Scientia Africanа, Vol. 23 (No. 4), October, 2024. Pp 157-162
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FIXED POINT OF DISCRETE DYNAMICAL SYSTEM OF LOTKA VOLTERRA MODEL

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Received: 25-09-2024
Accepted: 16-10-2024

https://dx.doi.org/10.4314/sa.v23i4.14

Journal Homepage: http://www.scientia-african.uniportjournal.info
Publisher: *Faculty of Science, University of Port Harcourt.*

ABSTRACT

Fixed point theorem is one of the theories in mathematics that has make many proofs been in existence. Lotka-Volterra model is a widely used pair of first-order nonlinear differential equations used to interpret the dynamics of two species that is a predator and a prey. The paper employs the contraction mapping and the Banach Fixed point theory on the Discrete Dynamical type of the Lotka-Volterra to see the outcome of its behavior. The Banach Fixed Theory is used in determining the fixed point of discrete dynamical system of Lotka Volterra model. The following solutions $(0,0)$, $(-\frac{\delta}{\gamma},\frac{\alpha}{\beta})$ *and* $\left(\frac{(\alpha-\beta y_n-1)}{t},\frac{(\delta+\gamma x_n-1)}{p}\right)$ *have all be discovered and they are fixed points of Lotka Volterra Model. The fixed point serving as the limiting behavior of Lotka Volterra is continuous and convergent.*

Keywords: Dynamical System, Discrete Lotka-Volterra, Fixed Point

INTRODUCTION

Many researchers have investigated dynamical systems. Din (2013) articulates the dynamics of a discrete Lotka-Volterra model. Farrukh Mukamedov and Mansoor Saburov (2014) state the dynamics of Lotka-Volterra (LV) type operators defined in a finite-dimensional simplex.

Zhifang Bi and Shuxia Pan (2018), aimed at the dynamics of a Predator-Prey system with three species. In population dynamics, predator-prey systems have been widely studied due to their importance as well as plentiful dynamical behaviors. A mathematical formulation of any fixed rule that is time-dependently represented is called a dynamic system. Any of the three number systems—integers, real numbers, and complex numbers—can be used to measure time. A dynamical system that follows a fixed rule and whose state changes over a state space in discrete time steps is called a discrete dynamical system. (Sara Fernandes et al. 2018)

Hang Deng, et al. (2019) investigated the existence and stability of all possible equilibria of the system. Their studies also show that cannibalism has both positive and negative effect on the stability of the system, it depends on the dynamic behavior of the original system. The Lotka-Volterra (LV) systems typically model the time evolution of conflicting species in biology. The use of Lotka-Volterra discrete time systems is a well-known subject of applied mathematics.

AmrElsonbaty and A. A. Elsadany (2021) aimed at introducing a new discrete fractional order model based on Lotka-Volterra prey-predator model with logistic growth of prey species.

MATERIALS AND METHODS

Definition 1: (Sara Fernandes et al. 2018)

A dynamical system is a system in which a function describes the time dependence of a point in an ambient space, such as parametric curve. In general, a system of n first-order differential equations in the space R^n is called a dynamical system of dimension n which determines the time behavior of evolution process.

Definition 2: (Sara Fernandes et al. 2018) (Carrasco-Gutierrez, et al., 2019)

A discrete dynamical system is a system with a state that only change at a sequence of instants $\{t_0, t_{1,} t_2, \ldots\}$. A one-dimensional discrete dynamical system $x_{n+1} = f(x_n)$ is given by iterating a map f, which we assume is smooth.

Definition 3: (Qamar Din, 2013)

Discrete Lotka-Volterra model is given by $x_{n+1} = \frac{\alpha x_n - \beta x_n y_n}{1+\gamma x_n}$, $y_{n+1} = \frac{\delta y_n + \epsilon x_n y_n}{1+\eta y_n}$ where parameters $\alpha, \beta, \gamma, \delta, \epsilon, \eta \in R^+$ and initial conditions x_0, y_0 are positive real numbers.

Definition 4:(Layek, 2015): LINEAR STABILITY ANALYSIS

A fixed point, x_0 is said to be stable if for a given $\varepsilon > 0$, there exist a $\delta > 0$ depending upon ε such that for all $t \geq t_0$, $\|x(t) - x_0(t)\| < \varepsilon$, whenever $\|x(t_0) - x_0(t_0)\| < \delta$, where $\|.\|: R^n \to R$ denotes the norm of a vector in R^n.

