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EFFICIENCY OF MODIFIED CENTRAL COMPOSITE DESIGNS WITH FRACTIONAL FACTORIAL REPLICATES FOR FIVE-VARIABLE NON-STANDARD MODELS

Iwundu, M. P and Nwoshopo, D. E.
Department of Mathematics and Statistics, University of Port Harcourt, Nigeria.
mary.iwundu@uniport.edu.ng ; dennisejike3@gmail.com

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ABSTRACT

The efficiencies of standard central composite designs are compared with modified central composite designs on non-standard models using D- and G-efficiency criteria. Diagonal elements of the Hat matrix are utilized in the construction of the modified central composite designs. Fractional factorial replicates are used to maintain manageable design sizes. Results show that D-efficiencies of the designs decline for standard CCDs as the number of missing quadratic terms increases but increase with modified CCDs for increased number of missing quadratic terms. Similarly, G-efficiencies of the designs decline for standard CCDs as the number of missing quadratic terms increases but increase with modified CCDs for increased number of missing quadratic terms.

Keywords: Standard CCDs, Modified CCDs, Non-Standard Model, Hat Matrix, Fractional Factorial Replicates, Design Efficiencies.

INTRODUCTION

Response Surface Methodology (RSM) strives to relate an output or a response variable to a number of input or predictors variables that affect it. Usually, the form of such a relationship is always not known, but low-order polynomial may be used as approximating models. Fractional factorial designs allow the study of a large number of factors with relatively few experimental trials. A well selected subset, or fraction, of the treatments can be employed with manageable loss of information about the main effects and key low-order interactions. At a later stage of experimentation, higher order design may be used to reflect the presence of curvature in the model.

3^k full factorial designs have been useful in estimating all parameters in regression models. However, the number of treatment combinations required by 3^k design increase rapidly with increased number of factors.Central Composite Designs as well as other optimal response surface designs, serve purposely to truncate the challenges imposed by 3^k factorial designs. There are three basic components of Central Composite Designs, namely; the factorial portion, the axial portion and the centre portion. The factorial component of a CCD addresses estimation of linear main effects and all the factor interaction effects. The $2k$ axial (star) portion addresses the estimation of all quadratic effects. The center portions accounts for estimation of model's lack of fit.

Box and Hunter (1957) considered the subject of Rotatability. The rationale behind rotatability is that the scaled prediction variance at any two locations from the design center should be the same.

As in Myers *et al.* (2009), Rotatability offers guidelines for the choice axial distance, α as well as the number of center runs, n_c.

Fundamental Model for the use of Second-order Response Surface Designs

The model that allows the use of second-order designs is

$$Y = \beta_0 + \sum_{i=1}^{k} \beta_i x_i + \sum_{j=i+1}^{k} \sum_{i=1}^{k-1} \beta_{ij} x_i x_j + \sum_{i=1}^{k} \beta_{ii} x_{ii}^2 + \varepsilon \qquad (1)$$

having $p = \frac{(k+1)(k+2)}{2}$ model parameters.Y is the measured response;β's are model coefficients; x_i's are the input variables and ε is an error term associated with Y. The second-order model may be represented in matrix form as

$$Y = X\beta + \varepsilon \qquad (2)$$

with solution

$$\hat{\beta} = (X^{'}X)^{-1}X^{'}Y \qquad (3)$$

provided that $(X^{'}X)^{-1}$ is non-singular.

The unbiased estimate of the parameters, $\hat{\beta}$, is such that

$$\mathrm{E}(Y) = \hat{Y} = X\hat{\beta} = X(X'X)^{-1}X'Y = HY \qquad (4)$$

where

$$H = X(X'X)^{-1}X' \qquad (5)$$

is called the hat matrix.

The diagonal entries h_{ii} of the hat matrix are are such that $0 \leq h_{ii} \leq 1$ and $\sum_{i=1}^{n} h_{ii} = p$; where p is the number of model coefficients and n is the number of observations or runs in the design.

