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ITERATIVE APPROXIMATION OF COMMON SOLUTION OF THREE CLASSES OF NONLINEAR PROBLEMS

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ABSTRACT

We introduce an iterative algorithm and prove that the sequence of the algorithm strongly converges to a common element of the set of fixed points of asymptotically nonexpansive mapping, the set of solutions of generalized mixed equilibrium problem and the set of solutions of variational inequality problem in a real Hilbert space. Furthermore, we applied the result obtained to solve an optimization problem.

INTRODUCTION

Let K be a nonempty subset of a real Hilbert space H. A mapping $A: K \to H$ is called: (i) monotone if

$$\langle Ax - Ay, x - y \rangle \geq 0 \ \forall x, y \in K \tag{1.1}$$

(ii) $\lambda-$inverse-strongly monotone .See, for example, Liduka and Takahashi (2005). If there exists a positive real number λ such that

$$\langle Ax - Ay, x - y \rangle \geq \lambda \| Ax - Ay \|^2 \ \forall \ x, y \in K$$

(iii) relaxed (λ, γ) - cocoercive if there exists $\lambda, \gamma > 0$ such that

$$\langle Ax - Ay, x - y \rangle \geq -\lambda \| Ax - Ay \|^2 + \gamma \| x - y \|^2 \ \forall \ x, y \in K$$

(iv) nonexpansive if $\| Ax - Ay \| \leq \| x - y \| \ \forall \ x, y \in H$

(v) $\mu-$Lipschitzian if there exists $\mu > 0$ such that

$$\| Ax - Ay \| \leq \mu \| x - y \| \ \forall \ x, y \in H$$

Let $A: K \to H$ be a nonlinear mapping. The variational inequality problem is to find $x^* \in K$ such that

$$\langle Ax^*, y - x^* \rangle \geq 0 \forall \ y \in K \tag{1.2}$$

See, for example, Blum and Oettli (1994) andBruck (1977). We shall denote the set of solutions of the variational inequality problem (1.2) by V I(K, A).

A point $x \in K$ is called a fixed point of T if Tx = x. The set $F(T) := \{x \in K : Tx = x\}$ is called the set of fixed points of T. Existence and approximation of fixed points of nonexpansive maps and their generalizations have been studied extensively in the literature due it its vital roles in solving different real world problems .See egChidume (2009).One important generalization of the class of nonexpansive mappings is the class of asymptotically nonexpansive mappings introduced by Goebel and Kirk (1972). Let K be a nonempty subset of a real normed linear space E. A mapping $T : K \to E$ is called asymptotically nonexpansive .Goebel and Kirk,(1972) if there exists a sequence $\{k_n\}, k_n \geq 1$, such $\lim_{n\to\infty} k_n = 1$ and $\| T^n x - T^n y \| \leq k_n \| x - y \|$ holds for each $x, y \varepsilon K$ Many authors have studied the problem of approximation of fixed points of asymptotically nonexpansive maps (see, e.g.Berinde (2007), Chidume (2009), Chidume*etal*(2003),Kang *etal* (2009),Schu (1991) and the references therein).

A monotone mapping $A: H \to H$ is said to be maximal if the graph G(A) is not properly contained in the graph of any other monotone map, where $G(A) := \{(x, y) \varepsilon H \times H : y \in Ax\}$ for a multi-valued mapping A. It is alsoknown that A is maximal if and only if for $(x, y) \in H \times H, < x - y, f - g > \geq 0$

for every $(x, y) \in G(A)$ implies $f \in Ax$. Let A be a monotone mapping defined from K into H and N_K q be a normal cone to K at $q \in K$ i.e.,

$N_K q = \{p \in H : \langle q - u, p \rangle \geq 0, \forall u \in K\}$ Define a mapping M by

$$Mq = \begin{cases} Aq + NKq, q \in K \\ \phi, q \notin K \end{cases}$$

Then, M is maximal monotone. Furthermore, $x^* \in M^{-1}(0) \Leftrightarrow x^* \in VI(K, A)$ see, for example, Rockafella (1976).Computation of fixed points is important in the study of many problems including inverse problems. For instance, it

can be shown that the split feasibility problem and the convex feasibility problem can both be formulated as a problem of finding fixed points of certain operators. In particular, construction of fixed points of nonexpansive mappings is applied in image recovery, signal processing and in transition operators for initial value problems of differential inclusions (see, for example, Byrne (2004)).

Finding a common element of the set of fixed points of nonexpansive mapping and the set of solutions of variational inequality problem has been studied extensively in the literature (see, for example, Iiduka and Takhashi (2005)) and the references contained therein).

Let $\varphi : K \to \Re \cup (+\infty)$ be a real-valued, proper, lower semi-continuous and convex function and $A: K \to H$ be a nonlinear mapping. Suppose $F : K \times K \to \Re \cup \{+\infty\}$ is an equilibrium bifunction, that is, $F(x, x) = 0 \forall x \in K$. The generalized mixed equilibrium problem is to find $x \varepsilon K$, (see, e.g., Zhang (2009)) such that

$$F(x, y) + \varphi(y) - \varphi(x) + \langle Ax, y - x \rangle \geq 0 \forall y \in K \qquad (1.3)$$

We shall denote the set of solutions of generalized mixed equilibrium problem by GMEP. This

$$\text{GMEP: } = \{x^* \in K : F(x^*, y) + \varphi(y) - \varphi(x^*) + \langle Ax^*, y - x^* \rangle \geq 0 \forall y \in K\}.$$

The generalized mixed equilibrium problem is a unifying problem for many important problems such as fixed point problems, optimization problems, variational inequality problems, Nash

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equilibrium problems and so on (see e.g Blum and Oettli(1994),Wu and Cheng (2013), Combettes and Hirstoaga (2005),Liu (2009),Moudafi (2008),Punpaeng and plubtieng (2008),Takahashi andTakahashi (2008),Wangkeeree (2008)) and the references therein). Some methods have been proposed to solve the generalised mixed equilibrium problem (see e.g., Zhang (2009) and the references therein).

For solving the generalized mixed equilibrium problem for a bifunction $F : K \times K \to \Re$, let us assume that F, φ and K satisfy the following conditions:

(A1) $F(x, x) = 0$ for all $x \in K$;

(A2) F is monotone, i.e., $F(y, x) + F(x, y) \le 0$ for all $x, y, \in K$;

(A3) For each $x, y \in K$, $\limsup_{t \uparrow 0} F(tz + (1-t)y, x) \le F(x, y)$;

(A4) For each $x \in K$, $y \mapsto F(x, y)$ is convex and lower semicontinuous;

(B1) For each $x \varepsilon H$ and $r > 0$ there exist a bounded subset D_x of K and $yx \in K$ such that for any $z \in K \setminus D_{x,}$

$$F(z, y) + \varphi(y) - \varphi(z) + \frac{1}{r}\langle yx - z, z - x\rangle < 0; \tag{1.4}$$

(B2) K is a bounded set.

Several weak convergence results have been proved for finding a common element of the set of fixed points of nonex-pansive mapping and either the set of solutions of equilibrium problem or the set of solutions of generalised mixed equilibrium problem in certain Banach spaces (see e.g.,Wattanawitoon*etal*(2010),Sombutand plubtieng (2010) and the references therein).

