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A REVIEW OF TENSOR INTERACTION IN THE THEORY OF POTENTIAL MODELS

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ABSTRACT

In this paper, we review the use of tensor potential by different researchers in the removal of degeneracies in the theory of potential models. The interaction of different tensor potentials and its combination such as Coulomb, Hulthèn, Yukawa and generalized potential with different potential models under Dirac formalism were reviewed.

Key words: Tensor potential and degeneracy.

INTRODUCTION

The dynamics of particles in quantum physics are obtained using standard wave equations which include Schrödinger equation, Dirac equation, Klein-Gordon equation, Bethe-Salpeter equation and Duffin-Kemmer-Petiau (DKP) equation.

The non-relativities spineless particle dynamics is described using the wave equation (SWE). klein-Gordon equation is used in describing relativistic spineless particles such as mesons while the dynamic description of relativistic spin-half particles like electrons, protons and neutrons use Dirac equation. The DKP equation is suitable for the description of spin-one and spin-zero particle's dynamics while Bethe-salpeter equation describes the bound states of a two body (particles) quantum filed theoretical system in a relativistically covarian formalism. Some examples of the two body system include positronium (bound state of an electron-position pair), excitons (bound states of an electron-hole pair) and mesons (as quark-antiquark bound states) (Mauriza *et al*., 2004).

In the last two decades different researchers have investigated the energy spectra and their associated wave functions with various potentials including tensor-potentials of various kind. The effect of tensor interactions in the degenerate states have been highly investigated.

This paper show case the review of some of the results of the tensor interactions on the degenerate states obtained by investigators. Tensor interaction has been successfully used by researchers to remove degeneracy (Ikot *et al*., 2013, Hassanabadi *et al.*, 2013a).

However, the mathematical framework of the problem determines the choice of the tensor interaction to be used and in most cases the coulomb-like or Cornell interaction is often used (Hassanabadi *et al.*, 2012 and Ikot *et al.*, 2014).

The coulomb tensor interaction effect has been investigated by many researchers with different potential regime including Hassanabadi *et al.* (2011), Ikot *et al*. (2013b), Obong *et al*. (2015a) and Maghsoodi *et al.* (2012). By noticing that Hulthèn potential is similar to the coulomb-like interaction when r tends to zero, Hassanabadi *et al.*(2013a) therefore introduced the Hulthèn potential as tensor interaction for the first time. The most widely used tensor interactions include coulomb, Yukawa, Hulthèn potential and generalized tensor.

REVIEWED PAPERS

Ikhdair and Sever (2010) in their paper obtain approximately the energy spectrum equations for the attractive scalar S(r) and repulsive vector V(r) Hulthèn potentials with a coulomb-like tensor potential for arbitrary spin-orbit quantum number κ. By considering the bound state of the spin symmetry solution, Ikhdair and Sever (2010) take

$$\Sigma(r) = -\frac{V_0}{e^{\frac{r}{r_0}}} \; and \; U(r) = -\frac{H}{r} \tag{1}$$

as the Hulthèn and coulomb tensor potential respectively. By replacing the spin-orbit tension in the Dirac upper component of the radial equation with an approximation in line with Ikhdair (2008) transformed the radial part of the Dirac equation into an hypergeometric equation which on solving in line with the NU parametric standard equation of Tezcan and Sever (2009) obtain the transcendental energy spectrum equation as

$$\left(E_{n,k} - M\right)\left(E_{n,k} + M - c_s\right) = \frac{d_0(k+H)(k+H+1)}{r_0^2} - \frac{1}{4}\left[\frac{r_0\left(E_{n,k} + M - c_s\right)V_0}{(n+k+H+1)} - \frac{(n+k+H+1)}{r_0}\right]^2 \tag{2}$$

Following the same steps as in the spin symmetry Case using

$$\Delta(r) = -\frac{V_0}{e^{\frac{r}{r_0}} - 1} \; and \; \sum(r) = c_{ps}, \tag{3}$$

they deduce the energy spectrum equation for the pseudospin symmetry as

$$\left(E_{n,k} + M\right)\left(E_{n,k} - M - c_{ps}\right) = \frac{d_0(k+H)(k+H-1)}{r_0^2} - \frac{1}{4}\left[\frac{r_0\left(E_{n,k} - M - c_{ps}\right)V_0}{(n+k+H)} - \frac{(n+k+H)}{r_0}\right]^2 \tag{4}$$

Lastly, Ikhdair and Sever (2010) showed numerically that degeneracy between the doublet states in the spin and pseudospin symmetris were removed with the introduction of the tensor interactions.

