*Scientia Africana, Vol. 21 (No. 1), April, 2022. Pp 27-36* https://dx.doi.org/10.4314/sa.v21i1.3
 *ISSN 1118 – 1931*

# A REVISED MATHEMATICAL SOLUTION TO THE PENNE BIOHEAT EQUATION FOR DEEP ROOTED TISSUES USING THE VARIABLE SEPARATION METHOD

**[1]Akpolile, A.F. [1]Mokobia, C.E. [2]Ikubor, J.E. [3]Ugbede, F.O. and [1] Agbajor, G.K.**
[1] Department of Physics, Delta State University, P.M.B 1, Abraka.
[2] Department of Radiology, Delta State University Teaching Hospital, Oghara.
[3] Department of Physics with Electronics, Evangel University, Akaeze, Ebonyi State, Nigeria.
**Corresponding author email:** anita.franklin@yahoo.com

*Received: 10-02-2022*
*Accepted: 24-03-2022*

## ABSTRACT

*The temperature distribution of tissues under the skin surface was obtained using Penne's bio-heat equation and the separation of variables technique. The resulting solution indicated a slight temperature difference between the recorded skin temperature and the temperature of the tissue of interest (TOI), which is a direct indication of the TOI's temperature. According to the graphical analysis, temperature is proportionally related to time and tissue distance from the skin surface. As a result, the given analytic solution may be utilized to easily analyze deep-rooted tissues while accounting for thermal properties.*

**Keywords:** Tissues, temperature, heat transfer, bio-heat.

## INTRODUCTION

The spatial distribution of temperature and heat transmission in tissues is critical in thermoregulation and many physiological processes in living creatures since there is a link between disease and temperature. The disease condition causes an elevation in temperature as a result of enhanced vascularity, hormonal activity, and metabolic activity. Researchers have created many bio-heat transfer models to estimate the temperature distribution in biological live tissues and achieve precise responses (Kabiri and Talaee, 2021). These models are aimed at improving temperature regulation during hypothermia and hyperthermia, predicting temperature distributions in core organs where direct temperature measurement is intrusive, and determining changes in organ temperature as a result of changes in bodily activities. Because of its simplicity, Penne's bio-heat equation is one of the most well-known models for studying heat exchange in tissues. Heat transport is aided by thermal conduction, convection, blood perfusion, and metabolic heat production in tissue (Ciesielski and Mochnacki, 2012). When the temperature of the blood differs from the temperature of the tissue through which it flows, convective heat transmission occurs, affecting the temperatures of both the blood and the tissue. Many physiological processes rely on perfusion-based heat transport interactions, such as thermoregulation and inflammation. Several factors influence the blood/tissue thermal interaction, including perfusion rate and vascular anatomy, which vary substantially between tissues, organs, and diseases. Furthermore, using the traditional Fourier's rule of heat conduction, the Pennes bioheat equation may be used to define the energy conservation equation for biological heat transfer (Lillicrap et al., 2017). In fact, key bioheat transfer results have been published in recent decades such as Pennes bioheat transfer equation (Pennes, 1948) and further microstructure bioheat transfer models (Weinbaum et al.,1984; Chen and Holmes, 1980). Modeling heat-related processes such as bioheat transfer and heat-induced stress aid in the development of biological and biomedical technologies such as thermotherapy, hyperthermia cancer treatment, thermal diagnostics, cryogenic surgery, and other uses (Hassain and Mohammadi, 2013). Quantitative and exact measurements of bioheat transfer are essential to completely know and anticipate the biological system's heat transfer mechanism. The intricacy of the accurate thermal analysis process of living tissues is due to its heterogeneity and anisotropy, as well as conduction, convection, and radiation heat flow, cell metabolism, and blood perfusion, among other factors. Because of its heterogeneity and anisotropy, as well as conduction, convection, and radiation heat flow, cell metabolism, and blood perfusion, among other aspects, the precise thermal analysis procedure of living tissues is complicated. As a result, building precise thermal models is highly challenging, and the majority of the suggested bio-heat equations are extremely complex, albeit not impossible to solve analytically. The analytical solutions to these equations are important in bio-heat transfer research because they reflect the equations' genuine physical properties and can be used to evaluate numerical results and establish the correctness of the in-vitro model analysis. In the existing literature, several methods for deriving analytical solutions to these equations have been provided (Al-Humedia and Al- Saadawi, 2021; Damor et al., 2015;Grysa and Marciag,2019;Sakar et al.,2015;Zhou and Chen,2009; Zhou and Chen,2007;Tsu-Ching et al.,2007;Gutierrez,2007;Romero et al.,2009;Damor et al.,2013; Roca Oriaa et al.,2019). The general heat equation for conduction with extra elements for heat sources is the linear bio-heat transfer equation for tissue (Lakhssassi et al.,2010). The primary goal of this study is to develop an analytical solution for the temperatures of tissues deep beneath the skin utilizing the well-known bio-heat equation while taking into account the thermal properties of the tissues such as density, conductivity, and thermal diffusivity.

