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FIXED POINT THEORY WITH PARTIALLY ORDERED METRIC SPACES AND HOMOTOPY

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ABSTRACT

The importance of fixed point theory and for that matter metric spaces have significantly manifested in sciences and in many branches of mathematical analysis over the years. The paper presents fixed point theory on a complete metric space with contractive mappings. Provision of some fixed point results of contraction mappings on a partially metric spaces with example. However, generalization of findings using the contraction mappings on partially metric spaces with application to Homotopy theory is provided.

Keyword: Metric spaces, fixed point, continuity, completeness, weakly contractive mappings

INTRODUCTION

Fixed point theorem has gained a wide range of applications. The contraction mapping principle Banach (1922) and in Pant (1994) is mainly used in classical functional analysis and generally accepted to be the origin of metric fixed point theory. Some generalization of fixed point theory have been applied in many fields (Petrov & Bisht, 2023; Petrov, 2023; Dehici et al., 2019; Kumari et al., 2023) as in differential equations, functional analysis, nonlinear analysis, engineering, game theory, operator theory etc. The study of fixed points of mappings satisfying certain metric contractive conditions have been attracted by many researchers, see (Rao et al., 2020; Ege & Alaca, 2015; Pant, 1994; Gholidahneh et al., 2017). This paper is aimed at providing prove to unique fixed point theorem for contractive conditions in partially ordered metric spaces (Beg et al., 2024; Rao et al. 2020; Rao et al. 2021; Prasad et al. 2018; Kyukim et al., 2017; Sedghi et al., 2015; Fadakar et al., 2021 and Mathews, 1994). Also, application using the technique in (Ege & Alaca, 2015; Ciric et al., 2011) and to Homotopy theory is highlighted. In algebraic topology, two continuous functions say u, v: $X \to Y$ are homotopic, if there exists a continuous family of functions $H: X\times[0, 1] \to Y$, such that $H(x, 0)=u(x)$ and $H(x, 1)=v(x)$. Where this is represented as $u\simeq v$. The homotopy theory studies spaces and maps up to continuous deformation called homotopy. The theory has also been applied to solving linear and nonlinear equations such as differential and integral equations as in (Ahmed et al., 2023) homotopy analysis method which has been use to solve the nonlinear system of volterra integral equations. Here are some definitions and lemma.

DEFINITION

Definition 1. Mathews (1994). A metric space (X, d) is a set X and a function d: X×X → [0, ∞) such that

i. d(x, y) ≥0 or d(x, y) = 0 if and only if x=y for all x, y ∈X.
ii. d(x, y) = d(y, x) for all x, y ∈ X
iii. d(x, y) ≤ d(x, z) + d(z, y) for all x, y, z ∈ X.

Definition 2.(Aggarwal & Kohli, 2022) Let u be a point in a metric space (X, d), with a positive, real number r. An open ball centered at u with radius r is a set

B(u; r) = {x∈X : d(x, u)<r} ∀ u∈X

And a closed ball centered at u with radius r is the set

B[u; r] = {x∈X : d(x, u)≤r} ∀ u∈X.

Definition 3. Let (X, d) be a metric space. A sequence (x_n) in X converges to the limit u as n→ ∞, where

x_n →u or $\lim_{n\to\infty} x_n = u$

and for every ε >0, there exist N∈ ℕ such that $|x_n - u| < \varepsilon$∀ n≥N.

Definition 4. Given a metric space (X, d). A sequence (x_n) in X is said to be Cauchy sequence if for every ε >0, there exist for m, n ≥ N as N∈ ℕ such that $|x_m - x_n| < \varepsilon$.

Definition 5. A function g :ℝ → ℝ is continuous at some point u∈ ℝ if

$$\lim_{x\to u} g(x) = g(u)$$

Definition 6. A function g :ℝ → ℝ has the limit u as x→a, we write

g(x) →L or
$\lim_{x\to a} g(x) = L$

If for every ε >0, there exist δ >0 such that $|g(x) - L| < \varepsilon$ and for $|x - a| < \delta$.