In modelling second-order response surface, test of significance of regression coefficients may reveal that some model components are not significant thus leading to such coefficients being removed from the full model. A second-order model with less than $p = \frac{(k+1)(k+2)}{2}$ model parameters is regarded as reduced or non-standard model. The ideal designs for second-order response surface models include the classical central composite designs, Box-Behnken designs, computer-aided designs and so many others that are contained in most literatures on Response Surface Methodology. Iwundu and Otaru (2019) constructed Hat-Matrix aided composite designs for Second-Order models. These designs were comparable with Standard Response Surface Methodology (RSM) designs and Computer-Generated designs. Their optimality and efficiency properties were further presented.

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A limitation in using classical response surface designs is that they naturally assume that all factors are equally easy to manipulate, by that giving way for complete randomization of experimental run order. In practice, the size of an experiment can become prohibitively large when a large number of factors is to be studied and cannot be completely randomized. However, these experiments are often conducted in a manner that restricts the randomization, which proceeds to split–plot structure. Some studies that investigate response surface experiments with split-plot structures include Letsinger *et al.* (1996), Vining *et al.* (2005) and Kowalsky *et al.* (2006).

In Iwundu (2018), steps are provided for constructing modified central composite designs for non-standard models. Unfortunately, the algorithm becomes difficult to use when $k \geq 5$. A way to handle this difficulty is to controllably reduce the number of experimental runs. For simplicity, it is logical to use fractional factorials instead of the full factorial. The aim of this research is to construct Modified Central Composite Design (MCCD) for non-standard models with fractional factorial replicates with the intents of comparing the performance of the modified central composite design with the standard central composite designs for non-standard models. For illustrative purpose, half fraction of full factorial designs in five variables is considered and three centre runs are employed in the design. An axial distance of $\alpha = F^{1/4}$ is used, where F is the number of fractional factorial points of Resolution V or higher. Thus, if *N* denotes the total number of experimental trials in the planned CCD, $N = 2^{k-q} + 2k + n_c$.

Fundamental Algorithm for Modified Central Composite Design

In constructing modified central composite designs with fractional factorial replicates, we rely on the procedure of Iwundu (2018) whose non-standard second-order model is formed using the Hierarchical method of Borkowski and Valeroso (1997). The standard central composite design is employed with fractional factorial replicates at the factorial portion. The diagonal elements of the hat matrix are $d_{v1}, d_{v2}, \ldots, d_{v2^k}$ for the factorial portion, $d_{\alpha 1}, d_{\alpha 2}, \ldots, d_{\alpha 2^k}$ for the axial portion and $d_{c1}, d_{c2}, \ldots, d_{cn_c}$ for the center portion. We search out for locations of the design points in each standard CCD portion where the uniqueness of diagonal elements has been lost. The design points in each portion of the standard CCD having the least diagonal elements are deleted from the CCD. Results show that the efficiency of the Modified Central Composite Design (MCCD) exceeds that of the classical or standard Central Composite Design (CCD).

The full second-order model in five design variables is given in expanded form as

$$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + \beta_4 x_4 + \beta_5 x_5 + \beta_{12} x_1 x_2 + \beta_{13} x_1 x_3 + \beta_{14} x_1 x_4 + \beta_{15} x_1 x_5 + \beta_{23} x_2 x_3 + \beta_{24} x_2 x_4 + \beta_{25} x_2 x_5 + \beta_{34} x_3 x_4 + \beta_{35} x_3 x_5 + \beta_{45} x_4 x_5 + \beta_{11} x_1^2 + \beta_{22} x_2^2 + \beta_{33} x_3^2 + \beta_{44} x_4^2 + \beta_{55} x_5^2 + \varepsilon \quad (6)$$

Following the hierarchical formation of non-standard models, the cases considered in this research involve the absence of one, two or three quadratic terms in the model.

