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SPINLESS TWO BODY BETHE-SALPETER EQUATION WITH SHIFTED HULTHÉN POTENTIAL USING SUSYQM

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ABSTRACT
In this paper, the SUSYQM formalism is used to solve the two body spinless Salpeter equation. We obtain approximately the energy eigenvalues and the corresponding wave function in a closed form for any arbitrary l state.

Key words: shifted Hulthen potential, Supersymmetric quantum mechanics, Bethe-Salpeter equation **PACS** numbers: 03.65Ca, 03.65Pm, 03.65Db

INTRODUCTION
The Bethe-Salpeter equation is the semi-relativistic equation that describes the bound states of a two body quantum field system in a relativistic covariant formalism [20]. However, this relativistic field equation is usually refer to as the spinless Salpeter equation (SSE) after some approximations and spin degree freedom have been neglected [22]. In the recent times, there have been increasing interests in finding the analytical solutions of wave equations in relativistic and non-relativistic quantum mechanics such as Schrodinger, Klein-Gordon, Dirac, Duffin Kemmer-Petiau (DKP) and spinless Bethe-Salpeter equations with different potential models [2,7-10,14, 16 ,17]. The SSE is a generalization of Schrodinger equation in the quantum relativistic regime [11]. The solution of SSE is a very difficult task because of its nonlocal nature [13]. Consequently, many authors have resorted to approximate techniques to deal with the problems arising from the SSE [15].

In this work, the supersymmetric quantum mechanics (SUSYQM) [1, 3,12] is use in solving the SSE with shifted Hulthén potential. The Hulthén potential is one of the important short-range potentials in physics which behaves like a Coulomb potential for small values of r and decreasing exponentially for a larger values of r. Many authors have investigated the Hulthén potential in both relativistic and non-relativistic quantum regimes [5, 18, 19, 21].

MATERIALS AND METHODS
The SSE for two body particles interacting in a spherically symmetric potential in the centre of mass system appears as [20, 22],

$$\left\{\sum_{i=1,2}\left(\sqrt{-\Delta+m_i^2}-m_i\right)+V(r)-E_{n,l}\right\}\chi(r)=0, \Delta=\nabla^2, \quad (1)$$

Where $\chi(r)=R_{n,l}(r)Y_{l,m}(\theta,\phi)$. For heavy interacting particles and using appropriate transformation (see refs.[11, 20]), one can recast SSE of Eq.(1) as[11],

$$\left[-\frac{1}{2\mu}\frac{d^2}{dr^2}+\frac{l(l+1)}{2\mu r^2}+W_{nl}(r)-\frac{W_{nl}^2(r)}{2\tilde{m}}\right]\psi_{nl}(r)=0 \qquad (2)$$

Where,

$$W_{nl}(r)=V(r)-E_{nl}, \qquad (3)$$

$$\mu=\frac{m_1m_2}{m_1+m_2}, \qquad (4)$$

$$\eta=\mu\left(\frac{m_1m_2}{m_1m_2-3\mu^2}\right)^{\frac{1}{3}}, \qquad (5)$$

$$\tilde{m}=\frac{\eta^3}{\mu^2}=\frac{\left(m_1m_2\mu\right)}{\left(m_1m_2-3\mu^2\right)} \qquad (6)$$

In this work, we consider the shifted Hulthén potential defined as [6],

$$V(r)=-\frac{\left(V_0+\frac{4}{b^2}\right)e^{-\frac{2r}{b}}}{\left(1-e^{-\frac{2r}{b}}\right)^2} \qquad (7)$$

where V_0 is the strength of the potential and $\frac{1}{b}$ is the screening parameter. Substituting Eq.(7) into Eq.(2) yields,

$$-\frac{d^2\psi_{nl}}{dr^2}+\left(\frac{l(l+1)}{r^2}-2\mu\left(\frac{(V_0+\frac{4}{b^2})e^{-\frac{2r}{b}}}{\left(1-e^{-\frac{2r}{b}}\right)}+E_{nl}\right)-\frac{\mu}{\tilde{m}}\left(\frac{(V_0+\frac{4}{b^2})e^{-\frac{2r}{b}}}{\left(1-e^{-\frac{2r}{b}}\right)}+E_{nl}\right)^2\right)\psi_{nl}(r)=0 \qquad (8)$$

It can be seen that equation (8) cannot be solved analytically because of the centrifugal term. Thus, we use the following Pekeris approximation for the centrifugal term as[4],

$$\frac{1}{r^2}\approx\frac{\left(\frac{2}{b}\right)^2e^{-\frac{2r}{b}}}{\left(1-e^{-\frac{2r}{b}}\right)^2} \qquad (9)$$