Definition 7. Mathews (1994). Let X be a nonempty set. A partial metric on X is a function d: X×X → ℝ for all x, y, z∈X

i) x=y if and only if d(x, x)=d(y, y)=d(x, y)
ii) d(x, y)=d(y, x)
iii) d(x, y)≤d(x, z) + d(z, x) – d(z, z)

hence the ordered pair (X, d) is called a partial metric space

Lemma 8: (Prasad et al., 2018**).** Let (X, d) be a metric space and ϕ: ℝ → ℝ. Suppose that $\{x_n\}$ is convergent to x, then we have ϕd(x, x_n) $\leq \lim_{n\to\infty} \inf d(x, x_n) \leq \lim_{n\to\infty} \sup d(x, x_n)$ ≤ d(x, x_n) for all x∈X,

The result is proved, given x_n= x as $\lim_{n\to\infty} d(x_n$, x) = 0

MATERIALS AND METHODS

Theorem 1. Given a complete metric space (X, d). Let g: X →X be a map such that

d(g(x), g(y)) ≤d(x, y) - ϕd(g(x), g(y)) (1.1)

and ϕ : ℝ → ℝ is a real function satisfying t< ϕ(t) for t>0 (1.2)

Then, the map g has a unique fixed point y∈X and g^nx → y

as n → ∞ for each x∈X

RESULTS AND DISCUSSION

Then we have, for m > n

α_n =d(g^m(x) , g^n(x))

≤d(g^{m-1}(x) , g^{n-1}(x)) −ϕ (d(g^m(x) , g^n(x)))

≤d(g^{m-1}(x) , g^{n-1}(x)) − (d(g^m(x) , g^n(x)))

≤ $\frac{1}{2}$d(g^{m-1}(x) , g^{n-1}(x))

$$\leq \frac{1}{2}\alpha_{n-1}$$

Reading from the last statement, it is clearly shown that $\alpha_{n-1} > \alpha_n$. We can therefore conclude that $\{\alpha_n\}$ is a non – increasing sequence and has a limit α.

Considering $\alpha > 0$ and continuity of ϕ, we have

$\phi(\alpha)$< lim inf$\phi(\alpha_n)$ < lim sup $\phi(\alpha_n) < \alpha$

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but this contradicts (1.2) and we get

$\lim_{n\to\infty} d(g^{m-1}(\mathrm{x}), g^{n-1}(x)) = 0.$

Let now show that $\{g^n x\}$ is Cauchy but on the contrary let's show $\{g^n x\}$ is not Cauchy. We have

$\varepsilon > 0$ for every $n \in \mathbb{N}$ and m > n such that,

$$\mathrm{d}(g^n x, g^m x) \geq \varepsilon$$

Choosing the smallest integer of m for which x hold, we obtain

$$\mathrm{d}(g^n x, g^{m-1} x) < \varepsilon$$

By triangular inequality, we have,

$$\begin{aligned}
\varepsilon &\leq \mathrm{d}(g^n x, g^m x)\\
&\leq \mathrm{d}(g^n x, g^{m-1} x) + \mathrm{d}(g^{m-1} x, g^m x) - \mathrm{d}(g^{m-1} x, g^{m-1} x) - \phi \mathrm{d}(g^n x, g^m x)\\
&= \mathrm{d}(g^n x, g^{m-1} x) + \mathrm{d}(g^{m-1} x, g^m x) - \phi \mathrm{d}(g^n x, g^m x)\\
&\leq \mathrm{d}(g^{n-1} x, g^{m-1} x) - \phi \mathrm{d}(g^n x, g^m x) \leq \mathrm{d}(g^{n-1} x, g^{m-1} x) - \mathrm{d}(g^n x, g^m x)\\
&\leq \frac{1}{2} \mathrm{d}(g^{n-1} x, g^{m-1} x)\\
&\leq \phi(\varepsilon)
\end{aligned}$$

And as n, m $\to \infty$, the right hand side of inequality moves to zero, for t$< \phi$(t).

so there exist y∈X such that $g^n x \to$y, as n→ ∞, x∈X, we have

$$\mathrm{d}(\mathrm{y}, \mathrm{x}) = \lim_{n\to\infty} d(y, g^n x) = \lim_{n,m\to\infty} d(g^n x, g^m x) = 0 \quad (1)$$

At this point, we observe that by (1), $\lim_{n\to\infty} \mathrm{d}(g^n x, \mathrm{y}) = 0$, we are required to demonstrate that y is a fixed point of g, we have

$\mathrm{d}(g^n x, g^m x) \leq \mathrm{d}(g^n x, \mathrm{y})$

and since $\mathrm{d}(g^n x, \mathrm{y}) = 0$ means $\mathrm{d}(g^n x, g^m x) = 0$ as $\mathrm{d}(y, \mathrm{x}) = 0$. Thus, we have

$\mathrm{d}(g^n x, g^m x) = \mathrm{d}(g^n x, \mathrm{y}) = \mathrm{d}(\mathrm{y}, \mathrm{y})$

Since ϕ is continuous and by lemma (8), we have

$$\begin{aligned}
\phi(\varepsilon) &\leq \lim_{n,m\to\infty} inf\, \mathrm{d}(g^n x, g^m x)\\
&\leq \lim_{n,m\to\infty} sup\, \mathrm{d}(g^n x, g^m x)\\
&\leq \varepsilon
\end{aligned}$$

And this statement contradicts with (1.2). Since $\{g^n x\}$ is a Cauchy and X is complete. Then we have for $\{g^n x\}$ converging to y ∈X. As g is continuous from (1.1) and (1.2),

$\mathrm{g}(x_0) = \lim_{n\to\infty} g^{n+1} x = \lim_{n\to\infty} g(g^n x) = \mathrm{g}\lim_{n\to\infty}(g^n x) = \mathrm{y}$

we have $\mathrm{g}x_0$ = y, hence this limit point y of $\{\mathrm{g}x_0\}$ is a fixed point of g.