The model when x_1^2 is absent is given as

$$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + \beta_4 x_4 + \beta_5 x_5 + \beta_{12} x_1 x_2 + \beta_{13} x_1 x_3 + \beta_{14} x_1 x_4 + \beta_{15} x_1 x_5 + \beta_{23} x_2 x_3 + \beta_{24} x_2 x_4 + \beta_{25} x_2 x_5 + \beta_{34} x_3 x_4 + \beta_{35} x_3 x_5 + \beta_{45} x_4 x_5 + \beta_{22} x_2^2 + \beta_{33} x_3^2 + \beta_{44} x_4^2 + \beta_{55} x_5^2 + \varepsilon \quad (7)$$

The model when x_1^2 and x_2^2 are absent is given as

$$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + \beta_4 x_4 + \beta_5 x_5 + \beta_{12} x_1 x_2 + \beta_{13} x_1 x_3 + \beta_{14} x_1 x_4 + \beta_{15} x_1 x_5 + \beta_{23} x_2 x_3 + \beta_{24} x_2 x_4 + \beta_{25} x_2 x_5 + \beta_{34} x_3 x_4 + \beta_{35} x_3 x_5 + \beta_{45} x_4 x_5 + \beta_{33} x_3^2 + \beta_{44} x_4^2 + \beta_{55} x_5^2 + \varepsilon \quad (8)$$

The model when x_1^2, x_2^2 and x_3^2 are absent is given as

$$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + \beta_4 x_4 + \beta_5 x_5 + \beta_{12} x_1 x_2 + \beta_{13} x_1 x_3 + \beta_{14} x_1 x_4 + \beta_{15} x_1 x_5 + \beta_{23} x_2 x_3 + \beta_{24} x_2 x_4 + \beta_{25} x_2 x_5 + \beta_{34} x_3 x_4 + \beta_{35} x_3 x_5 + \beta_{45} x_4 x_5 + \beta_{44} x_4^2 + \beta_{55} x_5^2 + \varepsilon \quad (9)$$

Two optimality criteria are employed in this research namely D-Optimality criterion and G-Optimality criterion. D-optimality criterion, which seeks to maximize the determinant of the moment matrix, has been so well studied as evident in prior and current researches such as in Kiefer and Wolfowitz (1959), Fedorov (1972), Atkinson and Donev (1992), Eze and Ngonadi (2018). Very recently, Nwanya *et al.* (2020) considered the performance of D-, G- and A-optimality criteria for reduced second-order models having no quadratic and no interaction terms for five variations of Central Composite Design. Wanyonyi *et al.* (2021) explored D-, A-, I-, and G- optimality criteria and their efficiency in determining an optimal split-plot design in mixture modelling. Jaja etal. (2021a) considered A-efficiency (a criterion closely related to D-efficiency) as a measure of the performance of CCDs and MCCDs constrained by missing observations for non-standard models. Jaja etal. (2021b) studied the robust properties of the designs in Jaja etal. (2021a). Iwundu and Oko (2021) utilized A-, D- and G-optimality criteria in studying the efficiency and optimal properties of four varieties of replicated Central Composite Design with full factorial portions. The efficiency of an experimental design can be quantified in interpretable form using one or more efficiency criteria. As in Goos and Jones (2011), D-efficiency of a design compares the determinant of the information matrix of that design to an "ideal" determinant associated with an orthogonal design.

D-efficiency of a design is mathematically given as

$$\text{D-efficiency} = \left(\frac{|X'X|}{N^p}\right)^{\frac{1}{p}} = \frac{|X'X|^{\frac{1}{p}}}{N} \quad (10)$$

where p is the number of parameters of the model.

G-Optimality criterion is concerned with designs whose scaled prediction variances have good prediction at a particular location in the design space. G-optimality criterion is symbolically written as

$$\text{minmax}\{Nf'(x)(X'X)^{-1}\ f(x)\} = \text{minmax}\{Nvar[\hat{y}(x)]\} \quad (11)$$

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G-efficiency, which has gained competitive usage with D-efficiency, is defined as

$$\text{G-efficiency} = \frac{p}{V(\underline{x})_{\max}} \qquad (12)$$

where p is the number of parameters in the model and $V(\underline{x})_{\max}$ is the maximum scaled variance of prediction.

Construction of MCCD in five design variables

Consider the construction of modified central composite design for non-standard model having the absence of one quadratic term, x_1^2, of the full second-order model. The factorial portion of the CCD shall consist of 2^{5-1} half fractional factorial of Resolution V with the Defining Relation $I = +ABCDE$.