In order to obtain strong convergence results for finding a common element of the set of solutions of equilibrium problem (or generalised mixed equilibrium problem), variational inequality problem and fixed point problem, many authors have obtained their results using the hybrid method of CQ algorithm and viscosity approximation methods (see e.g., Kang *et al*, Wattanawitoon(2010), Takahashi and Takahashi (2007) and the references therein). The CQ method involves the computation, at each step of the iteration process, two convex subsets C and Q of H, computation of $C \cap Q$ and projecting the initial vector onto $C \cap Q$. This is certainly not convenient to implement in application.

In Shehu (2011) , the author introduced an algorithm which does not involve either the CQ algorithm or the viscosity approximation method and proved strong convergence of the scheme to a common element of the fixed points set of a nonexpansive mapping, the set of solutions of a variational inequality problem for a $\mu-$Lipschitzian, relaxed $(\lambda, \gamma)-$cocoercieve mapping and the set of solutions of a GMEP in the framework of Hilbert spaces. He proved the following theorem.

Theorem (Sh (2011)☺ Let K be a closed convex subset of a real Hilbert space. H. let F be a bifunction from K x K satisfying $(A1)-(A4),\varphi : K \to = \cup\{+\infty\}$ be a proper lower semicontinuous and convex function with assumption (B1) or (B2), let A be a μ -Lipschitzian, relaxed (λ,γ) - cocoercive mapping of K into H and ψ be an α -inverse, strongly monotone mapping of K into H. Let T be a nonexpansive mapping of K into H such that $F := F(T)\bigcap VI(A,K)\bigcap GMEP \neq \phi.$ *Let* $\{x_n\}_{n=1}^{\infty}$ be generated by $x_1 \in K,$

$$\begin{cases} y_n = P_k[(1-\alpha_n)x_n] \\ U_n = T_{r_n}^{(F,\varphi)}(y_n - r_n\psi y_n) \\ x_{n+1} = (1-\beta_n)_{x_n} + \beta_n TP_k(U_n - S_n Au_n) \end{cases} \tag{1.5}$$

for all $n \geq 1$, *where* $\{\alpha_n\}_{n=1}^{\infty}$, *are sequencesin* $[0,1]$, $(s_n)_{n=1}^{\infty}$, $\{r_n\}_{n=1}^{\infty} \subset (0,\infty)$ *satisfying*:

$(i)\ \lim_{n\to\infty}\alpha_n = 0, \sum_{n-1}^{\infty}\alpha_n = \infty$

$(ii)\ 0 < \liminf_{n\to\infty}\beta_n < 1\ (iii)\ 0 < c \leq r_n \leq d < 2\alpha, \lim_{n\to\infty} |r_{n+1-}r_n| = 0$

$(iv)\ 0 < a \leq s_n \leq b < \dfrac{2(\gamma - \lambda\mu^2)}{\mu^2}, \lim_{n\to\infty}|s_n + 1 - s_n| = 0,$

Then $\{x_n\}_{n=1}^{\infty}$ *convergesstronglyto z* ε *F.*

We observe that under the hypothesis of theorem, the map $I - s_n$ A is a strict contraction. Consequently, the map $P_K\ (I - s_n A)$ is also a strict contraction, by the Banach contraction mapping principle, $P_K (I - s_n A)$ has a unique fixed point.

Furthermore, it can be shown that A is δ - strongly monotone, for some $\delta > 0$, using the assumptions on A. Hence A is $\mu -$ Lipschitzian and δ - strongly monotone. It is well known that with such a map, $x^* \varepsilon VI(K,A) \Leftrightarrow x^* = x^* = P_K(I - s_n A)x^*.$ Hence, the solution of VI(K,A), under the setting of theorem sh is unique. So VI (K, A) is a singleton which implies that F is a singleton and there are simpler algorithms than the one studied Shehu for approximating such a solution, (see e.g., [6], chapter 7).

In this paper, we introduce an iterative scheme for the class of asymptotically nonexpansive maps and prove strong convergence of the sequence to a common element of the set of fixed points of asymptotically nonexpansive maps, the set of solutions of variational inequality problem, and the set of solution of generalized mixed equilibrium problem in a real Hilbert space.

Our result improve significantly the theorem of Peng et al, Wu and Cheng (2013) (Corollary 3.3), extend those of Shehu (2011) , pubtieng and sombut (2010), and Sombut (2010), and a host of other authors from the class of nonexpansive maps to the more general class of

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asymptotically nonexpansive maps. Moreover, the condition $\lim|s_{n+1}-s_n|=0$ imposed in theorem Sh is dispensed with. Furthermore, our iterative scheme does not involve the CQ method, and the conditions imposed on the operator A do not make the set VI(K, A) a singleton, which is the case in theorem Sh.

Preliminaries

Let H be a real Hilbert space with inner product <.,.> and norm $\|.\|$ and let K be a nonempty closed convex subset of H. For any point $u \varepsilon H$, there exists a unique point $P_K u \in K$ such that

$$\|u-P_K u\| \le \|u-y\| \forall\ y \in K. \tag{2.1}$$

$P_K u$ is called the metric projection of H onto K. we know that $P_K u$ is a nonexpansive mapping of H onto K. It is also known that P_K satisfies the following inequality:

$$\langle x-P_K x, P_K x-y\rangle \ge 0 \tag{2.3}$$

For all $y \in K$.

In the context of the variational inequality problem

$$x^* \varepsilon\, VI\,(K,A) \Leftrightarrow x^* = P_k\left(x^*-\lambda A x^*\right) \forall \lambda > 0.$$

In what follows, we shall make use of the following lemmas.

Lemma 2.1 (Suzuki (2005), let $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ be bounded sequences in a Banach space E and let $\{\delta_n\}_{n=1}^{\infty}$ be a sequence in [0,1] with $0<\liminf \delta_n \le \limsup \delta_n < 1$. Suppose $x_{n+1}=\delta_n x_n+(1-\delta_n)y_n$ for all integers $n \ge 0$ and $\limsup(\| y_{n+1}-y_n\|-\|x_{n+1}-x_n\|) \le 0.$ Then $\lim\|y_n - x_n\| = 0.$

Lemma 2.2. (Xu, (2002) *Let* $\{a_n\}_{n=1}^{\infty}$ be a sequence of nonnegative real numbers such that

$$a_{n+1} \le (1-\delta_n)a_n + \delta_n \sigma_n, +\gamma_n,\ n \in \mathrm{N}$$

Where

(i) $\{\delta_n\}_n \subset (0,1), \lim_{n\to\infty} \delta_n = 0, \sum_{n\to\infty}^{\infty} \delta_n = \infty$

(ii) Lim sup $_n\sigma_n \le 0$;

(iii) $\gamma_n \ge 0, n \ge 1, \Sigma\, \gamma_n < +\infty.$ *Then,* $\lim_{n=\infty} a_n = 0.$

Lemma 2.3 (Chang, Zhou *etal* (2001)) Let E be a uniformly convex Banach space, K be a nonempty closed convex subset of E and T: K→ K be anasymptotically nonexpansive mapping, then $(I-T)$ is demiclosed at zero.

