checking. If the plot shows no sign of large jumps or volatility, then one can simply follow the above implementation procedure.

2.2 The Jiang–Oomen Test

The Jiang–Oomen test is a jump detection test over a fixed time interval $\left[0,T\right]$. It is based on Itô's lemma for semimartingales. Barndorff-Nielsen & Shephard (2006) proposed a test which is based on the difference of the realised quadratic variation and bipower variation. As pointed out by Lee & Mykland (2008), the Jiang–Oomen test and the Barndorff-Nielsen and Shephard test share a similar approach. Therefore, we only review the Jiang–Oomen test here.

2.2.1 Assumptions of the Jiang–Oomen Test

Let $X_t = \ln S_t$, where S_t is the underlying asset process. The Jiang–Oomen test assumes that:

(1) X_t is a semimartingale represented by

$$dX_t = \left(\alpha_t - \lambda_t m_t - \frac{1}{2}\sigma_t\right)dt + \sqrt{\sigma_t}\,dB_t + J_t d\tilde{N}_t, \tag{1}$$

where

α_t is the instantaneous drift process,
σ_t is the volatility process,
B_t is a standard Brownian motion,
$\tilde{N}_t$ is a counting process with intensity, and
λ_t is a non-zero jump size random variable satisfying $m_t = E\left[e^{J_t} - 1\right]$;

(2) the drift process α_t is assumed to be predictable and of locally bounded variation;

(3) the volatility process σ_t is assumed to be a positive càdlàg process with $\int_0^T \sigma_t dt < +\infty$ for any $T > 0$;

(4) X_t are observed at every $\Delta = \dfrac{1}{M}$ units of time over $\left[0,T\right]$. Thus, the sampled observations are $\left\{X_0, X_\Delta, X_{2\Delta},\ldots,X_{N\Delta}\right\}$, where $N=MT$; and

(5) there is no market microstructure noise.

2.2.2 Intuition

The intuition behind the Jiang–Oomen test can be described as follows. If there are no jumps of the asset prices, then one can replicate a variance swap using a perfectly hedged log contract. (A log contract pays $\ln S_T / S_0$ at maturity T.) But if jumps are present, then such a replicating strategy will yield stochastic and hedging errors whose discrete-time version equals the term $SV_M\left(T\right) - RV_M\left(T\right)$ in the three test statistics below. When we have large high-frequency data, this discrete-time version of errors will converge to the true errors. Since this quantity is zero if there are no jumps, one can create a test statistic based on it.

2.2.3 The Test Statistic

We wish to test H_0: there is no jump over $\left[0,T\right]$ versus H_1: there is a jump over $\left[0,T\right]$. From equation (1), it is clear that the hypothesis is equivalent to $H_0 : \lambda_t = 0$ for all $t \in \left[0,T\right]$ versus $H_0 : \lambda_t \neq 0$ for some $t \in \left[0,T\right]$. There are three associated test statistics:

(1) the difference test statistic:

$$T_1 = \frac{N}{\sqrt{\hat{\Omega}_{SV}^{(4)}}}\left(SV_M\left(T\right) - RV_M\left(T\right)\right);$$

(2) the logarithmic test statistic:

$$T_2 = \frac{\hat{V}_{(0,T)}N}{\sqrt{\hat{\Omega}_{SV}^{(4)}}}\left(\ln\left(SV_M\left(T\right) / RV_M\left(T\right)\right)\right);$$

(3) the difference test statistic:

$$T_3 = \frac{\hat{V}_{(0,T)}N}{\sqrt{\hat{\Omega}_{SV}^{(4)}}}\left(\frac{SV_M\left(T\right) - RV_M\left(T\right)}{RV_M\left(T\right)}\right);$$

where

$$SV_M\left(T\right) = 2\left(\sum_{j=1}^{N}\frac{S_{(j)\Delta} - S_{(j-1)\Delta}}{S_{(j-1)\Delta}} - \ln\left(S_T / S_0\right)\right),$$