The 16 factorial points to include in the design are as follows;

-1	-1	-1	-1	1
1	-1	-1	-1	-1
-1	1	-1	-1	-1
1	1	-1	-1	1
-1	-1	1	-1	-1
1	-1	1	-1	1
-1	1	1	-1	1
1	1	1	-1	-1
-1	-1	-1	1	-1
1	-1	-1	1	1
-1	1	-1	1	1
1	1	-1	1	-1
-1	-1	1	1	1
1	-1	1	1	-1
-1	1	1	1	-1
1	1	1	1	1

The axial portion of the CCD comprises 10 axial points; $(\pm\alpha,\ 0,\ 0,\ 0,\ 0)$, $(0,\ \pm\alpha,\ 0,\ 0,\ 0)$, $(0,\ 0,\ \pm\alpha,\ 0,\ 0)$, $(0,\ 0,\ 0,\ \pm\alpha,\ 0)$, $(0,\ 0,\ 0,\ 0,\ \pm\alpha)$
where $\alpha = 2$. The center portion shall comprise n_c= 3 center runs defined by $(0,\ 0,\ 0,\ 0,\ 0)$, $(0,\ 0,\ 0,\ 0,\ 0)$, $(0,\ 0,\ 0,\ 0,\ 0)$

With $\alpha = 2$ and *N*=29 design points associated with the one-half fractional factorial runs, 10 axial runs and n_c= 3 center runs yields the design matrix

$X =$

1	-1	-1	-1	-1	1	1	1	1	-1	1	1	-1	1	-1	-1	1	1	1	1
1	1	-1	-1	-1	-1	-1	-1	-1	-1	1	1	1	1	1	1	1	1	1	1
1	-1	1	-1	-1	-1	-1	1	1	1	-1	-1	-1	1	1	1	1	1	1	1
1	1	1	-1	-1	1	1	-1	-1	1	-1	-1	1	1	-1	-1	1	1	1	1
1	-1	-1	1	-1	-1	1	-1	1	1	-1	1	1	-1	-1	1	1	1	1	1

1	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	-1	1	-1	1	1	1	1
1	-1	1	1	-1	1	-1	-1	1	-1	1	-1	1	-1	1	-1	1	1	1	1
1	1	1	1	-1	-1	1	1	-1	-1	1	-1	-1	-1	-1	1	1	1	1	1
1	-1	-1	-1	1	-1	1	1	-1	1	1	-1	1	-1	1	-1	1	1	1	1
1	1	-1	-1	1	1	-1	-1	1	1	1	-1	-1	-1	-1	1	1	1	1	1
1	-1	1	-1	1	1	-1	1	-1	-1	-1	1	1	-1	-1	1	1	1	1	1
1	1	1	-1	1	-1	1	-1	1	-1	-1	1	-1	-1	1	-1	1	1	1	1
1	-1	-1	1	1	1	1	-1	-1	-1	-1	-1	-1	1	1	1	1	1	1	1
1	1	-1	1	1	-1	-1	1	1	-1	-1	-1	1	1	-1	-1	1	1	1	1
1	-1	1	1	1	-1	-1	-1	-1	1	1	1	-1	1	-1	-1	1	1	1	1
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
1	-2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	-2	0	0	0	0	0	0	0	0	0	0	0	0	0	4	0	0	0
1	0	2	0	0	0	0	0	0	0	0	0	0	0	0	0	4	0	0	0
1	0	0	-2	0	0	0	0	0	0	0	0	0	0	0	0	0	4	0	0
1	0	0	2	0	0	0	0	0	0	0	0	0	0	0	0	0	4	0	0
1	0	0	0	-2	0	0	0	0	0	0	0	0	0	0	0	0	0	4	0
1	0	0	0	2	0	0	0	0	0	0	0	0	0	0	0	0	0	4	0
1	0	0	0	0	-2	0	0	0	0	0	0	0	0	0	0	0	0	0	4
1	0	0	0	0	2	0	0	0	0	0	0	0	0	0	0	0	0	0	4
1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

The design points and the associated diagonal elements of the hat matrix are presented in Table 1. The diagonal entries associated with the factorial portion are a constant $d_v = 0.8750$, the axial portion has a maximum diagonal value $d_{\alpha max} = 0.5883$ at eight points and a minimum diagonal element $d_{\alpha min} = 0.3667$ at two points. Associated with the 3 centre points is a unique diagonal element $d_c = 0.2000$. The points $(2,0,0,0,0)$ and $(-2,0,0,0,0)$ are dropped from the design and a modified design is formed.