In another related development, Ikot *et al.* (2013b) with the introduction of a generalized tensor interaction (GTI) using the Nikiforov Uvarov (NU) technique were able to remove more of the degeneracy between the doublet states in the spin and pseudo pin symmetries.

The energy spectra of the bound states and the corresponding radial wave spinors for the spin and pseudospin symmetries were approximately calculated by the researchers. Also the effects of the GTI on the degenerate states were presented.

Ikot *et al.* (2013b) considered the relativistic Dirac equation coupled with tensor potential U(r) in the presence of the two Lorentz potentials given by Ginocchio (1997) as

$$\left\{\vec{\alpha}\cdot\vec{p} + \beta\left(M + \delta(r)\right) - i\beta\vec{\alpha}\cdot\hat{r}U(r)\right\}\Phi(r) = \left\{E - V(r)\right\}\Phi(r) = 0 \tag{5}$$

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In the light of Aydogdu *et al*. (2023), Ikot *et al*. (2013b) take $\Sigma(r)=c_{ps}$ and the Hulthen potential as

$$\Delta(r)=-\frac{ze^2\delta e^{-2\delta r}}{1-e^{-2\delta r}}, \qquad (6)$$

with the GTI (U(r)) given as

$$U(r)=U_C(r)+U_Y(r) \qquad (7)$$

Where

$$U_C(r)=-\frac{H}{r}=-\frac{z_a z_b e^2}{4\pi\varepsilon_0 r},\ \ U_Y(r)=\frac{V_1 e^{-\mu r}}{r}. \qquad (8)$$

Substituting equations (6) and (7) in the decoupled lower component of equation (5), Ikot *et al.* (2013b) obtained eqn. 9 which is the dynamic equation for the Dirac pseudospin symmetry as;

$$\left\{\begin{array}{l}\frac{d^2}{dr^2}-\frac{k(k-1)}{r^2}-\frac{2k\left(1+V_1e^{\mu f}\right)}{r^2}+\frac{H}{r^2}+\frac{\delta V_1 e^{-\mu r}}{r}+\frac{V_1e^{-\mu r}}{r^2}-\frac{H^2}{r^2}\\ -\frac{2HV_1e^{-\mu r}}{r^2}-\frac{V_1^2e^{-2\mu r}}{r^2}-\left(M+E_{n,k}^{ps}\right)\left(M-E_{n,k}^{ps}+c_{ps}\right)\left(\frac{e^{-2\mu r}}{1-e^{-2\mu r}}\right)\end{array}\right\}G_{n,k}(r)=0 \qquad (9)$$

The researchers eliminate the spin-orbit term $\frac{k(k-1)}{r^2}$ using the following approximation (Aydogdu *et al.*, 2013; Pekeris (1934).

$$\frac{1}{r^2}\approx\frac{4a^2e^{-2\mu r}}{\left(1-e^{-2\mu r}\right)^2}\ ,\qquad \frac{1}{r}\approx\frac{4ae^{-\mu r}}{\left(1-e^{-2\mu r}\right)^2} \qquad (10)$$

Using the transformation variable $s=e^{-2\mu r}$ together with the approximations (eqn. 10), equation (9) transformed to hypergeometric type second-order differential equation of the form (Ikot *et al.*, 2013b)

$$\frac{d^2G_{n,k}^{ps}(s)}{ds^2}+\frac{1}{s}\frac{(1-s)}{(1-s)}\frac{dG_{n,k}^{ps}(s)}{ds}+\frac{1}{s^2(1-s)^2}\left[\lambda_1^{ps}s^2+\lambda_2^{ps}s-\lambda_3^{ps}\right]G_{n,k}^{ps}(s)=0 \qquad (11)$$

By putting equation (11) side by side with the NU parametric equation of the form (Tezcan and sever, 2009).

$$\left(\frac{d^2}{ds^2}+\frac{(a_1-a_2s)}{s(1-a_2s)}\frac{d}{ds}+\frac{1}{s^2(1-a_1s)^2}\left[\zeta_1s^2+\zeta_2s-\zeta_3\right]\right)\varphi(s)=0 \qquad (12)$$

