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ON THE DERIVATION AND IMPLEMENTATION OF INVERSE FOUR STAGE RUNGE-KUTTA METHODS FOR SOLVING INITIAL VALUE PROBLEMS (IVPs) IN ORDINARY DIFFERENTIAL EQUATIONS.

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ABSTRACT
A four-stage inverse Runge-Kutta method for solving the numerical solution of initial value problems (IVPs) in ordinary differential equations (ODEs) is initiated. Numerical experiments were implemented with the aid of the computer application software Maple 18 to determine the level of performance, accuracy and consistency of the method. The resulting numerical evidences show that this method gives satisfactory results.

Key Words: Runge-Kutta methods, inverse methods, four-stage.

INTRODUCTION

One of the core areas of modern mathematics is differential equations. After Newton (1665-1666) and Leibnitz (1672-1675) invention of calculus, Bernoulli (1654-1705) and Euler (1733) and others began to consider key areas in mathematical physics such as the wave and heat equations. Differential equation were solved by Newton itself in the study of planetary motion and in his analysis of optics in 1704 (see Erasmus, 2013).

All of the above speaks of modeling. Modeling is what connects the subject of differential equations to real life situation. Basically, differential equations can be implemented to model real-life questions, and graphs and computer calculations provides the real-life answers. However, in most real-life situations, the differential equation that models the problem is often complicated to solve. Thus, most of these problems are solved or analyzed along certain conditions known as " initial value conditions" which the stated problem must satisfy. Due to the complexity associated in seeking the solution of differential equations, Burden and Faires (2011) stated that their solutions can be obtained in two approaches. The first approach is by transforming the given problem to a solvable form so as to be solved by an existing analytic methods. And the second method is seeking numerical methods in the approximation of the analytic solution of the given problem.

In mathematics, an initial value problem (otherwise called the Cauchy problem by some authors) is an ordinary differential equation (ODE) subject to a constraint, called the initial function at a point in the domain of the solution of the unknown function (Simmon and Krantz, 2013). Mostly, a system in physics and other sciences always amount to an initial value problem (IVP) which on solving yields the solution.

There are few known analytics methods for seeking the solution of initial value problems (IVPs) such as the d-expansion method, the perturbation method, the Lyapunov parameter method (Njoseh and Mamadu, 2017), etc. A major disadvantage of these methods is the complexity associated in handling the problem, and does not give a precise and a compact solution form. Thus, iterative techniques have been developed and implemented by various researchers over the years to effectively solve these problems. Basically, iterative methods for initial value problems are categorized into two groups. The first are the single step methods (Euler method, improved Euler method, Heun method, the Runge-Kutta methods, etc), and the second are the multi step methods (Adams-Bashforth methods, Adams-Moulton methods, Milne's method, etc).

The earliest and most famous method of Euler was first published in the year 1768 to 1770 in his three-volume work "institutions Calculi integralis", which was republished in the year 1913 (Euler, 1913). Euler's method is built on a simple principle. If a particle is moving at time t_0 such that its position equals y_0 and, if the velocity at that time t_0 is known to be x_0, then in a short period of time (so short) that the change in position will be equal (approximately) to the change in time in multiple of x_0.

However, the quest of generalizing the Euler's method, by allowing evaluations (number of evaluation) of the derivatives at a step is attributed to Runge (1895). Further works were made by Kutta (1901) and Heun (1900). The classical Runge-Kutta methods completely characterized the work of Hairer and Wanner (1981,1982). Also, special iterative methods for computing second order ordinary differential equation were poised by Nystrom (1925), who is also generally attributed for developing method for first order ordinary differential equations. The work of Huta (1956, 1957) is generally attributed for developing sixth order methods.

An ordinary differential equation may have unique solution, infinitely many solution or no solution at all. Thus, the quest for existence and uniqueness of solution became paramount in analyzing the solution of an initial value problem. The most popular theorem on the existence and uniqueness of solution for IVPs is the "Picard existence and uniqueness theorem". The result of Picard is of fundamental significance. It can easily be used to formulate theorems of existence and uniqueness for differential equation of all orders in different context (Simmon and Krantz, 2013).