$$RV_M\left(T\right) = \sum_{i=1}^{N} r_{\Delta,i}^2,$$

$$\hat{\Omega}_{SV}^{(4)} = \frac{\mu_6}{9}\frac{N^3 \mu_1^{-6}}{N-5}\sum_{i=1}^{N-6}\prod_{k=1}^{6}\left|r_{\Delta,\ j+k}\right|,$$

$$\hat{V}_{(0,T)} = \frac{1}{\mu_1^2} \sum_{i=1}^{T_1} \left| r_{\Delta,i+1} r_{\Delta,i} \right|,$$

$$\mu_p = \frac{2^{p/2}\Gamma\left(\left(p+1\right)/2\right)}{\sqrt{\pi}},$$

$$r_{\Delta,i} = X_{i\Delta} - X_{(i-1)\Delta}.$$

Jiang & Oomen (2008) showed that under H_0, the asymptotic distribution of these three test statistics is the standard normal distribution. Thus, testing can be carried out in a routine way once the significance level is given.

2.2.4 Advantages and Disadvantages

The Jiang–Oomen test has a fast rate of convergence to its asymptotic distribution. In the above discussion, we assumed that there is no market microstructure noise. But the test remains valid with a modified asymptotic variance when there is independent and identically distributed (i.i.d.) market microstructure noise; for details, see Jiang & Oomen (op. cit.). Unlike the Carr–Wu test, which uses price of options written on the asset, the Jiang–Oomen test uses asset prices. However, the Jiang–Oomen test relies on high-frequency data.

2.2.5 Suggestions for Implementation

One can see from the above that the implementation of the Jiang–Oomen test is relatively straightforward. However, the term $\hat{\Omega}_{SV}^{(4)}$ involves the summation of finite products, where each $r_{\delta,j+k}$ is a log difference of asset price and is usually very small. This means the Jiang–Oomen test is likely to be subject to numerical errors. One suggestion we have is to first multiple each finite product $\prod_{k=1}^{6} \left| \gamma_{\Delta,j+k} \right|$ by the coefficient $\frac{\mu_6}{9} \frac{N^3 \mu_1^{-6}}{N-5}$ and then carry out the summation. We also suggest that one use another jump test to validate the result.

2.3 The Lee–Mykland Test

The Lee and Mykland test is a nonparametric jump detection test that uses high-frequency data.

2.3.1 Assumptions of the Lee–Mykland Test

The Lee–Mykland test makes the following assumptions:

(1) the asset process S_t takes the following form:

$$d \ln S_t = \mu_t dt + \sigma_t dB_t + Y_t dN_t, \tag{2}$$

where μ_t and σ_t are the drift and volatility coefficients respectively,
B_t is the standard Brownian motion,
N_t is a counting process, and

Y_t is a predictable process which represents the jump size;
we also assume that $E\left[Y_t\right]$ (the mean process of Y_t) and $\sqrt{Var\left(Y_t\right)}$ (the standard deviation process of Y_t) are both predictable;

(2) Y_t, B_t, N_t are all stochastically independent;

(3) the process is observed at discrete time points $t = 1, 2, \ldots$ and $Y_1, Y_2, \ldots$ are i.i.d. random variables;

(4) observations of S_t are made at discrete times $0 \le t_0 < t_1 < \ldots < t_n = T$; and

(5) the drift and volatility do not change dramatically over a short period so that

$$\bigvee_i \bigvee_{t_i \le t \le t_{i+t}} \left|\mu_t - \mu_{t_i}\right| = O_p\left(\Delta t^{\frac{1}{2}-\in}\right),$$

$$\bigvee_i \bigvee_{t_i \le t \le t_{i+t}} \left|\sigma_t - \sigma_{t_i}\right| = O_p\left(\Delta t^{\frac{1}{2}-\in}\right),$$

where $\Delta t = \frac{T}{n}$ and O_p ("big oh P") is the standard stochastic O symbol. Specifically, for a sequence of random variables $\left(X_n\right)$, $X_n = O_p\left(1\right)$ means X_n is bounded in probability. If Y_n and R_n are two other sequences of random variables, then $X_n = O_p\left(Y_n\right)$ means $X_n = Y_n R_n$ and $R_n = O_p\left(1\right)$, where R_n can be considered as the rate at which the sequence X_n is bounded in probability. Interested readers can consult van der Vaart (2000) for more details.