**Mathematical Conceptualization**

Penne's bio-heat equation is given by (Akpolile et al., 2021);

$$\rho c\frac{\partial T}{\partial t} = \nabla(k\nabla T) + Q + w_b c_b(T_b - T) \quad (1)$$

where ρ, c and *k* are the density (kg/m$^3$ ), the specific heat (J/kg.K) and the thermal conductivity (W/m.K) of the tissue respectively. $w_b$ is the mass flow rate of blood per unit volume of tissue (kg/(s.m$^3$)), $C_b$ is the specific heat of the blood ; Q is the metabolic heat produced per unit volume from the tissue (W/m$^3$); $T_b$ denotes the temperature of the arterial blood (K); T represents the tissue temperature,$\partial T/\partial t$ denotes the rate of temperature rise.

*Scientia Africana, Vol. 21 (No. 1), April, 2022. Pp 27-36* https://dx.doi.org/10.4314/sa.v21i1.3
*© Faculty of Science, University of Port Harcourt, Printed in Nigeria* *ISSN 1118 – 1931*

The boundary conditions is described as follows;

$$\left.\begin{array}{ll}\text{(i)} & T(x,t)|_{t=0} = f(x) \text{ , where } f(x) \text{ is a characteristics of distance.} \\ \text{(ii)} & T(x,t)|_{x=0.001} = T_b + Q_0 \\ \text{(iii)} & \theta(x,t)|_{x=0.001} = 0 \\ \text{(iv)} & \theta(x,t)|_{t=0} = f(x)\end{array}\right\} \quad (2)$$

For a small volume of human tissue, assuming a one-dimensional situation, the equation 1 above can be expressed as;;

$$\rho c \frac{\partial T}{\partial t} = k \frac{\partial^2 T}{\partial x^2} + Q + w_b c_b (T_b - T) \quad (3)$$

For easy computation, the following variables are defined as (Akpolile et al., 2021);

(i) $Q = B_b Q_o$ (4)

(ii) $B_b = w_b C_b$ (5)

(iii) $T - T_b - Q_o = \emptyset$ (6)

(iv) $Q_o$ is a constant (7)

The equation 3 can be rewritten as;

$$\rho c \frac{\partial T}{\partial t} = k \frac{\partial^2 T}{\partial x^2} + B_b Q_o + B_b (T_b - T) \quad (8)$$

$$\rho c \frac{\partial T}{\partial t} = k \frac{\partial^2 T}{\partial x^2} + B_b Q_o + B_b T_b - B_b T \quad (9)$$

$$\rho c \frac{\partial T}{\partial t} = k \frac{\partial^2 T}{\partial x^2} - B_b T + B_b Q_o + B_b T_b \quad (10)$$

$$\rho c \frac{\partial T}{\partial t} = k \frac{\partial^2 T}{\partial x^2} - B_b (T - T_b - Q_o) \quad (11)$$