Lemma 2.4 (Lim and Xu, (1994) Suppose E is a Banach space with uniform normal structure, K a nonempty bounded

subset of E and $T: K \to K$ is uniformly $L-$Lipschitzian mapping with $L < N(E)^{\frac{1}{2}}$. Suppose also there exists a

nonempty closed convex subset A of K with the following property (P):

$x \in A$ implies that $\omega_w(x) \in A$, where $\omega_w(x)$ is the weak limit set of T at x. Then T has a fixed point in A.

Lemma 2.5. Let H be a real Hilbert space. Then

$$\|x+y\|^2 \le \|x\|^2 + 2(y, x+y), \forall\, x, y \in H.$$

Lemma 2.6. (Shioji and Takahashi, (1997))

Let $(a_0, a_1..) \in l^\infty$ *be such that* $\mu_n(a_n) \le 0$ *for all Banach* limit μ *and let* $\limsup_{n=\infty} (a_{n+1} - a_n) \le 0.$ Then, $\limsup_{n=\infty} a_n \le 0.$

Lemma 2.7 (Wangkeeree and Wangkeeree (2008)) Let K be a nonempty closed convex subset of H and let F be a bifunction of K x K into $\Re$ satisfying (A1)–(A4) and $\varphi: K \to \Re \cup \{+\infty\}$ be a proper lower semicontinuous and convex function. Assume that either (B1) or B2 holds. Let $r > 0.$ *Define a mapping* $T_r^{(F,\varphi)}: H \to 2^K$ *as follows*:

$$T_r^{(F,\varphi)}(x) := \left\{ z \in K : F\ (z, y) + \varphi(y) - \varphi(z) + \frac{1}{r}\langle y - z, z - x\rangle \ge 0 y \in K \right\}$$

For all $x\ \varepsilon\ H$. *Then, the following hold*:

(1) For each $x\ \varepsilon\ H, T_r^{F,\varphi}(x) \ne \phi;$

(2) $T_r^{(F,\varphi)}$ *is single valued*;

(3) $T_r^{(F,\varphi)}$ *is firmly nonexpansive, i.e., for each* $x, y, \varepsilon\ H,$

$$\left\|T_r^{(F,\varphi)}x - T_r^{(F,\varphi)}y\right\|^2 \le \left\langle T_r^{(F,\varphi)}x - T_r^{(F,\varphi)}y, x - y\right\rangle;$$

(4) $T_r^{(F,\varphi)} = GMEP\,(f);$

(5) *GMEP (F) is closed and convex.*

MAIN RESULTS

We now prove the main result of the manuscript.

Theorem 3.1. Let K be a closed convex subset of a real Hilbert space, H. Let F be a bifunction from K x K into $\Re$ satisfying (A1)- (A4), $\varphi: K \to \Re \cup \{+\infty\}$ be a proper lower semicontinous and convex function with assumption (B1) or (B2). Let A be a μ-inverse strongly monotone

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mapping of K into H and γ be an α - inverse strongly monotone mapping of K into H. Suppose T is an asymptotically nonexpansive mapping of K into K such that $F := F(T)\cap VI(A,K)\cap GMEP \neq \phi$ Let $\{x_n\}_{n=1}^{\infty}$, $\{y_n\}_{n=1}^{\infty}$ *and* $\{u_n\}_{n=1}^{\infty}$ be generated by $x_1 \varepsilon K$,

$$\begin{cases} y_n = P_K[1-\sigma_n x_n] \\ u_n = T_{r_n}^{(F,\varphi)}(y_n - r_n\psi y_n) \\ x_{n+1} = (1-\beta_n)x_n + \beta_n T^n P_K(u_n - s_n S u_n) \end{cases} \tag{3.1}$$

For all $n \geq 1$, *where* $\{\sigma_n\}_{n=1}^{\infty}, \{\beta_n\}_{n=1}^{\infty}$; *are sequencesin* $(0,1)$, $\{s_n\}_{n=1;}^{\infty}(r_n)_{n=1}^{\infty} \subset (0,\infty)$ *satisfying*:

(i). $\lim_{n\to\infty} a_n = 0, \sum_{n=1}^{\infty}\alpha_n = \infty, \sum_{n=1}^{\infty}\alpha_n^2 < \infty;$

(ii). $0 < c \leq r_n \leq d < 2\alpha, \lim_{n\to\infty}|r_{n+1} - r_n| = 0;$

(iii). $0 < a \leq s_n \leq b < 2\mu;$

(iv) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < (v) \sum_{n=1}^{\infty}(k_n^2 - 1) < \infty.$

Then, $\{x_b\}_{n=1}^{\infty}$ *convergesstronglyto* $u \in F$.

Proof.

Using (ii) and (iii), properties of A and ψ, respectively, it is clear that the operators $I - s_n A$ *and* $I - r_n\psi$ are nonexpansive for each $n \geq 1$.

Furthermore, from the recursion formula(3.1) we obtain:

$$\begin{aligned}
\|x_{n+1} - x^*\| &= \|(1-\beta_n)(x_n - x^*) + \beta_n[T^n P_K(I - s_n A)u_n - T^n P_K(I - s_n A)x^*]\| \\
&\leq (1-\beta_n)\|x_n - x^*\| + \beta_n k_n\|u_n - x^*\| \\
&\leq (1-\beta_n + \beta_n k_n(1-\alpha_n))\|x_n - x^*\| + \beta_n\alpha_n k_n\|x^*\| \\
&\leq [1-\beta_n + \beta_n k_n(1-\alpha_n) + \beta_n\alpha_n k_n] \max\{\|x_n - x^*\|, \|x^*\|\} \\
&\leq [1+\beta_n(k_n - 1)] \max\{\|x_n - x^*\|, \|x^*\|\} \\
&\leq \prod_{j=1}^{n}(1+(kj-1)) \max\{\|x_1 - x^*\|, \|x^*\|\} \\
&\leq \sum_{ej=1}^{\infty}{}^{(kj-1)} \max\{\|x_1 - x^*\|, \|x^*\|\} < \infty.
\end{aligned}$$

Hence, (x_n) is bounded. Consequently $(y_n), \{u_n\}$ and Au_n are bounded.

Step 1. We prove that $\lim_{n\to\infty}\|u_n - y_n\| = 0$. *Set* $P_n := P_K(u_n - s_n Au_n)$. *Then.*

$$\|P_{n+1} - P_n\| \leq \|u_{n+1} - u_n\|. \qquad (3.2)$$

$From\, u_n = T_{r_n}^{(F,\varphi)}(y_n - r_n \psi y_n)\, and\, u_{n+1} = T_{r_{n+1}}^{(F,\varphi)}(y_{n+1} - r_{n+1}\psi y_{n+1})$ and using Lemma 2.7 (2), we obtain:

$$F(u_{n,}y) + \varphi(y) - \varphi(u_n) + \langle \psi y_{n,} y - u_n \rangle + \qquad (3.3)$$

$$\frac{1}{r_n}\langle y - u_n, u_n - y_n \rangle \geq 0 \ \forall \ y \ \varepsilon \ K, \text{ and}$$

$$F(u_{n+1}, y) + \varphi(y) - \varphi(u_{n+1}) + \langle \psi y_{n+1}, y - u_{n+1} \rangle + \frac{1}{r_n}\langle y - u_{n+1}, u_n - y_n \rangle \geq 0. \forall y \in K \qquad (3.4)$$