The design points and the associated diagonal elements of the hat matrix for the Modified CCD are presented in Table 2. Table 3 gives the scaled prediction variances associated with standard CCD and modified CCD for five-variable non-standard model having one quadratic term of the model is removed. Table 4 gives the Summary Statistics on design efficiency when one quadratic term is removed from the full model. Table 5 gives the Summary Statistics on design efficiency when two quadratic terms are removed from the full model. Table 6 gives the Summary Statistics on design efficiency when three quadratic terms are removed from the full model.

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Table 1: Design points of 5-variable standard CCD and the h_{ii} values associated with the non-standard model having x_1^2 term absent.

Design point					h_{ii} value
-1	-1	-1	-1	1	0.8750
1	-1	-1	-1	-1	0.8750
-1	1	-1	-1	-1	0.8750
1	1	-1	-1	1	0.8750
-1	-1	1	-1	-1	0.8750
1	-1	1	-1	1	0.8750
-1	1	1	-1	1	0.8750
1	1	1	-1	-1	0.8750
-1	-1	-1	1	-1	0.8750
1	-1	-1	1	1	0.8750
-1	1	-1	1	1	0.8750
1	1	-1	1	-1	0.8750
-1	-1	1	1	1	0.8750
1	-1	1	1	-1	0.8750
-1	1	1	1	-1	0.8750
1	1	1	1	1	0.8750
-2	0	0	0	0	0.3667
2	0	0	0	0	0.3667
0	-2	0	0	0	0.5883
0	2	0	0	0	0.5883
0	0	-2	0	0	0.5883
0	0	2	0	0	0.5883
0	0	0	-2	0	0.5883
0	0	0	2	0	0.5883
0	0	0	0	-2	0.5883
0	0	0	0	2	0.5883
0	0	0	0	0	0.2000
0	0	0	0	0	0.2000
0	0	0	0	0	0.2000

Table 2: Design points of 5-variable MCCD and associated h_{ii} values in the absence of x_1^2 term

Design point					h_{ii} value
-1	-1	-1	-1	1	0.8958
1	-1	-1	-1	-1	0.8958
-1	1	-1	-1	-1	0.8958
1	1	-1	-1	1	0.8958
-1	-1	1	-1	-1	0.8958
1	-1	1	-1	1	0.8958
-1	1	1	-1	1	0.8958
1	1	1	-1	-1	0.8958

-1	-1	-1	1	-1	0.8958
1	-1	-1	1	1	0.8958
-1	1	-1	1	1	0.8958
1	1	-1	1	-1	0.8958
-1	-1	1	1	1	0.8958
1	-1	1	1	-1	0.8958
-1	1	1	1	-1	0.8958
1	1	1	1	1	0.8958
0	-2	0	0	0	0.5883
0	2	0	0	0	0.5883
0	0	-2	0	0	0.5883
0	0	2	0	0	0.5883
0	0	0	-2	0	0.5883
0	0	0	2	0	0.5883
0	0	0	0	-2	0.5883
0	0	0	0	2	0.5883
0	0	0	0	0	0.3333
0	0	0	0	0	0.3333
0	0	0	0	0	0.3333

Table 3: Scaled prediction variances for CCD and MCCD in absence of one quadratic model term