The advent of computers (digital) has emanated fresh interest on the Runge-Kutta methods. Much works were done for explicit Runge-Kutta methods by early researchers, interest has now shifted to entirely implicit Runge-Kutta methods. Although, a large number of researchers have shifted to the theory and development of particular methods, the Runge-Kutta methods (implicit) have being recognized as a more appropriate tool for evaluating stiff differential equations. For instance, Agbeboh and Omonkaro (2010) derived and implemented a new third-order rational Runge-Kutta method called the third-order inverse Runge-Kutta method for the numerical treatment of singular initial value problems in ordinary differential equations.

This present study considers the derivation and implementation of a new Runge-Kutta method called the four-stage inverse Runge-Kutta method for exploring the numerical solution of singular initial value problems in ordinary differential equations.

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MATERIALS AND METHODS

Inverse Runge-Kutta Methods Derivation

The general R-stage inverse Runge-Kutta family is given as

$$z_{i+1} = \frac{z_i}{1 + z_i \sum_{i=0}^{R} \psi_i c_i} \quad (2.1)$$

where $c_1 = hf\left(x_i, z_i\right)$,

$c_i = hf\left(x_i + hk_i, z_i + h\sum_{t=1}^{i-1} a_{it} c_t\right), \ i = 2, \cdots, R,$

with $k_2 = a_{21}, \ a_{31} = k_3 - a_{32}$, $a_{41} = k4 - a42 - a43,$

$$f(x_i, y_i) = -y_i^2 g\left(x_i, z_i\right), \ y_i = \frac{1}{z_i}{}_i,$$

$$\sum_{i=1}^{R} \psi_i = 1.$$

Applying the binomial expansion theorem on (2.1), we have

$$z_{i+1} = z_i\left(1 + z_i \sum_{i=0}^{R} w_i c_i\right)^{-1}$$

$$= z_i\left(1 - z_i \sum_{i=0}^{R} \psi_i c_i\right) + \cdots$$

$$\therefore z_{i+1} = z_i - z_i^2 \sum_{i=0}^{R} \psi_i c_i + \cdots \quad (2.2)$$

Let $R = 1$ in (2.2), we have that

$$\therefore z_{i+1} = z_i - z_i^2 \psi_1 c_1$$

From the constraints, we have that $w_1 = 1$ giving rise to

$$\therefore z_{i+1} = z_i - z_i^2 c_1 = z_i - z_i^2 hf_i \quad (2.3)$$

Rewriting (2.3) in the form (2.1), we have

$$z_{i+1} = \frac{z_i}{(1 + z_i hf_i)} \quad (2.4)$$

But, $f(x_i, y_i) = -y_i^2 g(x_i, z_i), \ y_i = \frac{1}{z_i},$

which implies that

$$f_i = -\frac{g_i}{z_i^2} \quad (2.5)$$

Substituting (2.5) into (2.4), we get

$$z_{i+1} = \frac{z_i}{1 - h\frac{g_i}{z_i}} = \frac{z_i^2}{z_i - hg_i}$$

Thus, the first - stage inverse Runge-Kutta method is given as

$$z_{i+1} = \frac{z_i^2}{z_i - hg_i} \quad (2.6)$$

Similarly, when R=2 in (3.2), we obtain

$$z_{i+1} = z_i - z_i^2(\psi_1 c_1 + \psi_2 c_2) \quad (2.7)$$

which is the second - stage inverse Runge-Kunge methods. The method is derived as follows'

Our interest in this paper is to develop a fourth –stage inverse Runge-Kutta methods. In a similar way to (2.3) and (2.7) by substituting R=4 in (2.2) yields

$$z_{i+1} = z_i - z_i^2(\psi_1 c_1 + \psi_2 c_2 + \psi_3 c_3 + \psi_4 c_4) \quad (2.8)$$

Which is the fourth-stage inverse Runge-Kunge methods. The method is derived as follows with the aid of Maple 18 software:

Now, expanding c_2 , c_3 and c_4 in a Taylor series through term of $o(h^4)$ to get

$$c_2 := hf + k_2 h^2 f_x + h^2 k_2 ff_u + h^3 k_2^2 ff_{xu} + \frac{1}{2} h^3 k_2^2 f_{xx} + \frac{1}{2} h^3 k_2^2 f^2 f_{uu} + \frac{1}{2} h^5 k_2^3 f^2 f_{xuu} + \frac{1}{2} h^4 k_2^3 ff_{xxu} + \frac{1}{6} h^4 k_2^3 f_{xxx} + \frac{1}{6} h^4 k_2^3 f^3 f_{uuu} \quad (2.9)$$

$$c_3 := hf + k_3 h^2 f_x + h\Big((k_3 - a_{32})hf + a_{32}\Big(hf + k_2 h^2 f_x + h^2 k_2 ff_u + h^3 k_2^2 ff_{xu} + \frac{1}{2}h^3 k_2^2 f_{xx} + \frac{1}{2}h^3 k_2^2 f^2 f_{uu} + \frac{1}{2}h^5 k_2^3 f^2 f_{xuu} + \frac{1}{2}h^4 k_2^3 ff_{xxu} + \frac{1}{6}h^4 k_2^3 f_{xxx} + \frac{1}{6}h^4 k_2^3 f^3 f_{uuu}\Big)\Big) f_u + h^2 k_3\Big((k_3 - a_{32})hf + a_{32}\Big(hf + k_2 h^2 f_x + h^2 k_2 ff_u + h^3 k_2^2 ff_{xu} + \frac{1}{2}h^3 k_2^2 f_{xx} + \frac{1}{2}h^3 k_2^2 f^2 f_{uu} + \frac{1}{2}h^5 k_2^3 f^2 f_{xuu} + \frac{1}{2}h^4 k_2^3 ff_{xxu} + \frac{1}{6}h^4 k_2^3 f_{xxx} + \frac{1}{6}h^4 k_2^3 f^3 f_{uuu}\Big)\Big) f_{xu} + \frac{1}{2}h^3 k_3^2 f_{xx} + \frac{1}{2}h\Big((k_3 - a_{32})hf + a_{32}\Big(hf + k_2 h^2 f_x + h^2 k_2 ff_u + h^3 k_2^2 ff_{xu} + \frac{1}{2}h^3 k_2^2 f_{xx} + \frac{1}{2}h^3 k_2^2 f^2 f_{uu} + \frac{1}{2}h^5 k_2^3 f^2 f_{xuu} + \frac{1}{2}h^4 k_2^3 ff_{xxu} + \frac{1}{6}h^4 k_2^3 f_{xxx} + \frac{1}{6}h^4 k_2^3 f^3 f_{uuu}\Big)\Big)^2 f_{uu} + \frac{1}{2}h^2 k_3\Big((k_3 - a_{32})hf + a_{32}\Big(hf + k_2 h^2 f_x + h^2 k_2 ff_u + h^3 k_2^2 ff_{xu} + \frac{1}{2}h^3 k_2^2 f_{xx} + \frac{1}{2}h^3 k_2^2 f^2 f_{uu} + \frac{1}{2}h^5 k_2^3 f^2 f_{xuu} + \frac{1}{2}h^4 k_2^3 ff_{xxu} + \frac{1}{6}h^4 k_2^3 f_{xxx} + \frac{1}{6}h^4 k_2^3 f^3 f_{uuu}\Big)\Big)^2 f_{xuu} + \frac{1}{6}h\Big((k_3 - a_{32})hf + a_{32}\Big(hf + k_2 h^2 f_x + h^2 k_2 ff_u + h^3 k_2^2 ff_{xu} + \frac{1}{2}h^3 k_2^2 f_{xx} + \frac{1}{2}h^3 k_2^2 f^2 f_{uu} + \frac{1}{2}h^5 k_2^3 f^2 f_{xuu} + \frac{1}{2}h^4 k_2^3 ff_{xxu} + \frac{1}{6}h^4 k_2^3 f_{xxx} + \frac{1}{6}h^4 k_2^3 f^3 f_{uuu}\Big)\Big)^3 f_{uuu} + h^3 \left(k_3^2\right)^{\frac{1}{2}(k_3 - a_{32})hf + \frac{1}{2}a_{32}\left(hf + k_2 h^2 f_x + h^2 k_2 ff_u + h^3 k_2^2 ff_{xu} + \frac{1}{2}h^3 k_2^2 f_{xx} + \frac{1}{2}h^3 k_2^2 f^2 f_{uu} + \frac{1}{2}h^5 k_2^3 f^2 f_{xuu} + \frac{1}{2}h^4 k_2^3 ff_{xxu} + \frac{1}{6}h^4 k_2^3 f_{xxx} + \frac{1}{6}h^4 k_2^3 f^3 f_{uuu}\right)} f_{xxu} + \frac{1}{6}h^4 k_3^3 f_{xxx}$$