Ikot *et al*. (2013b) deduced the equation for the energy spectra in the form

$$M^2+Mc_{ps}-\left(E_{n,k}^{ps}\right)^2+ME_{n,k}^{ps}+c_{ps}E_{n,k}^{ps}=-\delta^2\left\{\frac{\left(V_1+\frac{1}{2}\right)V_1}{2(n+\sigma)}-\frac{ze^2}{\delta(n+\sigma)}-(n+\sigma)\right\}^2 \qquad (13)$$

where

$$\sigma = \frac{1}{2}\left(1+\sqrt{4\eta_k\left(\eta_k-1\right)+4\left(V_1+2H+2k-1\right)V_1}\right). \quad (14)$$

Using the spin symmetry limit condition, the researchers obtained the energy spectra equation for the spin symmetry limit as

$$\left(E_{n,k}^s\right)^2 - M^2 - ME_{n,k}^s - Mc_s + ME_{n,k}^s - c_s E_{n,k}^s = -\delta^2 \left\{ \frac{\left(V_1-\frac{1}{2}\right)V_1}{2\left(n+\sigma_S\right)} - \frac{ze^2}{\delta\left(n+\sigma_S\right)} - \left(n+\sigma_S\right) \right\}^2 \quad (15)$$

where

$$\sigma_s = \frac{1}{2}\left(1+\sqrt{4\Lambda_k\left(\Lambda_k-1\right)+4\left(V_1+2H+2k+1\right)V_1}\right). \quad (16)$$

Ikot *et al*. (2013b) evaluated the energy Eigen values in the presence and absence of the GTI and high-lighted the occurrence of degeneracy in the pseudospin symmetry between the states.

$\left(1p_{\frac{3}{2}}, 2s_{\frac{1}{2}}\right), \left(1d_{\frac{5}{2}}, 2p_{\frac{3}{2}}\right)$ *and* $\left(1f_{\frac{7}{2}}, 2d_{\frac{5}{2}}\right)$ for k < 0 and k > 0 respectively. In the case of spin symmetry, Ikot *et al*. (2013b) showed that degeneracy exists in the states $\left(od_{\frac{5}{2}}, 1p_{\frac{3}{2}}\right), \left(0f_{\frac{7}{2}}, 1d_{\frac{5}{2}}\right)$ *and* $\left(0g_{\frac{9}{2}}, 1f_{\frac{7}{2}}\right)$ for k < 0 and k > 0 respectively.

Conclusively, with generalized tensor interaction the degeneracies were removed.

In 1933 manning demonstrated the usefulness associated with Manning Rosen (MR) potential in the study of diatonic molecules. Ikhdair and Hamzavi (2012) also show that MR being an empirical potential is used in analyzing the atomic interaction in diatomic materials close to the surface.

Hassanabadi *et al*. (2013a) analyzed Manning-Rosen Dirac equation with Hulthen tensor interaction using the parametric NU technique. They obtained both the energy spectra and the associated spinors with proper choice of approximation for the barrier centrifugal term. The role of degeneracy-removing power of the Hulthen tensor interactions was clearly demonstrated both numerically and graphically in their reports.

In another related development, Ikot *et al*. (2014) presented the analytic solutions for the Manning –Rosen Dirac equation using the parametric generalized Nikiforov-Uvarov formalism.

In their work the role of the generalized tensor interactions in removing degeneracy in the spin and pseudospin doublet states numerically is demonstrated.

Considering the case of pseudospin symmetry where $\Sigma(r) = c_{ps}$ = constant (Qiang and Dong, 2006 and Wei and Dong, 2010), they take $\Delta(r)$ as the Manning-Rosen potential (Hassanabadi *et al.* (2012), Manning-Rosen, (1933)) in the form

$$\Delta(r) = \frac{1}{2m/\hbar^2\beta^2}\left(\frac{\alpha\left(\alpha-1\right)e^{-4\mu r}}{\left(1-e^{-2\mu r}\right)^2} - \frac{Ae^{-2\mu r}}{\left(1-e^{-2\mu r}\right)}\right). \quad (17)$$

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The generalized tensor interaction is given as;

$$U(r) = -\frac{1}{r}\left(H_C + H_Y e^{-\delta r}\right), \quad H_C = \frac{z_a z_b e^2}{4\pi\varepsilon_0} \tag{18}$$

By applying eqns. 17 and 18 in the Dirac pseudospin symmetry equation with proper transformation Ikot *et al*.(2014) arrived at the second order differential equation with suitable approximation as;

$$\frac{d^2 G_{n,k}^{ps}(x)}{dx^2} + \frac{(1-x)}{x(1-x)}\frac{dG_{n,k}^{ps}(x)}{dx} + \frac{1}{x^2(1-x)^2}\left[-\gamma_1^{ps}x^2 + \gamma_2^{ps}x - \gamma_3^{ps}\right]G_{n,k}^{ps}(x) = 0 \tag{19}$$