2.3.2 INTUITION

The Lee–Mykland test allows one to tell whether the asset price process defined in equation (2) has a jump at a given time t_i. When such a jump exists, it is expected that the realised asset return would be greater than usual. However, when there is no jump but the volatility at t_i is high, one may also observe a greater realised asset return. To distinguish the two cases, it is natural to consider the ratio of the realised asset return to the instantaneous volatility at t_i. This leads to the test statistic given below.

2.3.3 THE TEST STATISTIC

2.3.3.1 We wish to test H_0: there is no jump at time t_i versus H_1: there is a jump at t_i. The test statistic is given by

$$T_{t_i} = \frac{\ln\left(S_{t_i} / S_{t_{i-1}}\right)}{\hat{\sigma}_{t_i}},$$

where

$$\hat{\sigma}_{t_i}^2 = \frac{1}{K-2} \sum_{j=i-K+2}^{i-1} \left|\ln\left(S_{t_j} / S_{t_{j-1}}\right)\right| \left|\ln\left(S_{t_{j-1}} / S_{t_{j-2}}\right)\right|,$$

and the constant K is the window size. It should be chosen to be large enough so that the effect of jumps on the instantaneous volatility vanishes. But K must be smaller than the number of observations n. Lee & Mykland (op. cit.) pointed out that $K = O_p\left(\Delta t^{\beta}\right)$ with

$-1 < \beta < -\frac{1}{2}$ is a suitable choice. They recommended that the optimal K for one-hour, 30-minute, 15-minute and 5-minute data are 78, 110, 156 and 270, respectively.

2.3.3.2 To test the above hypothesis, one may carry out the following:

(1) first, calculate $\frac{|T_{t_i}| - C_n}{S_n}$, where

$$C_n = \frac{\sqrt{2 \ln n}}{c} - \frac{\ln \pi + \ln(\ln n)}{2c\sqrt{2 \ln n}},$$

$$S_n = \frac{1}{c\sqrt{2 \ln n}},$$

$$c = \sqrt{\frac{2}{\pi}},$$

and n is the number of observations;

(2) for a given significance level α, calculate $\gamma = -\ln(-\ln(1-\alpha))$; and

(3) if $\frac{|T_{t_i}| - C_n}{S_n} > \gamma$, we reject H_0; otherwise, we accept H_0.

2.3.4 ADVANTAGES AND DISADVANTAGES

The Lee–Mykland test is a robust nonparametric test. The consistency of the test statistic at t_i is not affected by any jumps that occurred earlier. Also, Lee & Mykland (op. cit.) showed that the probability of misclassification is negligible when the frequency of observation is sufficiently high. On the other hand, the Lee–Mykland test relies on high-frequency data. We reiterate, nevertheless, one of the most significant advantages of this test, that it can be used to tell whether there is a jump at a given time t_i. When it is applied to a time series of asset prices, one would be able to tell not only whether there are jumps, but also where the jumps occur.

2.3.5 SUGGESTIONS FOR IMPLEMENTATION

The Lee–Mykland test is relatively simple to implement. It also allows one to tell whether a jump occurs at an instantaneous moment t. Compared with the three other jump tests reviewed in this paper, the Lee–Mykland test might stand out as a good choice. However, false discoveries can occur in any statistical test. For this reason, we still recommend that one use at least one other test to validate the result.

2.4 The Aït-Sahalia–Jacod Test

The Aït-Sahalia–Jacod test is another nonparametric jump test using high-frequency data.