Finally, Expanding c_4 We have

$$u[i+1] := u[i] - u[i]^2 \cdot \left(\sum_{i=1}^{4} c[i]\cdot w[i]\right)$$

RESULTS

Also, expanding u_{i+1} in a Taylor series through the terms of $o(h^4)$ to get

$$u_{i+1} := u_i + hf + \frac{1}{2}h^2\left(ffu + f_x\right) + \frac{1}{6}h^3\left(f_{xx} + 2ff_{xu} + f^2 f_{uu} + f_u\left(ff_u + f_x\right)\right) + \frac{1}{24}h^4\left(f^3 f_{uuu} + 4f^2 f_u f_{uu} + 3f^2 f_{uu} f_x + 5ff_u f_{xy} + f_u^2 f_x + 3ff_{xxu} + f_u f_{xx} + 3f_x f_{xu} + f_{xxx}\right) \quad (2.10)$$

$$z_{i+1} = z_i - z_i^2\left(\sum_{i=1}^{4}\psi_i c_i\right) \quad (2.11)$$

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$$z_{i+1} = \frac{z_i}{1 + z_i(\frac{1}{6}c_1 + \frac{1}{4}c_2 + \frac{1}{6}c_4)} \quad (2.12)$$

$$c_1 = hf(x_i, u_i),$$
$$c_2 = hf\left(x_i + \frac{2}{3}h, u_i + \frac{2}{3}c_1\right),$$
$$c_3 = hf\left(x_i + \frac{1}{2}h, u_i + \frac{5}{16}c_1 + \frac{3}{16}c_2\right),$$
$$c_4 = hf\left(x_i + h, u_i - \frac{1}{4}c_1 - \frac{3}{4}c_2 + 2c_3\right),$$
$$f(x_i, y_i) = -y_i^2 g(x_i, u_i),\ y_i = \frac{1}{u_i}.$$

Comparing (2.16) and (2.17) through $o(h^4)$, we get

$$A_1 := w_1 + w_2 + w_3 + w_4 = 1 \quad (2.13)$$
$$A_2 := \frac{2}{3}w_2 + \frac{1}{2}w_3 + w_4 = \frac{1}{2} \quad (2.14)$$
$$A_3 := \frac{4}{9}w_2 + \frac{1}{4}w_3 + w_4 = \frac{1}{3} \quad (2.15)$$
$$A_4 := \frac{8}{27}w_2 + \frac{1}{8}w_3 + w_4 = \frac{1}{4} \quad (2.16)$$
$$A_5 := \frac{2}{3}w_3 a_{32} + w_4\left(\frac{2}{3}a_{42} + \frac{1}{2}a_{43}\right) = \frac{1}{6} \quad (2.17)$$
$$A_6 := \frac{4}{9}w_3 a_{32} + w_4\left(\frac{4}{9}a_{42} + \frac{1}{4}a_{43}\right) = \frac{1}{12} \quad (2.18)$$
$$A_7 := \frac{1}{3}w_3 a_{32} + w_4\left(\frac{2}{3}a_{42} + \frac{1}{2}a_{43}\right) = \frac{1}{8} \quad (2.19)$$
$$A_8 := \frac{2}{3}w_4 a_{32} a_{43} = \frac{1}{24} \quad (2.20)$$

The system yields 10 unknowns satisfying only eight equations, the values of the three of them can be chosen arbitrarily provided the equations have solution. Thus, setting

$$k_2 := \frac{2}{3},\quad k_3 := \frac{1}{2},\quad k_4 := 1$$

DISCUSSIONS

In this section, we illustrate theoretical results given in section two, accuracies of the new method.

Problem 1:

Consider the initial value problem $z' = -2xz$ with $u(0) = 1$ and $h = 0.2$, in the interval (0,1), using the four-stage inverse Runge-Kutta method.