Major definition and Theorem

Definition 5: Contraction Mapping in Metric Space

Given (N, d) a metric space, a function $T: N \to N$ is said to be a contraction mapping if there is a constant q with $q < 1$ such that for all $x, y \in N$

$$d(T(x), T(y)) \leq q \cdot d(x, y)$$

Theorem: Banach Contraction Principle

Suppose (P, d) is a complete metric space and $H: P \to P$ is a mapping of contractions using the Lipschitz constant $k < 1$. Then, the fixed point $\in P$, for all $x \in P$ a unique point in H. That is; $\lim_{n\to\infty} H^n(x) = \omega$ Moreover, for each $x \in P$, we have $d(H^n(x), \omega) \leq \frac{k^n}{1-k} d(H(x), x)$, (Saleh, Qamrul and Khamsi, 2014).

FINDINGS AND DISCUSSIONS

Discrete Dynamical Type of the Lotka-Volterra

In this section, we investigate the discrete type of the Lotka Volterra to determine the roots, the solution, and the fixed point of each function.

$$x_{n+1} = \frac{\alpha x_n - \beta x_n y_n}{1+t x_n}, \; y_{n+1} = \frac{\delta y_n + \gamma x_n y_n}{1+p y_n} \qquad 1.1$$

The Zeros/Roots of Eq 1.1

For eq 1.1,

let$x_{n+1} = 0$

$\Rightarrow \frac{\alpha x_n - \beta x_n y_n}{1+t x_n} = 0$

$\Rightarrow \alpha x_n - \beta x_n y_n = (1 + t x_n) \times 0$

$$x_n(\alpha - \beta y_n) = 0$$

$x_n = 0$ and $\alpha - \beta y_n = 0 \Rightarrow \alpha = \beta y_n$, then $y_n = \frac{\alpha}{\beta}$

Also, let $y_{n+1} = 0$

$$\frac{\delta y_n + \gamma x_n y_n}{1 + p y_n} = 0$$

$\Rightarrow \delta y_n + \gamma x_n y_n = (1 + p y_n) \times 0$

$$y_n(\delta + \gamma x_n) = 0$$

$y_n = 0$and$\delta + \gamma x_n = 0 \Rightarrow \delta = -\gamma x_n$, then $x_n = -\frac{\delta}{\gamma}$

Scientia Africana, Vol. 23 (No. 4), October, 2024. Pp 157-162
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Therefore, the roots of eq 1.1 are (0,0)and $(-\frac{\delta}{\gamma}, \frac{\alpha}{\beta})$.

The Solution or Location of Eq 1.1

$$\frac{\alpha x_n - \beta x_n y_n}{1 + t x_n} = x_n$$

$\Rightarrow \alpha x_n - \beta x_n y_n = (1 + t x_n) \times x_n$

$$\alpha x_n - \beta x_n y_n = x_n + {x_n}^2 t$$

$${x_n}^2 t + x_n - \alpha x_n + \beta x_n y_n = 0$$

$$x_n[x_n t + 1 - \alpha + \beta y_n] = 0$$

$x_n = 0$and$x_n t + 1 - \alpha + \beta y_n = 0 \Rightarrow x_n t = \alpha - \beta y_n - 1$ then $x_n = \frac{\alpha - \beta y_n - 1}{t}$

Again,

$$\frac{\delta y_n + \gamma x_n y_n}{1 + p y_n} = y_{n+1}$$

$\Rightarrow \delta y_n + \gamma x_n y_n = (1 + p y_n) \times y_n$

$$\delta y_n + \gamma x_n y_n = y_n + {y_n}^2 p$$

$${y_n}^2 p + y_n - \delta y_n - \gamma x_n y_n = 0$$

$$y_n[y_n p + 1 - \delta - \gamma x_n] = 0$$

$y_n = 0$and$y_n p + 1 - \delta - \gamma x_n = 0 \Rightarrow y_n p = \delta + \gamma x_n - 1$ then $y_n = \frac{\delta + \gamma x_n - 1}{p}$

Hence, the solution of eq 1.1 is(0,0) and $(\frac{\alpha - \beta y_n - 1}{t}, \frac{\delta + \gamma x_n - 1}{p})$

Determination of the fixed point of Eq 1.1

In this section, we use the points of intersection of the function to determine the fixed point of eq 1.1

So, we determine the fixed point of eq 1.1

That is, $x_{n+1} = \frac{\alpha x_n - \beta x_n y_n}{1 + t x_n}$

At (0,0), that is $x_0 = 0$

$$f(0) = \frac{\alpha(0) - \beta(0) y_n}{1 + t(0)} = 0$$

Also, at $x_n = (\frac{\alpha - \beta y_n - 1}{t})$

$$f(\frac{\alpha - \beta y_n - 1}{t})$$

$$= \frac{\alpha(\frac{\alpha - \beta y_n - 1}{t}) - \beta(\frac{\alpha - \beta y_n - 1}{t}) y_n}{1 + t(\frac{\alpha - \beta y_n - 1}{t})}$$