By placing the parametric NU hypergeometric equation (Tezcan and Sever(2009)) side by side with eqn.19, Ikot *et al.* (2014) arrived at the energy eigenvalues equation of the form :

$$M^2 - ME_{n,k}^{ps} + Mc_{ps} - \left(E_{n,k}^{ps}\right)^2 + c_{ps}E_{n,k}^{ps}$$
$$= \frac{1}{4\delta^2}\left\{\frac{\left(H_Y + \frac{1}{2}\right)H_Y - \frac{1}{K\beta^2}\left(M - E_{n,k}^{ps} + c_{ps}\right)\left(A + \alpha(\alpha-1)\right)}{(n+\sigma)} - (n+\sigma)\right\}^2, \tag{20}$$

Where

$$\sigma = \frac{1}{2}\left(1 + \left[1 + 4\eta_k(\eta_k - 1) + 4\left(H_Y + 2k + 2H_C - 1\right)H_Y 4\alpha(\alpha-1)\left(M - E_{n,k}^{ps} + c_{ps}\right)\frac{1}{k\beta^2}\right]^{\frac{1}{2}}\right) \tag{21}$$

Using the same procedure in arriving at the pseudospin symmetry limit, Ikot *et al.* (2014) obtained the spin symmetry energy spectra by taking ∑(r) and Δ(r) as

$$\Sigma(r) = \frac{1}{k\beta^2}\left\{\frac{\alpha(\alpha-1)e^{-4\delta r}}{\left(1-e^{-2\delta r}\right)^2} - \frac{Ae^{-2\delta r}}{\left(1-e^{-2\delta r}\right)}\right\},$$
$$\Delta(r) = V(r) - S(r) = c. \tag{22}$$

Their spin energy spectra equation is

$$\left(E_{n,k}^{s}\right)^2 - M^2 + ME_{n,k}^{s} - Mc_s - ME_{n,k}^{s} - c_s E_{n,k}^{s} =$$
$$\frac{1}{4\delta^2}\left\{\frac{\left(H_Y - \frac{1}{2}\right)H_Y - \frac{1}{k\beta^2}\left(M + E_{n,s}^{s} - c_s\right)\left(A + \alpha(\alpha-1)\right)}{\left(n - \sigma_S\right)} - \left(n - \sigma_S\right)\right\}^2, \tag{23}$$

where

$$\sigma_s = \frac{1}{2}\left\{1+\begin{pmatrix} 1+4\Lambda_k\left(\Lambda_k-1\right)+4\left(H_Y+2k+2H_C-1\right)H_Y \\ -\frac{1}{4k\beta^2}\alpha\left(\alpha-1\right)\left(M+E^s_{n,k}-c_s\right)\end{pmatrix}^{\frac{1}{2}}\right\} \tag{24}$$

And $E^s_{n,k} \neq M - c_s$.

In conclusion, Ikot *et al*. (2014) numerically shows that degeneracies occur in the following;

States $\left(1p_{\frac{1}{2}}, 0p_{\frac{3}{2}}\right), \left(1d_{\frac{3}{2}}, 0d_{\frac{5}{2}}\right), \left(1f_{\frac{5}{2}}, 0f_{\frac{7}{2}}\right)$ *and* $\left(1g_{\frac{7}{2}}, 0g_{\frac{9}{2}}\right)$.

While in the spin symmetry limit the states $\left(1p_{\frac{1}{2}}, 0p_{\frac{3}{2}}\right), \left(1d_{\frac{3}{2}}, 0d_{\frac{5}{2}}\right), \left(1f_{\frac{5}{2}}, 0f_{\frac{7}{2}}\right)$ *and* $\left(1g_{\frac{7}{2}}, 0g_{\frac{9}{2}}\right)$ degenerate.

These degeneracies disappeared with the presence of the generalized tensor interactions while new degenerate states $\left(0d_{\frac{3}{2}}, 1d_{\frac{5}{2}}\right), \left(0f_{\frac{5}{2}}, 0f_{\frac{7}{2}}\right)$ *and* $\left(0d_{\frac{3}{2}}, 0g_{\frac{9}{2}}\right), \left(0d_{\frac{1}{2}}, 0f_{\frac{7}{2}}\right)$ appeared for the pseudospin symmetry and spin symmetry respectively as clearly reported by Ikot *et al* .(2014).