2.4.1 ASSUMPTIONS OF THE AÏT-SAHALIA–JACOD TEST

The Aït-Sahalia–Jacod test assumes the following:

(1) the underlying process X_t is an Itô semimartingale, i.e.,

$$X_t = X_0 + \int_0^t b_s ds + \int_0^t \sigma_s dB_s + \int_0^t \int_E K o \Delta(s,x)(\mu - \nu)(ds,dx)$$

$$+ \int_0^t \int_E K' o \Delta(s,x)\mu(ds,dx),$$

where B_t and μ are a Brownian motion and a Poisson random measure on $R_+ \times E, (E, \epsilon)$ is the auxiliary measurable space in the definition of random measures,
$\nu(ds,dx) = ds \otimes \lambda(dx)$ is the predictable projection/compensator of μ with $\lambda(dx)$ being a finite or σ-finite measure on (E, ϵ),
K is a continuous function with compact support with $K(x) = x$ on a neighbourhood of 0, and
$K'(x) = x - K(x)$.

The above equation is called the Grigelionis decomposition of X_t. For a detailed discussion of Itô semimartingales, see Jacod & Protter (op. cit.);

(2) the volatility process σ_t is also an Itô semimartingale and has the form

$$\sigma_t = \sigma_0 + \int_0^t \hat{b}_s ds + \int_0^t \hat{\sigma}_s dB_s + \int_0^t \hat{\sigma}'_s dB'_s + \int_0^t \int_E K o \hat{\Delta}(s,x)(\mu - \nu)(ds,dx),$$

$$+ \int_0^t \int_E K' o \hat{\Delta}(s,x)\mu(ds,dx),$$

where B' is another Wiener process which is independent of B;

(3) $\hat{B}_t$ is locally bounded, $b_t, \hat{\sigma}_t$ and $\hat{\sigma}'_t$ are càdlàg, Δ_t and $\hat{\Delta}_t$ are càdlàg, that is, left-continuous with right limits, Δ'_t are left-continuous with right limits on the stochastic interval $[0, \tau(\omega))$, $\sup_{x \in E} \frac{|\Delta(\omega,t,x)|}{\gamma(x)}$ and $\sup_{x \in E} \frac{|\tilde{\Delta}(\omega,t,x)|}{\gamma(x)}$ are locally bounded, where γ is a non-random non-negative function satisfying $\int_E (\gamma(x)^2 \wedge 1)\lambda(dx) < \infty$;

(4) $\int_0^t |\sigma_s| ds > 0$ for all $t \geq 0$;

(5) X_t are observed at time $i\Delta_n$ over $[0,T]$, where $\Delta_n = \frac{T}{n}$. In other words, we have observations $\{X_0, X_\Delta, X_{2\Delta}, \ldots, X_{n\Delta}\}$; and

(6) there is no market microstructure noise.

2.4.2 INTUITION

In probability theory, one uses the variance to measure the dispersion of a random variable. For a stochastic process *X*, an analogous concept is called the *p*-th variation $\hat{B}(p,\Delta_n)_t = \sum_{i=1}^{[t/\Delta_n]} \left| X_{i\Delta_n} - X_{(i-1)\Delta_n} \right|^p$, where $p \in \{2,3,\ldots\}$. The p-th variation measures the dispersion of a stochastic process. Aït-Sahalia & Jacod (2009) examined the asymptotic behaviour of the *p*-th variation and proposed a jump test. Their test is based on the following heuristic argument. When there are no jumps, $\hat{B}(p,\Delta_n)_t$ only sums some small increments. These small increments are natural for all stochastic processes, hence they are not considered as jumps. When there are jumps, $\hat{B}(p,\Delta_n)_t$ not only sums these small increments, it also sums all the jumps. In this case the power *p* determines the magnitude of the effect of jumps. In particular, when $p>2$, the effects from jumps far outweigh those of small increments. But there is no simple way to extract the effects of jumps from $\hat{B}(p,\Delta_n)_t$. One way to get around this difficulty is to take an appropriately chosen positive integer *k* and examine the ratio $\hat{B}(p,k\Delta_n)_t / \hat{B}(p,\Delta_n)_t$. Since the effects of jumps prevail in both $\hat{B}(p,k\Delta_n)_t$ and $\hat{B}(p,\Delta_n)_t$ but are of different magnitudes, this ratio allows one to detect jumps.