Substituting $y = u_{n+1}$ in (3.3) and $y = u_n$ in (3.4), we have

$$F(u_{n,+1}, u_n) + \varphi(u_n) - \varphi(u_{n+1}) + \langle \psi y_n, u_n - u_{n+1} \rangle + \frac{1}{r_n}\langle u_{n+1} - u_{n,}, u_{n+1} - y_{n+1} \rangle \geq 0. \qquad (3.6)$$

And hence,

$$0 \leq \left\langle u_n - u_{n+1,} r_n(\psi y_{n+1} - \psi y_n) + \frac{r_n}{r_{n+1}}(u_{n+1} - y_{n+1}) - (u_{n-y_n}) \right\rangle$$

$$= \left\langle u_{n+1} - u_{n,} u_n - u_{n+1} + \left(1 - \frac{r_n}{r_{n+1}}\right)(u_{n+1} - y_{n+1}) + (y_{n+1} - r_n \psi y_{n+1}) = (y_n - r_n \psi y_n) \right\rangle$$

It then follows, using nonexpansivity of $= (I - r_n \psi)$ that,

$$\|u_{n+1} - u_n\|^2 \leq \|u_{n+1} - u_n\| \left\{ 1 - \frac{r_n}{r_{n+1}} \|u_{n+1} - y_{n+1}\| + \|y_{n+1} - y_n\| \right\}$$

and so we have

$$\|u_{n+1} - u_n\| \leq \left| 1 - \frac{r_n}{r_{n+1}} \right| \|u_{n+1} - y_{n+1}\| + \|y_{n+1} - y_n\|,$$

and using condition (ii) of Theorem 3.1, we get

$$\begin{aligned} \|u_{n+1} - u_n\| &\leq \frac{1}{r_{n+1}} |r_{n+1} - r_n| \|u_{n+1} - y_{n+1}\| + \|y_{n+1} - y_n\| \\ &\leq M_{1c^{-1}} |r_{n+1} - r_n| + \|y_{n+1} - y_n\| \end{aligned} \qquad (3.7)$$

For $m \geq 1,\, set\, z_n = T^m p_{n,} n \geq 1.$ Then, using boundedness of $\{z_n\}$ and inequality 3.7, we get:

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$$\begin{aligned}\|z_{n+1}-z_n\| &= \|T^m P_{n+1} - T^m p_n\| \\ &\le k_m\|y_{n+1}-y_n\| + k_m M_{1c^{-1}}|r_{n+1}-r_n| \\ &\le k_m\|(1-\alpha_{n+1})x_{n+1}-(1-\alpha_n)x_n\| \\ &+ k_m M_{1c}{}^{-1}|r_{n+1}-r_n| \\ &\le k_m\|x_{n+1}-x_n\| + k_m\alpha_{n+1}\|x_{n+1}\| \\ &+ k_m\alpha_n\|x_n\| + k_m M_{1c}{}^{-1}|r_{n+1}-r_n|\end{aligned} \tag{3.8}$$

Hence. $\limsup_{n\to\infty}(\|z_{n+1}-z_n\|-\|x_{n+1}-x_n\|)\le 0.$ Using this and lemma 2.1, we get $\lim\|z_n - x_n\| = 0.$ Thus,

$$\lim_{n\to\infty}\|x_{n+1}-x_n\| = \lim\beta_n\|z_n - x_n\| = 0.$$

Using (3.7), we have, since $\|y_{n+1}-y_n\| \le \|(1-\alpha_{n+1})x_{n+1}-(1-\alpha_n)x_n\| \to 0\ as\ n\to\infty,$ that

$$\lim\|u_{n+1}-u_n\| = 0. \tag{3.9}$$

Consequently,

$$\lim\|p_{n+1}-p_n\| = 0. \tag{3.10}$$

Let $x^*\ \varepsilon\ F$ and let $\{T_{rn}\}_{n=1}^{\infty}$ be a sequence of mappings defined as in Lemma 2.7. Then we have $x^* = P_K(x^* - s\ Ax^*) = T_{rn}^{(F.\varphi)}(x^* - r_n\psi x^*)$. For each $n\ge 1$, using the fact that ψ and A are inverse strongly monotone, we obtain

$$\begin{aligned}\|u_n - x^*\|^2 &= \|T_{rn}^{(F,\varphi)}(y_n - r_n\psi y_n) - T_{rn}^{(F,\varphi)}(x^* - r_n\psi x^*)\|^2 \\ &= \le \|(I - r_n\psi)y_n - (I - r_n\psi)x^*\|^2 \\ &= \|y_n - x^*\|^2 - r_n(2\alpha - r_n)\|\psi y_n - \psi x^*\|^2\end{aligned}$$

And

$$\begin{aligned}\|p_n - x^*\|^2 &\le \|(I - s_n A)u_n - (I - s_n A)x^*\|^2 \\ &\le \|y_n - x^*\|^2 - s_n(2_\mu - s_n)\|Au_n - Ax^*\|^2\end{aligned}$$

Furthermore, using the convexity of $\|.\|^2$, we obtain

$$\begin{aligned}
\left\|x_{n+1}-x^*\right\|^2 &= \left\|(1-\beta_n)(x_n-x^*)+\beta_n\left(T^n P_n - x^*\right)\right\|^2 \\
&\le (1-\beta_n)\left\|x_n x^*\right\|^2 + \beta_n k_n^2\left\|p_n - x^*\right\|^2 \\
&\le (1-\beta_n)\left\|x_n - x^*\right\|^2 + \beta_n k_n^2\left\|x_n - x^* - \alpha_n x_n\right\|^2 \\
&+ \beta_n k_n^2 s_n\left(\left(s_n - 2_\mu\right)\left\|Au_n - Ax^*\right\|^2\right) \\
&\le \left\|x_n - x^*\right\|^2 + \beta_n\left(k_n^2-1\right)\left\|x_n - x^*\right\|^2 + 2\alpha_n\beta_n k_n^2\left\|x_n - x^*\right\|\left\|x_n\right\| \\
&+ x_n^2\beta_n k_n^2\left\|x_n\right\|^2 - \beta_n k_n^2 s_n\left(2_\mu - s_n\right)\left\|Au_n - Ax^*\right\|^2.
\end{aligned}$$

This implies,

$$\beta_n k_n^2\left(\left(2_\mu - s_n\right)\left\|Au_n - Ax^*\right\|^2\right) \le \left\|x_n - x_{n+1}\right\| M_2 + \left(k_n^2 - 1\right)M_3 + \alpha_n M_4$$

Where

$$M_2 := \sup_n\left\{\left\|x_n - x^*\right\| + \left\|x_{n+1} - x^*\right\|\right\},\ M_3 := \sup_n\left\{\left\|x_n - x^*\right\|^2\right\}$$

And

$$M_4 := \sup_n\left\{2\beta_n k_n^2\left\|x_n - x^*\right\|\left\|x_n\right\| + \beta_n k_n^2\alpha_n\left\|x_n\right\|^2\right\},$$

Using condition (iii), we have

$$\beta_n a\left(\left(2_\mu - b\right)\left\|Au_n - Ax^*\right\|^2\right) \le \left\|x_n - x_{n+1}\right\| M_2 + \beta_n\left(k_n^2-1\right)M_3 + \alpha_n M_4 \tag{3.11}$$