To further demonstrate the degeneracy removing power of the GTI, Ikot *et al*. (2014a) solve the Eckart-Dirac equation in conjunction with the generalized tensor interaction to obtain the energy eigenvalue equations for the pseudospin and the spin symmetries using the Nikiforov-Uvarov method. Thepseudospin symmetry limit case was achieved by Ikot *et al.* (2014a) by taking $\Sigma(r) = c_{ps} =$ constant and the scalar and vector potential difference $\left(\Delta(r)\right)$ as

$$\Delta(r) = \beta\frac{e^{-\frac{4r}{a}}}{\left(1-e^{-\frac{2r}{a}}\right)^2} - \alpha\frac{1+e^{-\frac{2r}{a}}}{1-e^{-\frac{2r}{a}}} \tag{25}$$

Using the Pekeris approximation (Aydogdu and Sever, (2011)

$$\frac{1}{r^2} \approx \frac{\left(\frac{2}{a}\right)^2 e^{-\frac{2r}{a}}}{\left(1-e^{-\frac{2r}{a}}\right)^2}, \quad \frac{1}{r} \approx \frac{\left(\frac{2}{a}\right)^2 e^{-\frac{r}{a}}}{\left(1-e^{-\frac{2r}{a}}\right)^2} \quad r \ll a \tag{26}$$

With the substitution of eqns.7 and 25 in equation (Ginocchio (2005) and Hamzavi *et al*., 2010)

$$\left\{\begin{matrix}\left[\frac{d^2}{dr^2} - \frac{k(k-1)}{r^2} + \frac{2k}{r}U(r) + \frac{dU(r)}{dr} - U^2(r)\right]G_{n,k}(r) \\ = \left[\left(M+E_{n,k}-\Delta(r)\right)\left(M-E_{n,k}+\Sigma(r)\right)\right]G_{n,k}(r)\end{matrix}\right\} \tag{27}$$

Where

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$k(k-1)=\bar{\ell}(\bar{\ell}+1)$ *and* $k(k+1)=\ell(\ell+1)$ with the transformation $y=e^{-\frac{2r}{a}}$ thereby obtaining a hypergeometric equation whose solution resulted in the energy eigenvalues equation (Ikot *et al*. (2014)) as

$$n^2+n+\frac{1}{2}+(2n+1)\left\{\sqrt{\frac{1}{4}+H^{ps}+L^{ps}-Q^{ps}}+\sqrt{H^{ps}}\right\}-Q^{ps}+ \\ 2H^{ps}+2\sqrt{H^{ps}\left(\frac{1}{4}+H^{ps}+L^{ps}-Q^{ps}\right)}=0. \tag{28}$$

The energy eigenvalues equation for the spin symmetry limit is obtain by Ikot *et al*. (2014a) using $\Delta(r)=c_s=$constant and the sum $\Sigma(r)$ as the Eckart potential in the form

$$\Sigma(r)=\gamma\frac{e^{-\frac{4r}{b}}}{\left(1-e^{-\frac{2r}{b}}\right)^2}-\eta\frac{1+e^{-\frac{2r}{b}}}{\left(1-e^{-\frac{2r}{b}}\right)}. \tag{29}$$

Invoking equations 7 and (26) into the equation (Hamzavi *et al*. (2010)

$$\left\{\begin{array}{l}\left[\frac{d^2}{dr^2}-\frac{k(k+1)}{r^2}+\frac{2k}{r}U(r)-\frac{dU(r)}{dr}-U^2(r)\right]F_{n,k}(r) \\ +\frac{\frac{d\Delta(r)}{dr}}{M+E_{n,k}-\Delta(r)}\left(\frac{d}{dr}+\frac{k}{r}-U(r)\right)F_{n,k}(r) \\ =\left[\left(M+E_{n,k}-\Delta(r)\right)\left(M-E_{n,k}+\Sigma(r)\right)\right]F_{n,k}(r)\end{array}\right\} \tag{30}$$

with the transformation $s=e^{-\frac{2r}{a}}$, Ikot *et al*. (2014a) obtained the Eckart Dirac solution for the spin symmetry energy eigenvalues equation as

$$n^2+n+\frac{1}{2}+(2n+1)\left\{\sqrt{\frac{1}{4}+H^s+L^s}-\left(\gamma+\sqrt{H^s}\right)\right\} \\ -k^s+2H^s+2\sqrt{H^s\left(\frac{1}{4}+H^s+L^s-Q^s\right)}=0. \tag{31}$$

Ikot *et al*. (2014a) from their numerical results shows that degeneracies occur in the doublet states of the spin and pseudospin symmetry and that with the presence of the tensor potential (GTI) these degeneracies were removed.