2.4.3 THE TEST STATISTIC

Suppose that we wish to test H_0: there are no jumps during $[0,t]$ versus H_1: there are jumps during $[0,t]$. The test statistic proposed by Aït-Sahalia & Jacod (op. cit.) is

$$\hat{S}(p,k,\Delta_n)_t = \frac{\hat{B}(p,k\Delta_n)_t}{\hat{B}(p,\Delta_n)_t}.$$

For $k \ge 2, p > 3, q \in \left(\frac{1}{2} - \frac{1}{p}, \frac{1}{2} \right)$ and an asymptotic level $\alpha \in (0,1)$, the rejection region is given by

$$\left\{ \hat{S}(p,k,\Delta_n)_t < k^{\frac{p}{2}-1} - z_\alpha \sqrt{\hat{V}_{n,t}} \right\},$$

where

$$\hat{V}_{n,t} = \frac{\Delta_n M(p,k) \hat{A}(2p,\Delta_n)_t}{\hat{A}(p,\Delta_n)_t^2},$$

$$\hat{A}(p,\Delta_n)_t = \frac{\Delta_n^{1-p/2}}{m_p} \sum_{i=1}^{[t/\Delta_n]} \left| X_{i\Delta_n} - X_{(i-1)\Delta_n} \right|^p 1_{\left\{ \left| X_{i\Delta_n} - X_{(i-1)\Delta_n} \right| \le \alpha \Delta_n^q \right\}},$$

$$M(p,k) = \frac{1}{m_p^2} \left(k^{p-2}(1+k) m_{2p} + k^{p-2}(k-1) m_p^2 - 2k^{p/2-1} m_{k,p} \right),$$

$$m_p = \frac{2^{p/2}}{\sqrt{\pi}} \Gamma\left(\frac{1+p}{2} \right),$$

$$m_{k,p} = E\left[\left|U\left(U+\sqrt{k-1}V\right)\right|^{p}\right].$$

2.4.4 ADVANTAGES AND DISADVANTAGES

The Aït-Sahalia–Jacod test statistic is scale-invariant. It assumes that the underlying process X_t is an Itô semimartingale which includes a wide class of processes. But the test does not depend on the law of X_t nor its coefficients, which is a good feature of the test. However, like the Lee–Mykland test, the Aït-Sahalia–Jacod test is only applicable to high-frequency data. It has three free parameters for users to choose which makes the test data-driven in some sense. Also, the Aït-Sahalia–Jacod test is not applicable when the microstructure noise is present. Later on, Aït-Sahalia, Jacod & Li (2012) gave a robustification of the Aït-Sahalia–Jacod test which allows the market microstructure noise to exist; see Aït-Sahalia, Jacod & Li (op. cit.). Another drawback of the Aït-Sahalia–Jacod test is that the significance level α may not be chosen arbitrarily; as Aït-Sahalia & Jacod (op. cit.) pointed out, an appropriate choice of α is between three and five times the average value of σ. This might restrict the use of the Aït-Sahalia–Jacod test in certain cases.

2.4.5 SUGGESTIONS FOR IMPLEMENTATION

To apply the Aït-Sahalia–Jacod test, one needs to choose several parameters. Aït-Sahalia & Jacod (op. cit.) pointed out that a large p emphasises large jumps and when p is too close to 3, a poor fit is more likely to appear. Thus, choosing $p=4$ seems to be a judicious decision. Moreover, one should not take k to be too large. Aït-Sahalia & Jacod (op. cit.) recommended that one take $k=2$ and q close to ½. The choice of α depends on σ which is unknown. We recommend that one first calculate the sample standard deviation $\hat{\sigma}$ and then set $\alpha = 4\hat{\sigma}$. However, if the desirable value of α falls far outside the interval $[3\hat{\sigma}, 5\hat{\sigma}]$, one should not use the Aït-Sahalia–Jacod test. Also, the calculation of the test statistic is relatively involved and numerical errors might occur; we recommend that one apply other tests to validate the result.