Taking limsupas $n\to\infty\, in$ (3.11) and using the fact that $\alpha_n \to 0\ as\ n\to\infty, \left\|x_{n+1}-x_n\right\| \to 0\ as\ n \to\infty, k_n^2 - 1 \to 0\ as\ n\to\infty$ and condition (iv), we get $\left\|Au_n - Ax^*\right\| \to 0,$ as $n\to\infty$ moreover,

$$\begin{aligned}
&\left\|x_{n+1}-x^*\right\|^2 \le (1-\beta_n)\left\|x_n - x^*\right\|^2 + \beta_n k_n^2\left\|p_n - x^*\right\|^2 \\
&\le (1-\beta_n)\left\|x_n - x^*\right\|^2 + \beta_n k_n^2\left(\left\|y_n - x^*\right\|^2 - r_n\left(2_\alpha - r_n\right)\left\|\psi y_n - \psi x^*\right\|^2\right) \\
&\le (1-\beta_n)\left\|s_n - x^*\right\|^2 + \beta_n k_n^2\left(\left\|x_n - x^*\right\|^2 + 2\alpha_n\left\|x_n - x^*\right\|\left\|x_n\right\| + \alpha_n^2\left\|x_n\right\|^2\right) - \beta_n k_n^2 r_n\left(2_\alpha - r_n\right)\left\|\psi y_n - \psi x^*\right\|^2 \\
&\le \left[1+\beta_n\left(k_n^2-1\right)\right]\left\|x_n - x^*\right\|^2 + \alpha_n M_4 - \left(2_\alpha - r_n\right)\beta_n k_n^2 r_n\left\|\psi y_n - \psi x^*\right\|^2.
\end{aligned}$$

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Hence, $\beta_n k_n^2 r_n (2_\alpha - r_n)\|\psi y_n - \psi x^*\|^2 \le \|x_{n+1} - x_n\| M_2 + (k_n^2 - 1)M_3 + \alpha_n M_4$. This implies, using condition (ii) that

$$\beta_n c(2\alpha - d)\|\psi y_n - \psi x^*\|^2 \le \|x_{n+1} - x_n\| M_2 + (k_n^2 - 1)M_3 + \alpha_n M_{4.} \tag{3.12}$$

Taking limsup as $n \to \infty$ in (3.12) and using the fact that $\alpha_n \to 0 \; as \; n \to \infty, \|x_{n=1} - x_n\| \to 0 \; as \; n \to \infty, k_n^2 - 1 \to 0 \; as \; n \to \infty$ and condition (iv), we get lim $\|\psi y_n - \psi x^*\| = 0.$ furthermore, using Lemma 2.7(3) and nonexpansivity of $(I - r_n \psi)$, we have:

$$\begin{aligned}
\|u_n - x^*\|^2 &\le \|T_{rn}^{(F,\varphi)}(y_n - r_n \psi y_n) - T_{rn}^{(F,\varphi)}(x^* - r_n \psi x^*)\|^2 \\
&\le \langle (y_n - r_n \psi x^*) - (x^* - r_n \psi x^*), u_n - x^* \rangle \\
&= \frac{1}{2}\left(\|(y_n - r_n \psi y_n) - (x^* - r_n \psi x^*)\|^2 + \|u_n - x^*\|^2 - \|(y_n - r_n \psi y_n) - (x^* - r_n \psi x^*) - (u_n - x^*)\|^2 \right) \\
&\le \frac{1}{2}\left(\|y_n - x^*\| + \|u_n - x^*\|^2 \right) - \frac{1}{2}\left(\|(y_n - u_n) - r_n(\psi y_n - \psi x^*)\|^2 \right) \\
&= \frac{1}{2}\left(\|y_n - x^*\|^2 + \|u_n - x^*\|^2 \right) - \frac{1}{2}\|u_n - y_n\|^2 + r_n \langle y_n - u_n, \psi x_n - \psi x^* \rangle - \frac{1}{2} r_n^2 \|\psi y_n - \psi x^*\|^2)
\end{aligned}$$

And hence,

$$\begin{aligned}
\|u_n - x^*\|^2 &\le \|y_n - x^*\|^2 - \|u_n - y_n\|^2 + 2r_n \langle y_n - u_n, \psi y_n - \psi x^* \rangle - r_n^2 \|\psi y_n - \psi x^*\|^2 \\
&\le \|y_n - x^*\|^2 - \|u_n - y_n\|^2 + 2r_n \|y_n - u_n\| \|\psi y_n - \psi x^*\|.
\end{aligned} \tag{3.13}$$

By convexity of $| \; |.| \; |^2$ and using inequality (3.23), we have

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &\le (1 - \beta_n)\|x_n - x^*\|^2 + \beta_n k_n^2 \|u_n - x^*\|^2 \\
&\le \|x_n - x^*\|^2 + (k_n^2 - 1)M_3 + \alpha_n M_4 - \beta_n k_n^2 \|u_n - y_n\|^2 \\
&\quad + 2r_n \|y_n - u_n\| \|\psi y_n - \psi x^*\|.
\end{aligned}$$

Consequently,

$$\begin{aligned}
&\beta_n \|u_n - y_n\|^2 \le \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 + (k_n^2 - 1)M_3 \\
&+ \alpha_n M_4 + 2d\|y_n - u_n\| \|\psi y_n - \psi x^*\|.
\end{aligned} \tag{3.14}$$

Taking limsup as $n \to \infty$ in (3.14) and using the fact that $\alpha_n \to 0 \; as \; n \to \infty, \|x_n - x_n\| \to 0 \; as \; n \to \infty, k_n^2 - 1 \to as \; n \to \infty$ and condition (iv), then $\lim_{n \to \infty} \|u_n - y_n\| = 0,$ completing step 1.

Step 2. We show that $\lim_{n\to\infty}\|u_n - p_n\| = 0.$ Using the nonexpasiveness of $(I - s_n A)$ and inequality (2.2) we have:

$$\begin{aligned}
\|p_n - x\|^2 &= \| P_K(u_n - s_n Au_n) - P_K(x^* - s_n Ax^*)\|^2 \\
&\leq \langle (u_n - s_n Au_n) - (x^* - s_n Ax^*) P_K(u_n - s_n Au_n) - x^* \rangle, \\
&= \frac{1}{2}\Big(\|(u_n - s_n Au_n) - (x^* - s_n Ax^*)\|^2 + \|P_K(u_n - s_n Au_n) - x^*\|^2 \\
&- \|(u_n - s_n Au_n)\|^2 + \|P_K(u_n - s_n - s_n Au_n) - x^*\|^2 \\
&\leq \frac{1}{2}\Big(\|u_n - x^*\|^2 + \|P_K(u_n - s_n Au_n) - x^*\|^2 \\
&- \|u_n - P_K(u_n - s_n Au_n)\|^2 + 2s_n (u_n - s_n Au_n), Au_n - Ax^* \rangle \\
&- s_n{}^2 \|Au_n - ax^*\|^2 \Big)
\end{aligned}$$

Therefore,

$$\begin{aligned}
\|p_n - x^*\|^2 &\leq \|u_n - x^*\|^2 - \|u_n - p_n\|^2 \\
+ 2s_n \langle u_n - P_K(u_n - s_n Au_n), Au_n - Ax^* \rangle &- s_n{}^2 \|Au_n Ax^*\|^2.
\end{aligned} \tag{3.15}$$