Using equation 6;

$$\rho c \frac{\partial \emptyset}{\partial t} = k \frac{\partial^2 \emptyset}{\partial x^2} - B_b \emptyset \quad (12)$$

$$\rho c \frac{\partial \emptyset}{\partial t} - k \frac{\partial^2 \emptyset}{\partial x^2} = -B_b \emptyset \quad (13)$$

Using method of separation of variables;

$$\emptyset(x,t) = X(x) T(t) \quad (14)$$

Differentiating equation 14 with respect to $x$ and t gives equations (15 – 17);

$$\emptyset_t = X(x) T'(t) \quad (15)$$

$$\emptyset_x = X'(x)\, T(t) \quad (16)$$

$$\emptyset_{xx} = X''(x)\, T(t) \quad (17)$$

The homogenous part of equation 13 is

$$\rho c \frac{\partial \emptyset}{\partial t} - k \frac{\partial^2 \emptyset}{\partial x^2} = 0 \quad (18)$$

Substituting equations (15 – 17) into 18; it will yield,

$$\rho c(X(x)T'(t)) - k(X''(x)\,T(t)) = 0 \qquad (19)$$

$$\rho c(X(x)T'(t)) = k(X''(x)\,T(t)) \qquad (20)$$

From equation 20, collect like terms;

$$\frac{\rho c\,(T'(t))}{T(t)} = \frac{k(X''(x))}{X(x)} \qquad (21)$$

Assume that equation 21 is equivalent to a constant, which we will call $-\lambda$.

$$\frac{\rho c\,(T'(t))}{T(t)} = \frac{k(X''(x))}{X(x)} = -\lambda \qquad (22)$$

The equation 22 implies that:

$$\frac{\rho c\,(T'(t))}{T(t)} = -\lambda \qquad (23)$$

$$\frac{k(X''(x))}{X(x)} = -\lambda \qquad (24)$$

Rearranging equation 23, we obtain:

$$\frac{T'(t)}{T(t)} = \frac{-\lambda}{\rho c} \qquad (25)$$

By integrating equation25, we obtain:

$$\int \frac{T'(t)}{T(t)} = \int \frac{-\lambda}{\rho c}\partial t \qquad (26)$$

$$\ln T(t) = \frac{-\lambda}{\rho c}t + C \qquad (27)$$

where C is the integration constant. Taking the natural logarithm of both sides of equation 27 will give;

$$T(t) = e^{\frac{-\lambda}{\rho c}t + C} = e^{\frac{-\lambda}{\rho c}t}.\,e^{C}$$

$$T(t) = A_0\, e^{\frac{-\lambda}{\rho c}t} \qquad (28)$$

where $A_0$ is a constant. Considering equation 24;

$$\frac{X''(x)}{X(x)} = \frac{-\lambda}{k} \qquad (29)$$

$$X''(x) = \frac{-\lambda}{k}X(x) \qquad (30)$$

$$X''(x) + \frac{\lambda}{k}X(x) = 0 \qquad (31)$$

Let $X$ = r; then equation 31 will be

$$r^2 + \frac{\lambda}{k}r = 0 \qquad (32)$$

$$r\left(r + \frac{\lambda}{k}\right) = 0 \qquad (33)$$

$$r_1 = 0 \text{ or } r_2 = -\frac{\lambda}{k} \qquad (34)$$

Substituting the roots of the equation, we have;

$$X = ae^{r_1 x} + be^{r_2 x} = ae^{0.x} + be^{-\frac{\lambda}{k}x}$$

$$X = a + be^{-\frac{\lambda}{k}x} \qquad (35)$$

where a and b are arbitrary constants.