3. EXAMPLES

In this section, we provide three examples to demonstrate actuarial applications of the above tests.

3.1 Example 1

Suppose an insurance firm is developing a 5-year equity-linked annuity (EIA) product on 1 June 2012. The product is a single premium contract with 95% of the premium being invested in a basket of fixed-interest securities and a 3% guaranteed rate of interest. The referred index is S & P 500. The contract pays off the benefit using the point-to-point method. The firm is currently in the stage of pricing this EIA. John, a seasoned pricing actuary, knows that there is a closed-form pricing formula for such a contract design; see, for example, Hardy (op. cit.). However, the formula does not allow jumps for the underlying index. To avoid modelling and pricing errors, John applies the Aït-Sahalia–Jacod test to see whether jumps are present. To run the test, John uses the 5-minute high-frequency data of iShares S&P500 Index from 9:30 am 1 May 2012 to 16:00 pm 31 May 2012. The dataset is from the Trade and Quotes (TAQ) database at Wharton Research Data Services (WRDS) from the Wharton School at the University of Pennsylvania. He samples the very first transaction that occurs every 5 minutes. The sample data contain 1,738 observations. He first calculates the sample standard deviation $\hat{\sigma}$. Then he follows the recommendation of Aït-Sahalia & Jacod (op. cit.) to choose $p=4$, $q=0.2$, $k=2$ and $\alpha = 4\hat{\sigma}$. Table 2 summarises the key values of the test. At the significance level $\alpha=12.15\%$, the test statistic is in the rejection region. Thus, John concludes that there are jumps in the data. This means that a closed-form formula may not be available for pricing; some numerical methods and simulation techniques might be called for.

TABLE 2. Key values of the Aït-Sahalia–Jacod test

p	k	q	$\hat{\sigma}$	α	$\hat{V}_{n,t}$	$\hat{S}(p,k,\Delta_n)_t$	$k^{\frac{p}{2}-1} - z_\alpha\sqrt{\hat{V}_{n,t}}$
4	2	0.4	0.0304	0.1215	0.6288	1.0138	1.0743

3.2 Example 2

As a scrupulous professional, the pricing actuary John in example 1 feels the significance level $\alpha=12.15\%$ used in the Aït-Sahalia–Jacod test is not very satisfactory; also, he feels it is a good practice to validate his conclusion using another jump test. Therefore, he decides to run the Jiang–Oomen test on the same dataset to cross-validate the result. The key values of the test are given in the following Table 3.

TABLE 3. Key value of the Jiang-Oomen test

α	Δ	N	$SM_V(T)$	$RV_M(T)$	$\hat{\Omega}_{SV}^{(4)}$	T_1
5%	5	1,738	2.0059×10^{-3}	2.0068×10^{-3}	1.6381×10^{-8}	–12.092

The asymptotic distribution of T_1 is the standard normal distribution and the Jiang–Oomen test is a two-sided test. Therefore, at significance level $\alpha=5\%$, John rejects the null hypothesis, confirming that there are jumps in the data. With peace of mind, John hands his analysis and conclusion to Megan, the chief actuary at the firm.

3.3 Example 3

Megan, the chief actuary in example 2, receives the report from John. She verifies John's analysis and finds everything is correct. Usually numerical methods and simulation would come into play if no closed-form formula is available. But Megan wants to make a further analysis to determine how many jumps there are and understand their characteristics of magnitude. To this end, she runs the Lee–Mykland test with $n=1{,}738$. Following the recommendation of Lee & Mykland (op. cit.), she chooses $K=270$. At the significance level