Aydogdu and Sever (2011) solved the Morse Dirac equation with coulomb-like tensor potential interactions in the framework of pseudospin and spin symmetry. Aydogdu and Sever (2011) solve the psendospin symmetry case by using $\Delta(r)$ from the work of Al-Dossary (2007) as follows:

$$\Delta(r) = D_e\left(e^{-2\mu(r-r_e)} - 2e^{-\mu(r-r_e)}\right),$$
$$\Sigma(r) = c_{ps} \; and \;\; U(r) = \frac{-\eta}{r} \tag{32}$$

Using Pekeris approximation for the barrier centrifugal term and the transformation $y = e^{-\mu r_e x}$, they arrived at the confluent hypergeometric-type of equation of the form

$$u\frac{d^2H(u)}{du^2} + (2\Lambda + 1 - u)\frac{dH(u)}{du} - \left(\Lambda + \frac{1}{2} - \frac{x}{2\sigma}\right)H(u) = 0 \tag{33}$$

On solving equation 33, Aydogdu and Sever (2011) 0btained the energy eigen values equation as

$$\left\{2\sqrt{E_{n,k}\left(E_{n,k} - c_{ps} - 2M\right) - \beta D_0} + \frac{\beta D_1 + 2E_{n,k}D_e}{\sqrt{E_{n,k}D_e - \beta D_2}}\right\}^2 + (1+2n)^2\mu^2 = 0 \tag{34}$$

$$where \quad \beta = \frac{(k+\eta)(k+\eta-1)}{r_0^2} \quad and \quad E_{n,k} \neq M + c_{ps} \tag{35}$$

However, Aydogdu and Sever (2011) removed the degeneracy between the doublet states of the porudospin with the tensor potential.

Obong *et al*. (2015) solve the multiparameter exponential type (MPET) Dirac equation with generalized tensor (GTI) interaction under the framework of spin and pseudospin symmetries.

They investigated the pseudospin symmetry limit case taking $\Sigma(r) = c_{ps} =$constant and $\Delta(r)$(Pena *et al*., 2014, Greene and Aldrich, 1976) as

$$\Delta(r) = \frac{Xe^{-2\beta r}}{\left(1-e^{-2\beta r}\right)} + \frac{Ye^{-2\beta r}}{\left(1-e^{-2\beta r}\right)^2} + \frac{Ze^{-4\beta r}}{\left(1-e^{-2\beta r}\right)^2} \tag{36}$$

While the GIT potential is taken as (Ikot *et al.*, 2014a)

$$U(r) = -\frac{1}{r}\left(H_C + H_Y e^{-\beta r}\right). \tag{37}$$

By substituting eqns. (36) and (37) into eqn. (27) with barrier centrifugal term being replaced in line with the approximation of Ikot *et al.* , 2013b, Obong *et al*. (2015) with the transformation $s = e^{-2\beta r}$ obtain the equation

$$\frac{d^2G_{n,k}^{ps}(s)}{ds^2} + \frac{(1-s)}{s(1-s)}\frac{dG_{n,k}^{ps}(s)}{ds} + \frac{1}{s^2(1-s)^2}\left[-\gamma_1^{ps}s^2 + \gamma_2^{ps}s - \gamma_3^{ps}\right]G_{n,k}^{ps}(s) = 0 \tag{38}$$

Comparing eqn. (38) with the standard parametric Nikiforov-Uvarov, Obong *et al*. (2015) obtained the energy eigenvalues equation

$$M^2 - \left(E_{n,k}^{ps}\right)^2 = \beta^2\left\{(n+\delta) + \left(\frac{\beta_{ps}(X-Z)}{4\beta^2} + H_Y\left(H_Y + \frac{1}{2}\right)\right)(n+\beta)^{-1}\right\}^2 - \left(M + E_{n,k}^{ps}\right)c_{ps} \tag{39}$$

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Where

$$\delta = \frac{1}{2}\left\{1+\left(1-\frac{\beta_{ps}\left(Y+Z\right)}{\beta^2}+H_Y\left(H_Y+2H_C+2k-1\right)^{\frac{1}{2}}\right)\right\} \quad (40)$$