$$= \frac{(\frac{\alpha^2 - \alpha\beta y_n - \alpha}{t}) - (\frac{\beta y_n \alpha - \beta^2 {y_n}^2 - \beta y_n}{t})}{1 + (\alpha - \beta y_n - 1)}$$

$$= \frac{(\frac{\alpha^2 - \alpha\beta y_n - \alpha - \beta y_n \alpha + \beta^2 {y_n}^2 + \beta y_n}{t})}{1 + \alpha - \beta y_n - 1}$$

$$= \frac{\alpha^2 - \alpha\beta y_n - \alpha - \beta y_n \alpha + \beta^2 {y_n}^2 + \beta y_n}{t(\alpha - \beta y_n)}$$

$$= \frac{\alpha(\alpha - \beta y_n - 1) - \beta y_n(\alpha - \beta y_n - 1)}{t(\alpha - \beta y_n)}$$

$$= \frac{(\alpha - \beta y_n)(\alpha - \beta y_n - 1)}{t(\alpha - \beta y_n)}$$

$$= \frac{(\alpha - \beta y_n - 1)}{t}$$

Hence $f\left(\frac{\alpha - \beta y_n - 1}{t}\right) = \frac{(\alpha - \beta y_n - 1)}{t}$

This shows a fixed point.

Also, for $\frac{\delta y_n + \gamma x_n y_n}{1 + p y_n} = y_{n+1}$

Then, at (0,0), that is$y_0 = 0$

$\Rightarrow f(0) = \frac{\delta(0) + \gamma x_n(0)}{1 + p(0)} = 0$

Also, at$y_n = \frac{\delta + \gamma x_n - 1}{p}$

$$\Rightarrow g\left(\frac{\delta + \gamma x_n - 1}{p}\right) = \frac{\delta(\frac{\delta + \gamma x_n - 1}{p}) + \gamma x_n(\frac{\delta + \gamma x_n - 1}{p})}{1 + p(\frac{\delta + \gamma x_n - 1}{p})}$$

$$= \frac{(\frac{\delta^2 + \gamma\delta x_n - \delta}{p}) + (\frac{\delta\gamma x_n + (\gamma x_n)^2 - \gamma x_n}{p})}{1 + \delta + \gamma x_n - 1}$$

$$= \frac{\frac{\delta^2 + \gamma\delta x_n - \delta + \delta\gamma x_n + (\gamma x_n)^2 - \gamma x_n}{p}}{\delta + \gamma x_n}$$

$$= \frac{\delta^2 + \gamma\delta x_n - \delta + \delta\gamma x_n + (\gamma x_n)^2 - \gamma x_n}{p(\delta + \gamma x_n)}$$

$$= \frac{\delta(\delta + \gamma x_n - 1) + \gamma x_n(\delta + \gamma x_n - 1)}{p(\delta + \gamma x_n)}$$

$$= \frac{(\delta + \gamma x_n)(\delta + \gamma x_n - 1)}{p(\delta + \gamma x_n)}$$

$$= \frac{(\delta + \gamma x_n - 1)}{p}$$

Hence $g\left(\frac{\delta+\gamma x_n-1}{p}\right) = \frac{(\delta+\gamma x_n-1)}{p}$

This also shows a fixed point.

FINAL RESULTS

Under this section we impose the contraction mapping and the Banach Fixed point theory on the Discrete Dynamical type of the Lotka-Volterra to see the outcome of its behaviour. From the definition of Banach fixed theorem, let (N, d) be a complete metric space then every contraction has a unique fixed point.

If $T(x) = x,\ T(y) = y$ then $d(x, y) = d\big(T(x), T(y)\big)$

$$\leq q \cdot d(x, y)$$

$q < 1$ so $d(x, y) = 0$ or $x = y$

Considering the x $q$uantity first:

$$x_{n+1} = \frac{\alpha x_n - \beta x_n y_n}{1 + t x_n}$$

$$x_{n+1} = \frac{(\alpha - \beta y_n) x_n}{1 + t x_n}$$

To show that fixed point exists, define a sequence $\{x_i\}_{i \in z^+}$ by setting $x_{n+1} = T(x_n)$

Expressing the contraction formula as

$$x_{n+1} = \frac{(\alpha - \beta y_n) T(x_n)}{1 + tT(x_n)}$$

From the Banach Fixed-point theorem:

$$d(x_{n+2}, x_{n+1}) \leq \frac{(\alpha - \beta y_n) d(x_{n+1},\ x_n)}{1 + tqd(x_{n+1},\ x_n)}$$

Or

$$d(x_{n+1}, x_n) \leq \frac{(\alpha - \beta y_n) q^n d(x_1,\ x_0)}{1 + tq^n d(x_1,\ x_0)}$$