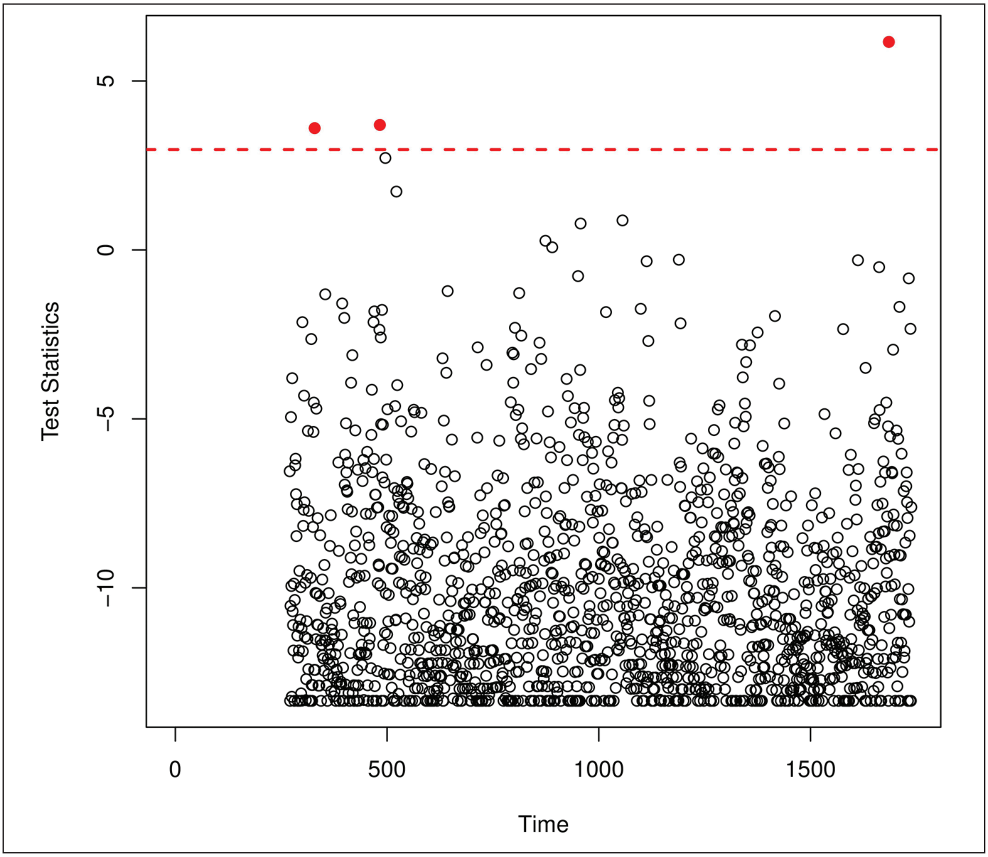

FIGURE 1. Test statistics with rejection threshold

$\alpha=5\%$, the threshold value is found to be 2.97 and the Lee–Mykland test shows that there are only three jumps. Figure 1 plots the jumps detected by this test; it shows that the majority of the data are far from the threshold and only three data points are detected as jumps. Then Megan looks closely at the magnitudes of these three jumps. A summary is given in Table 4. Since none of the three jumps is of significant size, Megan decides that a continuous semimartingale model might serve as a good approximation to the underlying true and unknown model. Therefore, she applies the formula in Hardy (op. cit.) to price the product. In addition, however, she applies the Markov Chain Monte Carlo method to verify her results.

TABLE 4. Jump times and sizes

Date	Time	Jump size (%)
7 May 2012	10:30	–0.269%
9 May 2012	10:10	–0.285%
31 May 2012	11:35	0.351%

4. SUMMARY

In the current actuarial literature, continuous semimartingales and the general semimartingales have been extensively used to model asset prices. Most research has focused on the pricing issues in insurance. No work has been introduced to justify the model selection using market data. In this review paper, we introduced four jump detection tests to actuaries: the Carr–Wu test, the Jiang–Oomen test, the Lee–Mykland test and the Aït-Sahalia–Jacod test. The Carr–Wu test uses short-term option price data while the other three tests use asset price data. These tests allow actuaries to choose between continuous models and models with jumps using a statistical test. We also demonstrated the actuarial applications of these tests. To the best of our knowledge, this is the first paper that introduces rigorous statistical tests for jumps to the actuarial community and reviews the major jump tests in the literature. We hope that this paper is useful for both researchers and practitioners in actuarial science, insurance and risk management.

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