*Scientia Africanа, Vol. 21 (No. 1), April, 2022. Pp 27-36* https://dx.doi.org/10.4314/sa.v21i1.3
 *ISSN 1118 – 1931*

$$\emptyset(x,t) = \left(a + be^{-\frac{\lambda}{k}x}\right) A_0\, e^{\frac{-\lambda}{\rho c}t} \quad (36)$$

The equation 36 is the homogenous solution. Then the particular solution for $\emptyset(x,t)$ is;

$$\emptyset(x,t) = -P_0 B_b e^{-\frac{\lambda}{k}x} \quad (37)$$

where $P_0$ is a constant. The general solution for $\emptyset(x,t)$ is;

$$\emptyset(x,t) = \left(a + be^{-\frac{\lambda}{k}x}\right) A_0\, e^{\frac{-\lambda}{\rho c}t} - P_0 B_b e^{-\frac{\lambda}{k}x} (38)$$

From equation 6, we have;

$$T = \emptyset + T_b + Q_o \quad (39)$$

Applying equation 4, we have;

$$T(x,t) = \emptyset(x,t) + T_b + \frac{Q}{B_b}$$

$$T(x,t) = \left(a + be^{-\frac{\lambda}{k}x}\right) A_0\, e^{\frac{-\lambda}{\rho c}t} - P_0 B_b e^{-\frac{\lambda}{k}x} + T_b + \frac{Q}{B_b}$$

Applying equation 5, we have;

$$T(x,t) = \left(a + be^{-\frac{\lambda}{k}x}\right) A_0\, e^{\frac{-\lambda}{\rho c}t} - P_0 w_b c_b e^{-\frac{\lambda}{k}x} + T_b + \frac{Q}{w_b c_b} \quad (40)$$

where a, b, $P_0$ and $A_0$ are arbitrary constants which can bo obtained by applying the boundary conditions.

Applying equation 2 (iii) to equation 40, we have;

$$\left(a + be^{-\frac{0.001\lambda}{k}}\right) A_0\, e^{\frac{-\lambda}{\rho c}t} = 0 \quad (41)$$

$$\Rightarrow A_0\, e^{\frac{-\lambda}{\rho c}t} \neq 0$$

$$a + be^{-\frac{0.001\lambda}{k}} = 0$$

$$a = -be^{-\frac{0.001\lambda}{k}} \quad (42)$$

Applying equation 2 (iv) to equation 50, we have;

$$\left(a + be^{-\frac{\lambda}{k}x}\right) A_0\, e^{\frac{-\lambda}{\rho c}.0} = f(x) \quad (43)$$

$$\left(a + be^{-\frac{\lambda}{k}x}\right) A_0 = f(x)$$

$$\Rightarrow A_0 = f(x) \quad (44)$$

$$a + be^{-\frac{\lambda}{k}x} = f(x) \quad (45)$$

Substituting equation 42 into 45, it becomes;

$$-be^{-\frac{0.001\lambda}{k}} + be^{-\frac{\lambda}{k}x} = f(x)$$

$$be^{-\frac{\lambda}{k}x} - be^{-\frac{0.001\lambda}{k}} = f(x)$$

$$b\left(e^{-\frac{\lambda}{k}x} - e^{-\frac{0.001\lambda}{k}}\right) = f(x) \quad (46)$$

Taking the Log of both sides of equation 46;

$$\log b\left(e^{-\frac{\lambda}{k}x} - e^{-\frac{0.001\lambda}{k}}\right) = \log f(x)$$

$$\text{b}\log e^{-\frac{\lambda}{k}x} - \text{b}\log e^{-\frac{0.001\lambda}{k}} = \log f(x)$$

$$\text{Fb.} -\frac{\lambda}{k}x\log e + \text{b}\,.\frac{0.001\lambda}{k}\log e = \log f(x)$$

$$\text{b}\left(-\frac{\lambda}{k}x + \frac{0.001\lambda}{k}\right) = \log f(x)$$

$$\frac{\lambda\text{b}}{k}(-x + 0.001) = \log f(x)$$

$$\lambda\text{b}(0.001 - x) = k\log f(x)$$

$$\text{b} = \frac{k\log f(x)}{\lambda(0.001 - x)} \qquad (47)$$

substituting equation 47 into equation 42

$$a = -\frac{k\log f(x)}{\lambda(0.001 - x)}e^{-\frac{0.001\lambda}{k}} \qquad (48)$$