Hence, we have using inequality (3.15) and the fact that $\|u_n - x^*\|^2 \leq \|x_n - x^* - \alpha_n x_n\|^2$, that

$$\begin{aligned}
\|x_{n+1} - x\|^2 &\leq (1 - \beta_n)\|x_n - x^*\|^2 + \beta_n k_n^2 \|p_n - x^*\|^2 \\
&\leq \|x_n - x^*\|^2 + \beta_n (k_n^2 - 1)\|x_n - x^*\|^2 \\
&+ 2\alpha_n \beta_n k_n^2 \|x_n - x^*\|\|x_n\| + \alpha_n^2 \beta_n k_n^2 \|x_n\|^2 - \beta_n k_n^2 \|u_n - p_n\|^2 \\
&+ 2s_n \beta_n k_n^2 \langle u_n - p_n,\ Au_n - Ax^* \rangle - \beta_n k_n^2 s_n^2 \|Au_n - Ax^*\|^2 \\
&\leq \|x_n - x^*\|^2 + \beta_n (k_n^2 - 1) M_3 + \alpha_n k_n^2 M_4 \\
&- \beta_n k_n^2 \|u_n - p_n\|^2 + 2s_n \beta_n k_n^2 \|u_n - p_n\| \|Au_n - Ax^*\|.
\end{aligned}$$

Since $\lim_{n\to\infty}\alpha_n = 0, \lim_{n\to\infty}\|x_{n+1} - x_n\| = 0, \lim_{n\to\infty}(k_n^2 - 1) = 0, \lim_{n\to\infty}\|Au_n - Ax^*\| = 0$ and condition (iv), we obtain $\lim_{n\to\infty}\|u_n - p_n\| = 0,$ completing step 2.

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Step 3. We show that $\lim_{n\to\infty}\|x_n - T^n x_n\| = 0.$

$$\begin{aligned}\|x_n - T^n x_n\| &\le \|x_n - x_{n+1}\| + \|x_{n+1} - T^n x_n\|\\ &\le \|x_n - x_{n+1}\| + (1-\beta_n)\|x_n - T^n x_n\| + \beta_n\|T^n P_n - T^n x_n\|\\ &\le \|x_n - x_{n+1}\| + (1-\beta_n)\|x_n - T^n x_n\| + \beta_n k_n\|p_n - x_n\|\\ &\le \|x_n - x_{n+1}\| + (1-\beta_n)\|x_n - T^n x_n\| + \beta_n k_n\|p_n - x_n\|\\ &+ \beta_n k_n\|y_n - x_n\|\\ &\le \|x_n - x_{n+1}\| + (1-\beta_n)\|x_n - T^n x_n\| + \beta_n k_n\|p_n - y_n\|\\ &+ \beta_n \alpha_n k_n\|x_n\|,\end{aligned}$$

So that:

$$\|x_n - T^n x_n\| \le \frac{1}{\beta_n}\{\|x_n - x_{n+1}\| + \beta_n k_n\|p_n - y_n\| + \beta_n\alpha_n\|x_n\|\}.$$

Since from step 1 and step 2, $\lim_{n\to\infty}\|p_n - u_n\| = 0, \, and \lim_{n\to\infty}\|u_n - y_n\| = 0,$ we have that $\lim_{n\to\infty}\|p_n - y_n\| = 0.$ Thus,

$\lim_{n\to\infty}\|x_n - Tx_n\| = 0$ completing step 3.

Step 4. We prove that $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$

$$\|x_{n+1} - T^n x_n\| \le \|x_{n+1} - x_n\| + \|x_n - T^n x_n\| \to 0,\, n \to \infty$$

$$\begin{aligned}\|x_{n+1} - Tx_n\| &\le \|x_{n+1} - T^n x_n\| + \|T^{n=1} x_n - x_n\|\\ &\le \|x_{n+1} - T^n x_n\| + \|T^{n-1} x_n - x_n\|\\ &\le \|x_{n+1} - T^n x_n\| + [\|T^{n-1} x_n - T^{n-1} x_{n-1}\|\\ &+ \|T^{n-1} x_{n-1} - x_{n-1}\| + \|x_{n-1} - x_n\|]\\ &\le \|x_{n+1} - T^n x_n\| + k_{n-1}\|x_n - x_{n-1}\|\\ &+ \|T^{n-1} x_{n-1} - x_{n-1}\| + \|x_{n-1} - x_n\|\end{aligned}$$

Thus, $\lim_{n\leftarrow\infty}\|x_{n+1} - Tx_n\| = 0.$ Hence

$$\|x_n - Tx_n\| \le \|x_{n+1} - x_n\| + \|x_{n+1} - Tx_n\| \to 0,\, as\, n \to \infty.$$

Completing step 4.

As $\{x_n\}_{n=1}^{\infty}$ is bounded, there exists a subsequence $\{xnj\}_{j=1}^{\infty}$ of $\{x_n\}_{n=1}^{\infty}$ such that $\{xnj\}_{j=1}^{\infty}$ converges weakly to some $u \, \varepsilon \, K$

We show that $u \varepsilon F$.

(1) We first show that $u \varepsilon GMEP$. Since $u_n := T_{rn}^{(F,\varphi)}(y_n - r_n\psi y_n), n \geq 1$, we have for any $y \varepsilon k$ that

$$F(u_{n,} y) + \varphi(y) - \varphi(u_n) + \langle \psi y_n, y - u_n \rangle + \frac{1}{r_n}\langle y - u_{n,} u_n - y_n \rangle \geq 0.$$

Furthermore, replacing n by nj in the least inequality and using (A2), we obtain:

$$\varphi(y) - \varphi(u_{nj}) + \langle \psi y_{nj}, y - u_{nj} \rangle + \frac{1}{r_{nj}}\langle y - u_{nj}, u_{nj} - y_{nj} \rangle \geq F(y, u_{nj}). \qquad (3.16).$$

Let $z_t := ty + (1-t)u$ for all $t \varepsilon (0,1)$ and $y \varepsilon K$. This implies that $z_t \varepsilon K$. Then, by (3.16), we have

$$\begin{aligned}
\langle z_t - u_{nj}, \psi_{zt} \rangle \geq (u_{nj}) - \varphi(z_t) + \langle z_t - u_{nj}, \psi z_t \rangle \\
- (z_t - u_{nj}, \psi y_{nj}) - \left\langle z_t - u_{nj}, \frac{u_{nj} - y_{nj}}{r_{nj}} \right\rangle + F(z_t, u_{nj}) \\
= \varphi(u_{nj}) - \varphi(z_t) + \langle z_t - u_{nj}, \psi_{zt} - \psi y_{nj} \rangle \\
+ \langle z_t - u_{nj}, \psi y_{nj} - \psi y_{nj} \rangle - \left\langle z_t - u_{nj}, \frac{u_{nj - y_{nj}}}{r_{nj}} \right\rangle \\
+ F(z_t, u_{nj}).
\end{aligned}$$