For uniqueness of the fixed point, let v be another fixed point of g. hence

$$\mathrm{d}(\mathrm{v}, x_0) \leq \mathrm{d}(\mathrm{gv}, \mathrm{g}x_0) \leq \phi \mathrm{d}(\mathrm{v}, x_0) < \mathrm{d}(\mathrm{v}, x_0)$$

is a contradiction, hence v $= x_0$, implies g has a unique fixed point.

Example 1: Given T : E → E and v, u ∈ $[0, \frac{1}{2}[$,we have

$$\mathrm{g}(\mathrm{x}) = \begin{cases} \frac{x}{13}, & if\ x \in [0, \frac{1}{2}[\\ \frac{1}{14}, & if\ x = \frac{1}{2} \end{cases}$$

Let v, u ∈ $[0, \frac{1}{2}[$. Thus
d(g(x), g(y)) ≤d(x, y) - ϕd(g(x), g(y)

$$|\mathrm{g}(\mathrm{x}) - g(y)| = \left|\frac{x}{13} - \frac{y}{13}\right| = \frac{1}{13}|x-y|$$

for
$|x-y|$ = d(x, y)
And

$$|x-y| - \frac{1}{13}|x-y|$$

Where
$\mathrm{x} - \frac{1}{13}\mathrm{x} = \frac{12}{13}\mathrm{x}$ and $-y + \frac{1}{13}\mathrm{y} = -\frac{12}{13}y$
Which implies that
$|\mathrm{g}(\mathrm{x}) - g(y)| = \frac{1}{13}|x-y| \leq \frac{12}{13}|x-y| =$ d(x, y) - ϕd(g(x), g(y))
Now, if x∈ $[0, \frac{1}{2}[$ and y= $\frac{1}{2}$, we get

$$|g(x) - g(y)| = \left|\frac{x}{13} - \frac{1}{14}\right|$$

for

$$|x - 1|$$

And

$|x - 1| - \left|\frac{x}{13} - \frac{1}{14}\right| = \left|\frac{12x}{13} - \frac{13}{14}\right|$

Consequently, we have x, y∈ $[0, \frac{1}{2}]$, and thus

$$|g(x) - g(y)| = \left|\frac{x}{13} - \frac{1}{14}\right| \leq \left|\frac{12x}{13} - \frac{13}{14}\right|$$

DISCUSSION

Theorem 2. Let (X, d) be a complete metric space and U be an open subset of X where V be closed in X such that U ⊂ V. Suppose we have an operator $H : V \times [0,1] \to X$ meet these conditions

i) Given $x \neq H(x,t)$ for $x \in V \setminus U$ where $t \in [0,1]$

ii) If $\phi : \mathbb{R} \to \mathbb{R}$ is a given non-decreasing continuous function as $t < \emptyset(t)$, we have $t \in [0,1]$ for each $x, y \in V$, then $d\big(H(x,t), H(y,t)\big) \leq d(x,y) - \phi d(x,y)$.

iii) Given $M \geq 0$, where,

$$d\big(H(x,t), H(x,s)\big) \leq M|\lambda - \mu|$$

For every $\lambda, \mu, t \in [0,1]$ and for every $x \in U$. Hence H(., 0) has a fixed point if and only if

$H(.,1)$ has a fixed point.

RESULTS AND DISCUSSION

Take set S = {λ ∈ [0, 1] such that x = (x, λ) for $x \in$ U} and given H (., 0) possesses a fixed point in U. And with $O \in S$ indicates S is a non – empty set.

At this stage we show S is open as well as closed in [0, 1] and by means of connectedness of [0,1], we get set S= [0,1]

First, we begin by showing S is open in [0,1].

Let $\{\lambda_n\}_{n=1} \subseteq S$ with $\lambda_n \to \lambda \in [0,1]$, as $n \to \infty$ for $x_0 \in U$ and $t \in S$ with $x_0 = H(x_0, \lambda_0)$.