Applying equation 2(ii) to equation 44; we have:

$$T(x,t)|_{x=0.001} = T_b + Q_0 = \left(a + be^{-0.001\frac{\lambda}{k}}\right)A_0\, e^{\frac{-\lambda}{\rho c}t} - P_0 w_b c_b e^{-0.001\frac{\lambda}{k}} + T_b + \frac{Q}{w_b c_b} \quad (49)$$

Applying equation 5 to equation 49

$$T_b + \frac{Q}{w_b c_b} = A_0\, e^{\frac{-\lambda}{\rho c}t}\left(a + be^{-0.001\frac{\lambda}{k}}\right) - P_0 w_b c_b e^{-0.001\frac{\lambda}{k}} + T_b + \frac{Q}{w_b c_b} \qquad (50)$$

$$A_0\, e^{\frac{-\lambda}{\rho c}t}\left(a + be^{-0.001\frac{\lambda}{k}}\right) - P_0 w_b c_b e^{-0.001\frac{\lambda}{k}} = 0 \qquad (51)$$

$$A_0\, e^{\frac{-\lambda}{\rho c}t}\left(a + be^{-0.001\frac{\lambda}{k}}\right) - P_0 w_b c_b e^{-0.001\frac{\lambda}{k}} \qquad (52)$$

$$\Rightarrow A_0\, e^{\frac{-\lambda}{\rho c}t} = -P_0 w_b c_b e^{-0.001\frac{\lambda}{k}} (53)$$

$$a + be^{-0.001\frac{\lambda}{k}} = -P_0 w_b c_b e^{-0.001\frac{\lambda}{k}} \qquad (54)$$

Considering equation 53;

$$P_0 = \frac{-A_0\, e^{\frac{-\lambda}{\rho c}t}}{w_b c_b e^{-0.001\frac{\lambda}{k}}} \qquad (55)$$

$$P_0 = \frac{-f(x)\, e^{\frac{-\lambda}{\rho c}t}}{w_b c_b e^{-0.001\frac{\lambda}{k}}} \qquad (56)$$

Substituting equations 56, 52, 51, 48 into 44;

$$T(x,t) = f(x)\, e^{\frac{-\lambda}{\rho c}t}\left(\left(-\frac{k\log f(x)}{\lambda(0.001 - x)}e^{-\frac{0.001\lambda}{k}}\right) + \frac{k\log f(x)}{\lambda(0.001 - x)}e^{--\frac{\lambda}{k}x}\right) - \left(\frac{f(x)\, e^{\frac{-\lambda}{\rho c}t}}{w_b c_b e^{-0.001\frac{\lambda}{k}}}w_b c_b e^{-\frac{\lambda}{k}x}\right) + T_b + \frac{Q}{w_b c_b}$$

$$T(x,t) = f(x)\, e^{\frac{-\lambda}{\rho c}t}.\frac{k\log f(x)}{\lambda(0.001 - x)}\left(\left(-e^{-\frac{0.001\lambda}{k}}\right) + e^{--\frac{\lambda}{k}x}\right) - \left(\frac{f(x)\, e^{-\lambda\left(\frac{t}{\rho c} + \frac{x}{k}\right)}}{e^{-0.001\frac{\lambda}{k}}}\right) + T_b + \frac{Q}{w_b c_b}$$

*Scientia Africana, Vol. 21 (No. 1), April, 2022. Pp 27-36* https://dx.doi.org/10.4314/sa.v21i1.3
 *ISSN 1118 – 1931*

$$T(x,t) = \frac{f(x) k \log f(x)}{\lambda(0.001 - x)}\, e^{\frac{-\lambda}{\rho c} t}\left(\left(-e^{-\frac{0.001\lambda}{k}}\right) + e^{--\frac{\lambda}{k}x}\right) - \left(\frac{f(x)\, e^{-\lambda\left(\frac{t}{\rho c}+\frac{x}{k}\right)}}{e^{-0.001\frac{\lambda}{k}}}\right) + T_b + \frac{Q}{w_b c_b} \quad (57)$$

where $f(x)$ is the source term, $T$ is the tissue temperature is (°K), $k$ is the thermal conductivity ($m^2/s$), λ is the tissue thickness (m), $x$ is the tissue's distance from the skin's surface (m), and $t$ is time (s). The equation (57) was used to model the temperature values for deep-seated human tissues applying the maple V18.0 software. The obtained values are presented in Table 1.