Since $\|y_{nj} - u_{nj}\| \to 0, j \to \infty$ step 1, we obtain $\|\psi y_{nj} - \psi u_{nj}\| \to \infty$. Furthermore, by the monotonicity of ψ, we obtain $\langle z_t - u_{nj}, \psi z_t - \psi u_{nj} \rangle \geq 0$. Also,

$$\|x_n - u_n\| \leq \|u_n - y_n\| + \|y_n - x_n\| \to 0$$

Implies that $\{u_{nj}\}$ converges weakly to u. then, by (A4) we obtain as $j \to \infty$,

$$\langle z_t - \mu, \psi z_t \rangle \geq \varphi(u) - \varphi(z_t) + F(z_{t,} u). \qquad (3.17)$$

Using (A1),)A4) and (3.17) we also obtain

$$\begin{aligned}
0 = F(z_t, z_t) + \varphi(z_t) - \varphi(z_t) \leq tF|z_t, y| + (1-t)F(z_t, u) + t\varphi(y) \\
+ (1-t)\varphi(u) - \varphi(z_t) + t\varphi z_t - t\varphi z_t \\
\leq t[F(z_t, y) + \varphi(y) - \varphi(z_t)] + (1-t)\langle z_t - u, \psi z_t \rangle \\
= t[F(z_t, y) + \varphi(y) - \varphi(z_t)] + (1-t)t\langle y - u, \psi z_t \rangle
\end{aligned}$$

And hence

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$$0 \le F(z_t, y) + \varphi(y) - \varphi(z_t) + (1-t)\langle y-u, \psi z_t\rangle$$

Letting $t \to 0$, we obtain, for each $y \, \varepsilon \, K$,

$$0 \le F(u, y) + \varphi(y) - \varphi(u) + \langle y-u, \psi u\rangle$$

This implies that $u \, \varepsilon \, G\,M\,E\,P$.

(ii) Next, we show $u \, \varepsilon \, VI\,(K, A)$. Put

$$M\omega = \begin{cases} Aw + N_N w, \, w \, \varepsilon \, k, \\ \phi, \, w \notin K. \end{cases}$$

Since A is μ - inverse strongly monotone, it is monotone. Thus, M is maximal monotone, (see Rockafelalr(1976)). Let $(w_1, w_2) \, \varepsilon \, G(M)$. Since $w_2 - Aw_1 \, \varepsilon \, N_k w_1$ and $p_n \, \varepsilon \, K$, we have

$$\langle w_1 - p_n, w_2 - Aw_1\rangle \ge 0.$$

On the other hand, from $p_n = P_k(I - s_n A)u_n$ and inequality (2.3), we have $\langle w_1 - p_n, p_n - (I - s_n A)u_n\rangle \ge 0$ and hence $\left\langle w_1 - p_n, \frac{p_n u_n}{s_n} + Au_n\right\rangle \ge 0.$ It follows from inequality (3.19) with n replaced with nj and the monotonicity of A that.

$$\begin{aligned}
\langle w_1 - p_{nj}, w_2\rangle &\ge \langle w_1 - p_{nj}, Aw_1\rangle \ge \langle w_1 - p_{nj}, A\omega_1\rangle \\
&\quad - \left\langle w_1 - p_{nj}, \frac{p_{nj} - u_{nj}}{s_{nj}}\right\rangle + Au_{nj} \\
&= \left\langle w_1 - p_{nj}, Aw_1 - \frac{p_{nj} - u_{nj}}{s_{nj}} - Au_{nj}\right\rangle \\
&= \langle w_1 - p_{nj}, A\omega_1 - Ap_{nj}\rangle + \langle w_1 - p_{nj}, Ap_{nj} - Au_{nj}\rangle \\
&\quad - \left\langle w_1 - p_{nj,} \frac{p_{nj} - u_{nj}}{s_{nj}}\right\rangle \\
&\ge \langle w_1 - p_{nj}, Ap_{nj} - Au_{nj}\rangle - \left\langle w_1 - p_{nj}, \frac{p_{nj} - u_n}{s_{nj}}\right\rangle,
\end{aligned}$$

Which implies by step 2 and $u_j \to u(j \to \infty)$, that $\langle w_1 - u, w_2\rangle \ge 0.$ So, we have $u \, \varepsilon \, M^{-1}0$ and hence $u \, \varepsilon \, VI(K, A)$.

(iii) We now show that $u \, \varepsilon \, F(T)$. Using lemma 2.3, we obtain, since T is asymptotically nonexpansive, x_{nj} converges weakly to u and $\|x_{nj} - Tx_{nj}\| \to 0 \; as \; n \to \infty$ that $u \, \varepsilon \, F(T)$.

(iv) Now we prove that $\limsup_{n\to\infty}\langle -u, x_n - u\rangle \le 0.$

Define a map

$\phi : H \to \Re by\ \phi(x) := \mu_n \|x_n - x\|^2\ \forall\ x\ \varepsilon\ H.$ *Then,* $\phi(x) \to \infty$ *as* $(x) \to \infty, \phi$ *is comptinous*

andconvex, so there exists $y^*\ \varepsilon\ H$ such that $\phi(y^*) = \min_{w\in H} \phi(w)$. Hence, the set

$K^* := \left\{ x\ \varepsilon\ H : \phi(x) = \min_{w\in H} \phi(w) \right\} \neq \phi$. We shall make use of lemma 2.4. if

$x\ \varepsilon\ K^*$ *and* $y* := w - T^{nj}x,$ for some $mj, j \to \infty,$ then using the weak lower semi-continuity of ϕ and $\lim\|x_n - Tx_n\| = 0,$ we have (since $\lim\|x_n - Tx_n\| = 0,$ implies that $\lim\|x_n - T^{rn}x_n\| = 0, m \ge 1$ (by induction):

$$\begin{aligned}
\phi(y^*) &\le \liminf_{j\to\infty} \phi\ \ (T^{rn}; x) \le \limsup_{m\to\infty} \phi(T^{rn}x) \\
&= \limsup_{m\to\infty}\left(\mu_n \|x_n - T^m x\|^2 \right) \\
&= \limsup_{m\to\infty}\left(\mu_n \|x_n - T^m x_n + T^m x_n - T^m x\|^2 \right) \\
&\le \limsup_{m\to\infty}\left(\|\mu_n - x\|^2 \right) = \phi(x) = \min_{w\in H} \phi(w)
\end{aligned}$$

By lemma 2.4, $K^* \cap F(T) \neq \phi$. Assume that $y^* = u\varepsilon K^* \cap F(T)$. Let $t\ \varepsilon\ (0,1)$. Then, it follows that $\phi(u) \le \phi(u - tu)$ and using Lemma 2.5, we obtain that

$$\|x_n - u + tu\|^2 \le \|x_n - u\|^2 + 2t\langle u, x_n - u + tu\rangle$$

Which implies that

$$\mu_n\langle -u, x_n - u + tu\rangle \le 0.$$

Furthermore, we obtain, as $t \to 0,$

Hence, for $\varepsilon > 0,$ there exists $\delta > 0$ such that $\forall t\varepsilon(0,\delta)$ and for all $n \ge 1,$

$$\langle -u, x_n - u\rangle < \mu_n\langle -u, x_n - u + tu\rangle + \varepsilon \le \varepsilon$$

Consequently,

$$\mu_n\langle -u, x_n - u\rangle < \mu_n\langle -u.x_n - u + tu\rangle + \varepsilon \le \varepsilon.$$