By taking the sum of the Lorentz potentials $\left(\Sigma\left(r\right)\right)$ and the tensor interaction (U(r)) respectively as

$$\Delta\left(r\right)=\frac{Xe^{-2\beta r}}{\left(1-e^{-2\beta r}\right)}+\frac{Ye^{-2\beta r}}{\left(1-e^{-2\beta r}\right)^2}+\frac{Ze^{-4\beta r}}{\left(1-e^{-2\beta r}\right)^2},\ \Delta\left(r\right)=c_s$$
$$U\left(r\right)=-\left(\frac{H_C}{r}+\frac{H_Y e^{-\beta r}}{r}\right) \quad (41)$$

and substituting eqn. (41) into eqn. (30) with the transformation $x=e^{-2\beta r}$ they obtained the second order differential equation of the form

$$\frac{d^2F^s_{n,k}\left(x\right)}{dx^2}+\frac{\left(1-x\right)}{x\left(1-x\right)}\frac{dF^s_{n,k}\left(x\right)}{dx}+\frac{1}{x^2\left(1-x\right)^2}\left[-\Omega^s_1x^2+\Omega^s_2x-\Omega^s_3\right]F^s_{n,k}\left(x\right)=0 \quad (42)$$

Obong *et al*. (2015) by comparing eqn. (42) with the standard parametric NU equation (Tezcan and Sever (2009)) obtained the energy eigen values equation for the spin symmetry limit as

$$M^2-\left(E^s_{n,k}\right)^2=\beta^2\left\{\left(n+\delta\right)+\left(\frac{\beta_s\left(X-Z\right)}{4\beta^2}+H_Y\left(H_Y-\frac{1}{2}\right)\right)\left(n+\delta\right)^{-1}\right\}^2+\left(M+E^s_{n,k}\right)c_s \quad (43)$$

Obong *et al*. (2015) shows the presence of degeneracy in the states $\left(1s_{\frac{1}{2}},0d_{\frac{3}{2}}\right),\left(1p_{\frac{3}{2}},0f_{\frac{5}{2}}\right),\left(1d_{\frac{5}{2}},0g_{\frac{7}{2}}\right)$ *and* $\left(1f_{\frac{7}{2}},0h_{\frac{9}{2}}\right)$ for the pseudospin symmetry limiting case, while in the spin limit, degeneracy occur in the following states; $\left(0p_{\frac{3}{2}},0p_{\frac{1}{2}}\right),\left(0d_{\frac{5}{2}},0d_{\frac{3}{2}}\right),\left(0f_{\frac{7}{2}},0f_{\frac{5}{2}}\right)$ *and* $\left(0g_{\frac{9}{2}},0g_{\frac{7}{2}}\right)$.

With the tensor interaction Obong *et al*. (2015) concluded that these degeneracies were removed with new degenerate states being established.

The eight parameter exponential-type (EPET) potential was further studied by Obong *et al*. (2016) with Coulomb-like tensor interaction. Obong *et al*. (2016) solved the EPET Dirac equation under pseudospin symmetry limit by taking $\Sigma(r)=c_{ps}=$constant, and $\Delta\left(r\right)$ as

$$\Delta\left(r\right)=A+\frac{B}{\left(q+e^{\alpha r}\right)}+\frac{C}{\left(q+e^{\alpha r}\right)^2}+\frac{Fe^{\alpha r_e}e^{i\theta}}{\left(q+e^{\alpha r}\right)}+\frac{Ge^{\alpha r_e}e^{2\alpha r}}{\left(q+e^{\alpha r}\right)^2} \quad (44)$$

While the coulomb-like tensor interaction term as

$$U\left(r\right)=-\frac{H_C}{r}\ . \quad (45)$$

By substituting eqns. (44) and (45) into eqn. (30) and replacing the centrifugal barrier coupling term with the Pekeris approximation (Pekeris, 1934; Aydogdu and Sever, 2010) as

$$\frac{1}{r^2} \approx \left(d_0 + d_1 \frac{1}{\left(q+e^{\alpha r}\right)} + d_1 \frac{1}{\left(q+e^{\alpha r}\right)^2} \right) \tag{46}$$