**Table 1: Thermal parameters of tissues**

| TISSUE | k(W/m.K) | $\rho_t$(kg/m$^3$) | $C_t$(J/kg/°K) | λ(m) | $x$(m) | Q(W/m$^2$) | $C_b$(J/kg/°K) | $w_b$x10$^{-3}$ (s$^{-1}$) |
|---|---|---|---|---|---|---|---|---|
| LIVER | 0.520 | 1060 | 3600 | 0.12 | 0.02 | 700 | 3770 | 15 |
| KIDNEY | 0.556 | 1060 | 3830 | 0.03 | 0.07 | 900 | 3770 | 61 |

**Source:** (Kabiri and Talaee,2021)

## RESULTS AND DISCUSSION

The temperature of deep-seated tissues could be determined using the defined analytical solution of Penne's bio-heat equation in equation 57. Thermal parameters from known works of literature were used in the computations, as stated in Table 1 above. The fluctuation of temperature (T) with time (t) and distance (x) is illustrated in Figures 1, 2, 3, and 4. They are all linear graphs, indicating that temperature is proportional to time. In the graphs below (1 and 3), an intercept at 311.2503 and 311.74037 on the T axis, respectively, was observed, and also, T increases in a constant proportion to t, but the temperature, T, is the same at all points in Figures 2 and 4 below. Also, from figures 2 and 4, a slight variation was observed for the temperature, suggesting that different tissues have different temperature values which could be attributed to the properties of the tissue under investigation. From the equation under study and the diagrams obtained, it is observed that tissue temperature is dependent on the metabolic heat produced in the tissue, tissue thickness, and distance of the tissue from the skin surface, suggesting that the skin temperature measured using the 19 thermometer placed under the armpit cannot be assumed to be the exact temperature value of the tissue under study, especially in cases of illnesses like cancer. The temperature profile appears to coincide with data from previous works of literature (Damor et al.,2013; Liu and Cheng. 2008), indicating that the same model was used under different situations.