Design points					Scaled prediction variance for standard CCD	Scaled prediction variance for Modified CCD
-1	-1	-1	-1	1	25.3750	24.1875
1	-1	-1	-1	-1	25.3750	24.1875
-1	1	-1	-1	-1	25.3750	24.1875
1	1	-1	-1	1	25.3750	24.1875
-1	-1	1	-1	-1	25.3750	24.1875
1	-1	1	-1	1	25.3750	24.1875
-1	1	1	-1	1	25.3750	24.1875
1	1	1	-1	-1	25.3750	24.1875
-1	-1	-1	1	-1	25.3750	24.1875
1	-1	-1	1	1	25.3750	24.1875
-1	1	-1	1	1	25.3750	24.1875
1	1	-1	1	-1	25.3750	24.1875
-1	-1	1	1	1	25.3750	24.1875
1	-1	1	1	-1	25.3750	24.1875
-1	1	1	1	-1	25.3750	24.1875
1	1	1	1	1	25.3750	24.1875
-2	0	0	0	0	16.9167	-
2	0	0	0	0	16.9167	-
0	-2	0	0	0	16.9167	15.7500

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0	2	0	0	0	16.9167	15.7500
0	0	-2	0	0	16.9167	15.7500
0	0	2	0	0	16.9167	15.7500
0	0	0	-2	0	16.9167	15.7500
0	0	0	2	0	16.9167	15.7500
0	0	0	0	-2	16.9167	15.7500
0	0	0	0	2	16.9167	15.7500
0	0	0	0	0	5.8000	9.000
0	0	0	0	0	5.8000	9.000
0	0	0	0	0	5.8000	9.000

Table 4: Summary Statistics when one quadratic term is removed; $k = 5, n_c = 3, \alpha = 2$

Design Type	Missing Coefficients	$\det\left(\frac{1}{N}(X'X)\right)$	D-efficiency %	Max SPV	G-efficiency %
Standard CCD on Reduced Model N=29 and p=20	x_1^2	7.7801e – 004	69.91	25.3750	78.82
	x_2^2	7.7801e – 004	69.91	25.3750	78.82
	x_3^2	7.7801e – 004	69.91	25.3750	78.82
	x_4^2	7.7801e – 004	69.91	25.3750	78.82
	x_5^2	7.7801e – 004	69.91	25.3750	78.82
MCCD on reduced model when N=27 and p=20	x_1^2	0.0013	71.73	24.1875	82.69
	x_2^2	0.0013	71.73	24.1875	82.69
	x_3^2	0.0013	71.73	24.1875	82.69
	x_4^2	0.0013	71.73	24.1875	82.69
	x_5^2	0.0013	71.73	24.1875	82.69

Table 5: Summary Statistics when two quadratic terms are removed; $k = 5, n_c = 3, \alpha = 2$

Design Type	Missing Coefficients	$\det\left(\frac{1}{N}(X'X)\right)$	D-efficiency %	Max SPV	G-efficiency %
Full CCD on reduced model with $N = 29$ and $p = 19$	x_1^2, x_2^2	8.6959e – 004	69.01	25.2934	75.12
	x_1^2, x_3^2	8.6959e – 004	69.01	25.2934	75.12
	x_1^2, x_4^2	8.6959e – 004	69.01	25.2934	75.12
	x_1^2, x_5^2	8.6959e – 004	69.01	25.2934	75.12
	x_2^2, x_3^2	8.6959e – 004	69.01	25.2934	75.12
	x_2^2, x_4^2	8.6959e – 004	69.01	25.2934	75.12
	x_2^2, x_5^2	8.6959e – 004	69.01	25.2934	75.12
	x_3^2, x_4^2	8.6959e – 004	69.01	25.2934	75.12
	x_3^2, x_5^2	8.6959e – 004	69.01	25.2934	75.12
	x_4^2, x_5^2	8.6959e – 004	69.01	25.2934	75.12
Modified CCD	x_1^2, x_2^2	0.0030	73.66	22.8860	83.02
	x_1^2, x_3^2	0.0030	73.66	22.8860	83.02

on reduced model when $N = 25$ and $p = 19$	x_1^2, x_4^2	0.0030	73.66	22.8860	83.02
	x_1^2, x_5^2	0.0030	73.66	22.8860	83.02
	x_2^2, x_3^2	0.0030	73.66	22.8860	83.02
	x_2^2, x_4^2	0.0030	73.66	22.8860	83.02
	x_2^2, x_5^2	0.0030	73.66	22.8860	83.02
	x_3^2, x_4^2	0.0030	73.66	22.8860	83.02
	x_3^2, x_5^2	0.0030	73.66	22.8860	83.02
	x_4^2, x_5^2	0.0030	73.66	22.8860	83.02