Since ε is arbitrary, we have

$$\mu_n\langle -u, x_n - u\rangle \le 0.$$

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Furthermore, since $\lim\|x_{n+1}-x_n\|=0,$ we also have:

$$\limsup_{n\to\infty}\left(\langle -u, x_n - u\rangle - \langle -u, x_{n+1}-u\rangle\right)\leq 0,$$

From the recursion formula (3.1), we have:

$$\begin{aligned}\|x_{n+1}-u\|^2 &\leq (1-\beta_n)\|x_n-u\|^2+\beta_n k_n^2\|p_n-u\|^2\\ &\leq (1-\beta_n)\|x_n-u\|^2+\beta_n k_n^2\|y_n-u\|^2\\ &\leq (1-\beta_n)\|x_n-u\|^2+\beta_n k_n^2\left[(1-\alpha_n)\|x_n-u\|^2\right.\\ &\left.+2\alpha_n\langle u,u-x_n\rangle+\alpha_n^2\|x_n\|^2\right]\\ &\leq \left(1-\alpha_n\beta_n k_n^2\right)\|x_n-u\|^2+\alpha_n\beta_n k_n^2\left[2\langle -u,x_n-u\rangle\right]\\ &+\left[\alpha_n^2+\left(k_n^2-1\right)\right]M,\end{aligned}$$

where $M:=\sup_{n\geq 1}\left\{k_n^2\|x_n\|^2+\beta_n\|x_n-u\|^2\right\}$ Using Lemma 2.2, we get that x_n converges strongly to $u\ \varepsilon\ F$. This completes the proof.

Corollary 3.2. let K be a closed convex subset of a real Hilbert space H. Let F be a bifunction from KxK into $\Re$ satisfying (A1) – (A4), $\varphi: K\to\Re\cup\{+\infty\}$ be a proper lower semicontinous and convex function with assumption (B1) or (B2), A be a μ-inverse strongly monotone mapping of K into H and ψ be an α -inverse strongly monotone mapping of K into H. Let T be a nonexpansive mapping of K into itself andlet $\{x_n\}_{n=1,}^{\infty}, \{y_n\}_{n=1}^{\infty}$ and $\{u_n\}_{n=1}^{\infty}$ be generated by $x_1\ \varepsilon\ K,$

$$\begin{cases} y_n = P_K\left[(1-\alpha_n)x_n\right]\\ u_n = T_{rn}^{(F,\varphi)}\left(y_n - r_n\psi y_n\right)\\ x_{n+1} = (1-\beta_n)x_n+\beta_n TP_K\left(u_n - s_n Au_n\right)\end{cases}\tag{3.20}$$

For all $n\geq 1,$ *where* $\{\alpha_n\}_{n=1}^{\infty}, \{\beta_n\}_{n=1}^{\infty}$ are sequences in [0,1], $\{s_n\}_{n=1}^{\infty}\subseteq(0,\infty)$ *satisfying*:

(i) $\lim_{n\to\infty}\alpha_n = 0, \sum_{n=1}^{\infty}\alpha_n=\infty;$

(ii) $0<c\leq r_n\leq d<2\alpha, \lim_{n\to\infty}|r_{n+1}-r_n|=0;$

(iii) $0<A\leq s_n\leq b<2\mu;$ $(iv)\ 0<\liminf_{n\to\infty}\beta_n\leq\limsup_{n\to\infty}\beta_n<1.$ *Then* $\{x_n\}_{n=1}^{\infty}$ converges strongly to $u\ \varepsilon\ F$.

Application

We now study the following optimization problem:

$$\min_{u\varepsilon K}\varphi(u)\tag{4.1}$$

Where K is a nonempty closed convex bounded subset of a real Hilbert space H and $\varphi: K \to = \cup\{+\infty\}$ is a proper lower semicontinuous and convex function. We denote the set of solutions of problem (4.1) by D. Let $F(x,y)=0, \forall x, y \,\varepsilon\, K, T \equiv I \; and \; \psi \equiv 0$ in Theorem 3.1. it then follows from Theorem 3.1 that the iterative sequence (x_n) generated by

$$\begin{cases} y_n = Pk[(1-\alpha_n)x_n] \\ u_n = \arg\min_{u\varepsilon k}\left[\varphi(u)+\dfrac{1}{2r_n}\|u-y_n\|^2\right] \\ x_{n+1} = (1-\beta_n)x_n P_k(u_n - s_n A u_n) \end{cases} \tag{4.2}$$

For all $n \geq 1, where \{\alpha_n\}_{n=1}^{\infty}, \{\beta_n\}_{n=1}^{\infty}$, are sequences in [0,1], $\{s_n\}_{n=1,}^{\infty}\{r_n\}_{n=1}^{\infty} \subset (0,\infty)$ satisfying:

(i) $\lim_{n\to\infty}\alpha_n = 0, \sum_{n=1}^{\infty}\alpha_n = \infty;$

(ii) $0 < c \leq r_n \leq d < 2\alpha, \lim_{n\to\infty}|r_{n+1}-r_n| = 0;$

(iii) $0 < a \leq s_n \leq b < 2\mu;$

(iv) $0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1$ converges strongly to an element of $VI(K,A) \cap D$

Furthermore, let $F(x,x) \equiv 0, \forall x \,\varepsilon\, K, T \equiv I, A \equiv 0 \; and \; \psi \equiv 0$ in Theorem 3.1, it follows from Theorem 3.1 that the iterative sequence (x_n)generated by

$$\begin{cases} y_n = Pk[(1-\alpha_n)x_n] \\ u_n = \arg\min_{u\varepsilon k}\left[\varphi(u)+\dfrac{1}{2r_n}\|u-y_n\|^2\right] \\ x_{n+1} = (1-\beta_n)x_n + \beta_n u_n \end{cases} \tag{4.3}$$

Where (i) $\lim_{n\to\infty}\alpha_n = 0, \sum_{n=1}^{\infty}\alpha_n = \infty;$

(ii) $0 < c \leq r_n \leq d < 2\alpha, \lim_{n\to\infty}|r_{n+1}-r_n| = 0.$

(iii) $0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1$converges strongly to an element of D.

Remark 4.1. Let A be a $\mu - Lipschitzian$ and relaxed λ, γ - cocoersive map with $\lambda < \frac{r}{\mu^2}$. Then A is σ - inverse strongly monotone with $\sigma := \left(\frac{r}{\mu^2} - \lambda\right)$. In this case, the assumption that A is σ - inverse strongly monotone is weaker than the assumptions that A is μ -Lipschitzian and relaxed (λ, γ) – cocoersive imposed in theorem Sh.

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Remark 4.2. All the results of this manuscript were obtained without using the CQ method which is not convenient to implement in any possible application, and has been studied by many authors to prove their results.

Prototypes. The prototypes of our iteration parameters are:

$$\alpha_n := \frac{1}{n}, n \geq 1; \beta_n := \frac{1}{2}\left(\frac{n}{n+1}\right), n \geq 1; r_n := \frac{\alpha}{4} + \frac{d}{2(n+2)}, \text{ and}$$

$$s_n := \frac{\mu}{4} + \frac{b}{2(n+1)}, n \geq 1.\ a = \frac{\mu}{4}, b = \mu, c = \frac{\alpha}{4}, d = \alpha.$$

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