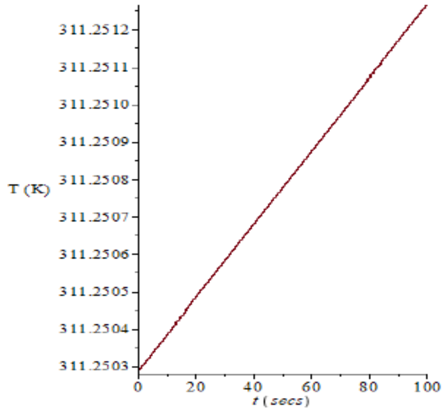


Figure 1: A plot of temperature (T) against time ( *t*) for liver tissue

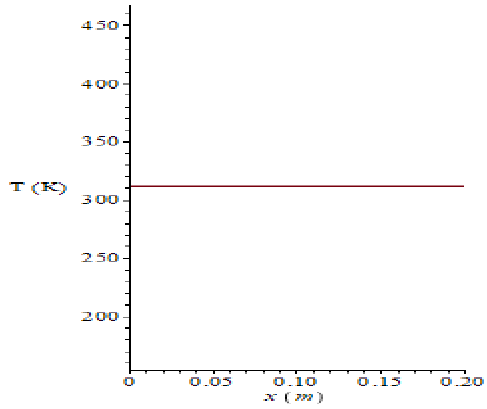


Figure 2: A plot of temperature (T) against distance (*x*) for liver tissue

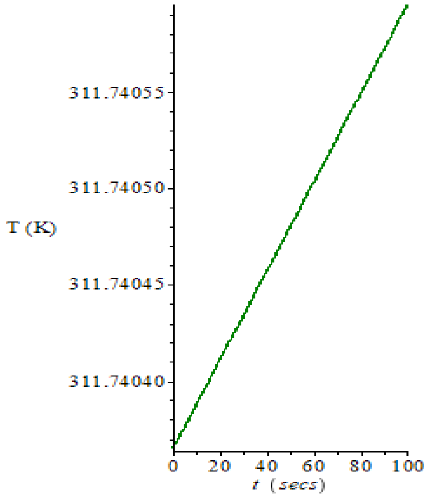


Figure 3: A plot of temperature (T) against time ( *t*) for kidney tissue

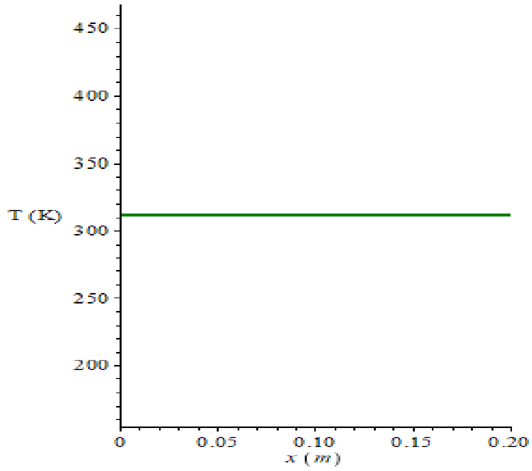


Figure 4: A plot of temperature (T) against distance (*x*) for kidney tissue

## CONCLUSION

The goal of this research is to use the approach of separable variables to solve Penne's bio-heat transfer equation analytically and to apply the above solution in estimating the temperature of deep-seated tissues in the body. Tissue thickness was successfully integrated into the bio-heat transfer model in this work, providing a solution that indicated that the temperature distribution had a proportionate increase in time and distance from the skin. The estimated temperature value for the tissue under examination was found to be slightly different from skin temperature in the theoretical analysis, especially in cases of the disease. The results obtained agreed with other investigations in comparable areas for medical applications such as thermographic imaging of lesions. There has been no mathematical derivation relating to this subject.

## REFERENCES

Akpolile, A.F., Mokobia, C.E. and Ikubor, J.E. (2021). Analytical Approach to the Penne's Bioheat Equation for the Evaluation of Temperature for Deep

*Scientia Africana, Vol. 21 (No. 1), April, 2022. Pp 27-36* https://dx.doi.org/10.4314/sa.v21i1.3
*© Faculty of Science, University of Port Harcourt, Printed in Nigeria* ISSN 1118 – 1931

Seated Tissues. Advances in Mathematics: Scientific Journal , 10(7), pp 2957–2976; ISSN: 1857-8365 (printed); 1857-8438 (electronic) https://doi.org/10.37418/amsj.10.7.4

Al-Humedia, H.O. and Al-Saadawi, F.A. (2021). The numerical solution of bioheat equation based on shifted Legendre polynomial. International Journal of Nonlinear Analysis and Application, 12(2), pp 1061-1070.

Chen, M.M. and Holmes, K.R. (1980). Microvascular Contributions in Tissue Heat Transfer. Annals of New York Academy of Science, 335, pp 137-150.

Ciesielski, M. and Mochnacki, B. (2012). Numerical Analysis of Interrelations between Skin Surface Temperature and Burn Wound Shape. Scientific Research of the Institute of Mathematics and Computer Science, 1(11), pp 15-22.

Damor, R.S., Kumar, S. and Shukla, A.K (2015). Parametric study of fractional bio-heat equation in skin tissue with sinusoidal heat flux. Fractional Differential Calculus, 5(1), pp 43-53.