With the transformation $s = qe^{-\alpha r}$, Obong *et al.*(2016) arrived at the equation

$$\frac{d^2 G_{n,k}^{ps}(s)}{ds^2} + \frac{(1-s)}{s(1-s)} \frac{dG_{n,k}^{ps}(s)}{ds} + \frac{1}{s^2(1-s)^2}\left[-\chi_1^{ps} s^2 + \chi_2^{ps} s - \chi_3^{ps}\right] G_{n,k}^{ps}(s) = 0 \tag{47}$$

They solved eqn. (47) by comparing it with the standard NU parametric equation to obtain the energy eigen values equation for the EPET potential under the pseudospin symmetry as

$$n(n+1) + \frac{1}{2} - (2n+1)\left\{ \begin{array}{l} \left(\sqrt{\frac{1}{4} + \chi_1^{ps} + \chi_3^{ps} + \chi_2^{ps}} - \sqrt{\chi_3^{ps}} \right) \\ + \chi_2^{ps} + 2\chi_3^{ps} - 2\sqrt{\chi_3^{ps}\left(\frac{1}{4} + \chi_1^{ps} + \chi_3^{ps} + \chi_2^{ps} \right)} \end{array} \right\} = 0 \tag{48}$$

Under the spin symmetry limit, Obong *et al.* (2016) take $\Delta(r) = c_s$ =constant and ∑(r) to be EPET potential and U(r) as the tensor potential i.e

$$\begin{aligned} \Sigma(r) &= A + \frac{B}{\left(q+e^{\alpha r}\right)} + \frac{C}{\left(q+e^{\alpha r}\right)^2} + \frac{Fe^{\alpha r_e} e^{i\theta}}{\left(q+e^{\alpha r}\right)} + \frac{Ge^{\alpha r_e} e^{2\alpha r}}{\left(q+e^{\alpha r}\right)^2}, \\ U(r) &= -\frac{H_C}{r} \end{aligned} \tag{49}$$

By involving the Pekeris approximation after substituting eqn. (49) into equation (30) with the transformation y = $qe^{-\alpha r}$ Obong *et al.* (2016) obtain the parametric hypergeometric equation of the form.

$$\frac{d^2 F_{n,k}^{s}(y)}{dy^2} + \frac{(1-y)}{y(1-y)} \frac{dF_{n,k}^{s}(y)}{dy} + \frac{1}{y^2(1-y)^2}\left[-\chi_1^{s} y^2 + \chi_2^{s} y - \chi_3^{s}\right] F_{n,k}^{s}(y) = 0 \tag{50}$$

This hypergeometric-type equation (Eqn. 50) is solved by the researchers using the parametric NU technique. Thus, Obong *et al.*, (2016) placing eqn. 50 side by side with eqn. 17 arrived at the energy eigen values equation of the EPET potential within the limit of the spin symmetry in the present of the coulomb-like tensor interaction as

$$n^2 + (n+\frac{1}{2}) - 2q\left(2n+\frac{1}{2}\right)\left\{ \begin{array}{l} \sqrt{\frac{1}{4} + \chi_1^{s} + \chi_3^{s} + \chi_2^{s}} + q\chi_2^{s} + 2\chi_3^{s} + \\ 2\left[\left(n+\frac{1}{2}\right) - q\sqrt{\frac{1}{4} + \chi_1^{s} + \chi_3^{s} + \chi_2^{s}}\right]\sqrt{\chi_3^{s}} \end{array} \right\} = 0 \tag{51}$$

Obong *et al*. (2016) reported the presence of degeneracy between the states without tensor interaction in the spin symmetry limit. These states, they reported are

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$\left(0p_{\frac{1}{2}},0p_{\frac{3}{2}}\right),\left(0d_{\frac{3}{2}},0d_{\frac{5}{2}}\right),\left(0f_{\frac{5}{2}},0f_{\frac{7}{2}}\right)$ *and* $\left(0g_{\frac{7}{2}},0g_{\frac{9}{2}}\right)$. However, these degeneracies disappeared with the presence of the coulomb tensor while new degeneracies resurface with the presence of the coulomb tensor.

CONCLUSION

Coulomb-like, Yukawa, and Hulthén potentials have been highly investigated as well as generalized tensor by various researchers. The incorporation of these tensor interactions in Dirac formalism has greatly enhanced the removal of degeneracies between the doublet states as reported in the reviewed papers. However, the generalized tensor interaction has shown a remarkable removal of degeneracies compared to single tensor potential interaction. In most cases, the tensor potential apart from removing the existing degeneracies, introduces new degeneracies which are often regarded as accidental degeneracy which may be due to the geometry of the Dirac framework.

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