Table 6: Summary Statistics when three quadratic terms are removed; $k = 5,\ n_c 3, \alpha = 2$

Design Type	Missing Coefficients	$\det\left(\frac{1}{N}(X'X)\right)$	D – Efficiency %	Max SPV	G- efficiency %
Full CCD On Reduced Model with N = 29 and p = 18	x_1^2, x_2^2, x_3^2	9.3716e – 004	67.88	25.2377	71.32
	x_1^2, x_2^2, x_4^2	9.3716e – 004	67.88	25.2377	71.32
	x_1^2, x_2^2, x_5^2	9.3716e – 004	67.88	25.2377	71.32
	x_1^2, x_3^2, x_4^2	9.3716e – 004	67.88	25.2377	71.32
	x_1^2, x_3^2, x_5^2	9.3716e – 004	67.88	25.2377	71.32
	x_1^2, x_4^2, x_5^2	9.3716e – 004	67.88	25.2377	71.32
	x_2^2, x_3^2, x_4^2	9.3716e – 004	67.88	25.2377	71.32
	x_2^2, x_3^2, x_5^2	9.3716e – 004	67.88	25.2377	71.32
	x_2^2, x_4^2, x_5^2	9.3716e – 004	67.88	25.2377	71.32
	x_3^2, x_4^2, x_5^2	9.3716e – 004	67.88	25.2377	71.32
MCCD On Reduced Model when N = 23 and p = 18	x_1^2, x_2^2, x_3^2	0.0082	76.58	21.6104	83.29
	x_1^2, x_2^2, x_4^2	0.0082	76.58	21.6104	83.29
	x_1^2, x_2^2, x_5^2	0.0082	76.58	21.6104	83.29
	x_1^2, x_3^2, x_4^2	0.0082	76.58	21.6104	83.29
	x_1^2, x_3^2, x_5^2	0.0082	76.58	21.6104	83.29
	x_1^2, x_4^2, x_5^2	0.0082	76.58	21.6104	83.29
	x_2^2, x_3^2, x_4^2	0.0082	76.58	21.6104	83.29
	x_2^2, x_3^2, x_5^2	0.0082	76.58	21.6104	83.29
	x_2^2, x_4^2, x_5^2	0.0082	76.58	21.6104	83.29
	x_3^2, x_4^2, x_5^2	0.0082	76.58	21.6104	83.29

DISCUSSION OF RESULTS

In comparing design's efficiencies for standard CCD and modified CCD on non-standard model having one missing quadratic term, the determinant value of the normalized information matrix associated with the modified central composite designs are larger than those associated with the standard central composite design. Consequently, D-efficiency values associated with the modified central composite designs are higher than those associated with the standard central composite designs. Specifically, the D-efficiency value associated with the standard CCD is 69.91% and the D-efficiency value associated with the Modified CCD is

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71.73%. The modified CCD also performs better than the standard CCD in term of G-optimality and efficiency. Specifically, G-efficiency associated with the standard CCD is 78.82% and that associated with the modified CCD is 82.69%.

For two missing quadratic terms, the D-efficiency values associated with the modified central composite designs are higher than those associated with the standard central composite designs. Specifically, the D-efficiency value associated with the standard CCD is 69.01% and the D-efficiency value associated with the Modified CCD is 73.66%. The maximum scaled prediction variance for standard CCD gives G-efficiency value of 75.12% and that associated with the Modified CCD gives G-efficiency value of 83.02%. Similarly, for three missing quadratic terms, the D-efficiency values associated with the modified central composite designs are higher than those associated with the standard central composite designs. In each case of absence of three quadratic terms, the D-efficiency value associated with the standard CCD is 67.88% and the D-efficiency value associated with the Modified CCD is 76.58%. As in one and two missing quadratic model terms, G-efficiency is higher for the Modified CCD. Specifically, G-efficiency for standard CCD is 71.32% while G-efficiency for modified CCD is 83.29%.

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