Damor, R.S., Kumar, S. and Shukla, A.K 2013). Numerical Solution of Fractional Bioheat Equation with Constant and Sinusoidal Heat Flux Condition on Skin Tissue. American Journal of Mathematical Analysis, 1(2), pp 20-24.

Grysa, K. and Maciag, A. (2019). Mathematical model of identification of the skin surface temperature caused by the temperature of tissue inflammation. Institute of electrical electronics engineers, Doi:10.1109/DJ.2019.8813329.

Gutierrez, G. (2007). Study of the Bioheat Equation with a Spherical Heat Source for Local Magnetic Hyperthermia. XVI Congress on Numerical Methods and their Applications, Cordoba, Argentina.

Hossain, S. and Mohammadi, F. (2013). One-dimensional Steady-state Analysis of Bioheat Transfer Equation: Tumour Parameters Assessment for Medical Diagnosis Application.In:Proceedings 6th international multi-conference on engineering and technological innovation (IMETI 2013), pp 26-30.

Kabiri, A. and Talaee, M.R. (2021). Analysis of hyperbolic Pennes bioheat equation in perfused homogeneous biological tissue subject to the instantaneous moving heat source. SN Applied Science,3: 398, https://doi.org/10.1007/s42452-021-04379-w

Lakhssassi, A., Kengne, E. and Semmaoui, H. (2010). Modified Pennes’ equation modeling bio-heat transfer in living tissues: analytical and numerical analysis. Natural Science, 2(12), pp 1375-1385.

Lillicrap,T., Tahtalı, M., Neely, A., Wang, X., Bivard, A. and Lueck, C.(2017). A model based on the Pennes bioheat transfer equation is valid in normal brain tissue but not brain tissue suffering focal ischaemia. Australasian Physics and Engineering Science in Medicine, 40, pp 841–850.

Liu, K. and Cheng, P.(2008). ‘Finite propagation of heat transfer in a multilayer tissue’. Journal of thermo-physics and heat transfer, 22, pp 775 – 782.

Pennes, H.H. (1948). Analysis of tissue and arterial temperatures in the resting human forearm. Journal of Applied Physiology, 1, pp 93-122.

Roca Oriaa, E.J., Bergues Cabrales, L.E and Bory Reyesc, J.(2019). Analytical solution of the bioheat equation for thermal response induced by any electrode array in anisotropic tissues with arbitrary shapes containing multiple-tumor nodules. Revista Mexicana de F´ısica, 65, pp 284–290.

Romero, R., Jimenez-Lozano, J.N., Shen, M. and Gonzalez, F.J.(2009). Analytical Solution of the Pennes Equation for Burn-Depth Determination from Infrared Thermographs. Mathematical Medicine and Biology, 27, pp 21-38, doi: 10.1093/imammb/dqp010, Advance Access publications.

Sakar, D., Haji-Sheikh, A. and Jain, A. (2015). Temperature distribution in multi-layer skin tissue in presence of a tumor. International Journal of Heat and Mass Transfer, 91, pp 602-610.

Tzu-Ching, S., Ping, Y., Win-Li, L. and Hong-Sen, K.(2007). Analytical Analysis of the Pennes' Bioheat Transfer Equation with Sinusoidal Heat Flux Condition on Skin Surface. IPEM, Medical Engineering & Physics, 29(9), pp 946-953.

Weinbaum, S., Jiji, L.M. and Lemons, D.E.(1984). Theory and experiment for the effect of vascular microstructure on surface tissue heat transfer. Part I. Anatomical foundation and model conceptualization. Journal of Biomechanical Engeering-T. ASME, 106, pp 321-330.

Zhou, M. and Chen, Q.(2007). Study of the surface temperature distribution of the tissue affected by the point heat source. Institute of Electrical and Electronics Engineering.

Zhou, M. and Chen, Q. (2009.). Estimation of Temperature Distribution in Biological Tissue by Analytic Solutions to Pennes' Equation. 2nd International Conference on Biomedical Engineering and Informatics (BMEI), Institute of Electrical and Electronics Engineering, Nanjing, China.