There exists $r > 0$ such that $B_x(x_0, r) \subseteq U$ as U is open in X. For

$$\phi(t) < t$$

Let $\varepsilon > 0$ and considering$\{\lambda \in [0,1] \mid M|\lambda - \lambda_0|\} \leq \varepsilon = B_x(x_0, r)$. We have for $x, y \in U$

$$d(H(x,\lambda), x_0) = d(H(x,\lambda), H(x_0, \lambda_0))$$

$$\leq d(H(x,\lambda), H(x,\lambda_0) + d(H(x,\lambda_0) + H(x_0,\lambda_0)$$

$$\leq d(x,y) - \phi d(x,y) + M|\lambda - \lambda_0|$$

$$\leq d(x,y) - d(x,y) + M|\lambda - \lambda_0|$$

$$= M|\lambda - \lambda_0| \quad \leq \varepsilon$$

And $H(x,\lambda) \in V$. Therefore $H(\cdot,\lambda) : V \to V$. As a result, since all the hypothesis of theorem 1 hold, $H(\cdot,\lambda)$ has a fixed point in V, where by connectedness of V *and* $U, H(\cdot,\lambda)$ has a fixed point in U. So $U \subseteq S$ and hence G is open in [0, 1].

Let's now show that $\mathbb{S}$ is closed in [0,1]. And let $\{\lambda_n\}_{n \epsilon \mathbb{N}}$ be a sequence in $\mathbb{S}$ for $\lambda_n \to \lambda_x \epsilon [0,1]$ as $n \to \infty$. Here, we are required to show

$\lambda_n \in \mathbb{S}$. There exists $x_n \in \mathbb{U}$ with $x_n = H(x_n, \lambda_n)$ for all $n \epsilon \mathbb{N}$, from the definition of $\mathbb{G}$.

Moreover,

$$d(x_n, x_m) = d(H(x_n, \lambda_n), H(x_m, \lambda_m)$$

$$\leq d(H(x_n, \lambda_n), H(x_n, \lambda_m) + d(H(x_n, \lambda_m), H(x_m, \lambda_m)$$

$$\leq d(x_n, \lambda_n) - \phi(d(x_n, \lambda_n) + \mathrm{M}\,|x_n - \lambda_m|$$

$$\leq d(x_n, \lambda_n) - d(x_n, \lambda_n) + M|x_n - \lambda_m|$$

$$= M|x_n - \lambda_m|$$

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$\leq \varepsilon$

And from the last statement, we have $d(x_n, x_m) \leq \varepsilon$ for all m, $n \epsilon \mathbb{N}$ and this gives rise to,

$$d(x_n, x_m) \leq M|x_n - \lambda_m|$$

By continuity of $M \geq 0$, and given convergence of $\{\lambda_n\}_{y\, n \epsilon \mathbb{N}}\}$ with n, $m \to \infty$, obtain,

$$\lim_{n,m\to\infty} d(x_n, \lambda_m) = 0.$$

Thus, we have $\{x_n\}_{n \epsilon \mathbb{N}}$ Cauchy sequence in X. As X is complete, there exist $x_x \in V$ such that

$$\lim_{n\to+\infty} d(x_x, x_n) = 0.$$

And by letting $n \to +\infty$, we have,

$$d(x_n, H(x_x, \lambda_x) = d(H(x_n, \lambda_n), H(x_x, \lambda_x)$$

$$\leq d(H(x_n, \lambda_n), H(x_n, \lambda_x) + d(H(x_n, \lambda_m) + H(x_x, \lambda_x)$$

$$\leq d(x_n, \lambda_n) - \phi d(x_n, \lambda_n) + M|x_n - \lambda_x|$$

$$\leq d(x_n, \lambda_n) - d(x_n, \lambda_n) + M|x_n - \lambda_x|$$

$$= M|x_n - \lambda_x|$$

Taking $\lim_{n\to+\infty} d(x_n, H(x_x, \lambda_x) = 0$

Hence

$$d(x_x, H(x_x, \lambda_x) = \lim_{n\to+\infty} d(x_n, H(x_x, \lambda_x) = 0$$

And as a result, $x_x = H(x_x, \lambda_x)$ and by (i), we have $x_x \in V$ and given $\lambda_x \in \mathbb{S}$.

Implies $\mathbb{S}$ is close in [0, 1] and H (., 1) has a fixed point in V.

CONCLUSION

In conclusion, the result of partially ordered metric spaces is investigated. The partially ordered metric spaces provide us with a powerful framework for delving into order relation and metric properties. Generalization of the findings with examples and to homotopy theory is provided. Where the homotopy allows for deformation of functions within spaces and for continuity of functions. The results are achieved using topological properties such as compactness and connectedness with regards to close and open unit interval.

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