Since $q^n < 1$

$$d(x_{n+1}, x_n) < (\alpha - \beta y_n) q^n d(x_1,\ x_0)$$

Assuming $n < m$

$$d(x_m, x_n) < (q^{m-n-1} + q^{m-n-2} + \cdots + q + 1) d(x_{n+1}, x_n)$$

$$d(x_m, x_n) < \left(\frac{1 - q^{m-n}}{1 - q}\right) d(x_{n+1}, x_n)$$

$$d(x_m,\ x_n) < \left(\frac{1 - q^{m-n}}{1 - q}\right) q^n (\alpha - \beta y_n) d(x_1, x_0)$$

Since $q^{m-n} < 1$

$$d(x_m, x_n) < \frac{q^n}{1 - q} (\alpha - \beta y_n) d(x_1, x_0)$$

Thus $\{x_i\}$ is Cauchy. This shows that (x_n) is Cauchy sequence in X.

Hence, (x_n) must be convergent, say $lim_{n \to +\infty} x_n = x$.
Since T is continuous, we have:

$$Tx = T\left(\lim_{n \to +\infty} x_n\right)$$
$$= \lim_{n \to +\infty} T(x_n)$$
$$= \lim_{n \to +\infty} x_{n+1}$$
$$Tx = x$$

Since the limit of x_{n+1} is the same as that of (x_n).
Thus, x is a fixed point of T.

Also considering the y quantity,

$$y_{n+1} = \frac{\delta y_n + \gamma x_n y_n}{1 + p y_n}$$

$$y_{n+1} = \frac{(\delta + \gamma x_n) y_n}{1 + p y_n}$$

Define a sequence $\{y_i\}_{i \in z^+}$ by setting $y_{n+1} = T(y_n)$
Expressing the contraction formula as

$$y_{n+1} = \frac{(\delta + \gamma x_n) T(y_n)}{1 + pT(y_n)}$$

From the Banach Fixed-point theorem

$$d(y_{n+2}, y_{n+1}) \leq \frac{(\delta + \gamma x_n) d(y_{n+1},\ y_n)}{1 + pd(y_{n+1},\ y_n)}$$

or

$$d(y_{n+1},\ y_n) \leq \frac{(\delta + \gamma x_n) q^n d(y_1,\ y_0)}{1 + pq^n d(y_1,\ y_0)}$$

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Since $q^n < 1$

$$d(y_{n+1,}y_n) < (\delta + \gamma x_n)q^n d(y_1,\ y_0)$$
$$d(y_{n+1,}y_n) < q^n(\delta + \gamma x_n)d(y_1,\ y_0)$$

Assuming $n < m$

$$d(y_m,\ y_n) < d(y_m,\ y_{m-1}) + d(y_{m-1},\ y_{m-2}) + \cdots + d(y_{n+1},\ y_n)$$
$$d(y_m,\ y_n) < q^{m-n-1} + q^{m-n-2} + \cdots + \mathrm{q} + 1)d(y_{n+1},\ y_n)$$
$$d(y_m,\ y_n) < \frac{1-q^{m-n}}{1-q} d(y_{n+1},\ y_n)$$
$$d(y_m,\ y_n) < \left(\frac{1-q^{m-n}}{1-q}\right) q^n d(y_1, y_0)$$

Since $q^{m-n} < 1$

$$d(y_m,\ y_n) < \frac{q^n}{1-q}(\delta + \gamma x_n)d(y_1,\ y_0)$$

Thus $\{y_i\}$ is Cauchy. This shows that $y_n$is Cauchy sequence in Y. Hence, (y_n) must be convergent, say $lim_{n\to+\infty} y_n = y$. Since T is continuous we have:

$$Ty = T\left(\lim_{n\to+\infty} y_n\right)$$
$$= \lim_{n\to+\infty} T(y_n)$$
$$= \lim_{n\to+\infty} y_{n+1}$$
$$Ty = y$$

Since the limit of y_{n+1} is the same as that of (y_n).
Thus, y is a fixed point of T.

CONCLUSION

Fixed point has been used to prove many theorems in mathematics. In this paper, fixed point has helped in determining the fixed point of discrete dynamical system of Lotka Volterra model. The following solutions (0,0), $(-\frac{\delta}{\gamma}, \frac{\alpha}{\beta})$ and$\left(\frac{(\alpha-\beta y_n-1)}{t}, \frac{(\delta+\gamma x_n-1)}{p}\right)$ have all be discovered and they are fixed points. Banach fixed theory was also explore on the discrete dynamical system of Lotka Volterra model to see the existence of the fixed point of discrete dynamical system of Lotka Volterra model.

Conflicts of Interest

The authors declare no conflicts of interest and that authors are responsible for coauthors declaring their interests.

Data Availability

No data were used to support this study.

Funding

This study did not receive fund in any form.

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