The Exact solution is $z(x) = \frac{1}{1+x^2}$.
Clearly, $g(x, z) = -2xz^2, x_0 = 0,\ z_0 = 1$.

Table 1: Showing the comparison of Results between the Four-stage Runge-Kutta method, three-stage Runge-Kutta method and exact solution for Problem 1

i	x_i	Exact solution	Four-stage inverse Runge-Kutta method	Error by Four-stage-inverse Runge-Kutta method	Three-stage inverse Runge-Kutta method	Error by Three-stage inverse Runge-Kutta method
0	0	1.0000	1.0000	0.0000E+00	1.0000	0.0000E+00
1	0.2	0.9615	0.9630	1.5000E-03	0.9625	1.005E-03
2	0.4	0.8621	0.8786	1.6500E-02	0.8957	3.3613E-02
3	0.6	0.7353	0.7877	5.2400E-02	0.8386	1.0328E-01
4	0.8	0.6098	0.7083	9.8500E-02	0.7876	1.7781E-01
5	1.0	0.500	0.6428	1.428E-01	0.7403	2.4033E-01

Table 1 above is also illustrated with figure 1 below.

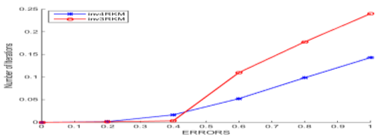

Figure 1: Comparison of Results between the Four-stage Runge-Kutta

Problem 2:

Given the initial value problem $z' = -2x,\ z(0) = 1,\ h = 0.2$, on the interval (0,1).
The exact solution is $z(x) = 1 - x^2$.

Table 2: Comparison of Results between the Four-stage Runge-Kutta method, three-stage Runge-Kutta method and exact solution for problem 2

i	x_i	Exact solution	Four-stage inverse Runge-Kutta method	Error by Four-stage-inverse Runge-Kutta method	Three-stage inverse Runge-Kutta method	Error by Three-stage inverse Runge-Kutta method
0	0	1	1	0.0000E+00	1	0.0000E+00
1	0.2	0.9600	0.9615	1.5000E-03	0.9615	1.5385E-03
2	0.4	0.8400	0.8621	2.2100E-02	0.8621	9.7931E-02
3	0.6	0.6400	0.7352	9.5200E-02	0.7353	2.2471E-01
4	0.8	0.36	0.6098	2.4980E-1	0.6098	3.5024E-01
5	1.0	0.00	0.5000	5.0000E-1	0.5000	4.6000E-01

The inverse four-stage Runge kutta methods compete favourably with other methods

Problem 3:

Consider the initial value problem $z' = 1 - \cos(x)$ with $z(0) = 1$ and $h = 0.2$, in the interval (0,1).

The Exact solution is $u(x) = -\sin(x) + x + 1$.

Clearly, $g(x, u) = 1 - \cos(x), x_0 = 0$, $u_0 = 1$.

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Table 3: Comparison of Results between the Four-stage Runge-Kutta method, three-stage Runge-Kutta method and exact solution for problem 3

i	x_i	Exact solution	Four-stage inverse Runge-Kutta method	Error by Four-stage-inverse Runge-Kutta method	Three-stage inverse Runge-Kutta method	Error by Three-stage inverse Runge-Kutta method
0	0	1	1	0.0000E+00	1	0.0000E+00
1	0.2	1.001330669	1.001332331	1.6620E-06	1.0013323	1.6620E-06
2	0.4	1.010581658	1.010694605	1.1295E-04	1.0106946	9.3639E-03
3	0.6	1.035357527	1.036653166	1.1296E-03	1.0366532	3.5322E-02
4	0.8	1.082643909	1.09008761	7.4440E-03	1.0900888	8.8758E-02
5	1.0	1.158529015	1.188394444	2.9865E-02	1.1883944	1.8706E-01

The derived four-stage inverse Runge-Kutta method have been implemented to solve initial value problems in ordinary differential equations as shown in the problems 1,2 and.3 above . Results obtained (as presented in tables) are compared with the three-stage inverse Runge-Kutta method. The results show that the approximate solution at some grid-points are very close to the exact solution as ⍰ tends to infinity. We obtained satisfactory results because the method compete favorably with the existing methods

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