Substituting equation(9) into Eq.(8) yields,

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$$-\frac{d^2\psi_{nl}}{dr^2}+\frac{1}{\left(1-e^{-\frac{2r}{b}}\right)^2}\left\{\begin{array}{l}\left(2\mu\left(V_0+\frac{4}{b^2}\right)-\frac{\mu}{\tilde{m}}\left(V_0+\frac{4}{b^2}\right)^2-\frac{2\mu E_{nl}\left(V_0+\frac{4}{b^2}\right)}{\tilde{m}}\right)e^{-\frac{4r}{b}}\\+\left(\begin{array}{l}\left(\frac{2}{b}\right)^2-2\mu\left(V_0+\frac{4}{b^2}\right)+\\\frac{2\mu E_{nl}\left(V_0+\frac{4}{b^2}\right)}{\tilde{m}}\end{array}\right)e^{-\frac{2r}{b}}\end{array}\right\}\psi_{nl}(r)$$

$$=\left(\frac{\mu E_{nl}^2}{\tilde{m}}+2\mu E_{nl}\right)\psi_{nl}(r) \qquad (10)$$

or more explicitly, we write

$$-\frac{d^2\psi_{nl}}{dr^2}+V_{eff}(r)\psi_{nl}(r)=\tilde{E}\psi_{nl}(r), \qquad (11)$$

Where,

$$V_{eff}(r)=\frac{De^{-\frac{4r}{b}}+Qe^{-\frac{2r}{b}}}{\left(1-e^{-\frac{2r}{b}}\right)^2} \qquad (12)$$

$$D=2\mu V_0+\frac{8\mu}{b^2}-\frac{\mu V_0^2}{\tilde{m}}-\frac{8\mu V_0}{\tilde{m}b^2}-\frac{16\mu}{\tilde{m}b^4}-\frac{2\mu V_0E_{nl}}{\tilde{m}}-\frac{8\mu E_{nl}}{\tilde{m}b^2}, \qquad (13)$$

$$Q=\left(\frac{2}{b}\right)^2 l(l+1)-2\mu V_0-\frac{8\mu}{b^2}+\frac{2\mu V_0E_{nl}}{\tilde{m}}+\frac{8\mu E_{nl}}{\tilde{m}b^2} \qquad (14)$$

$$\tilde{E}_{nl}=\frac{\mu E_{nl}^2}{\tilde{m}}+2\mu E_{nl} \qquad (15)$$

Supersymmetric Quantum Mechanics Method

In the SUSUQM one normally deal with the partner Hamiltonians [1, 3, 12]

$$H_{\pm}=\frac{p^2}{2m}+V_{\pm}(x), \qquad (16)$$

where

$$V_{\pm}(x)=\Phi^2(x)\pm\Phi'(x). \qquad (17)$$

In the case of good SUSY, i.e. $E_0=0$, the ground state of the system is obtained via

$$\phi_0^-(x)=Ce^{-U}, \qquad (18)$$

where C is a normalization constant and

$$U(x)=\int_{x_0}^{x}dz\Phi(z). \qquad (19)$$

Next, if the shape invariant condition

$$V_+(a_0,x)=V_-(a_1,x)+R(a_1), \tag{20}$$

where a_1 is a new set of parameters uniquely determined from the old set a_0 via the mapping $F:a_0\mapsto a_1=F(a_0)$ and $R(a_1)$ does not include x, the higher state solutions are obtained via

$$E_n=\sum_{s=1}^{n}R(a_s), \tag{21a}$$

$$\phi_n^-(a_0,x)=\prod_{s=0}^{n-1}\left(\frac{A^\dagger(a_s)}{[E_n-E_s]^{1/2}}\right)\phi_0^-(a_n,x), \tag{21b}$$

$$\phi_0^-(a_n,x)=C\exp\left\{-\int_0^x dz\Phi(a_n,z)\right\}, \tag{21c}$$

where

$$A_s^\dagger=-\frac{\partial}{\partial x}+\Phi(a_s,x). \tag{22}$$

Therefore, this condition determines the spectrum of the bound states of the Hamiltonian

$$H_s=-\frac{\partial^2}{\partial x^2}+V_-(a_s,x)+E_s. \tag{23}$$

and the energy eigenfunctions of

$$H_s\phi_{n-s}^-(a_s,x)=E_n\phi_{n-s}^-(a_s,x)\quad,\qquad n\geq s \tag{24}$$

are related via [14-16]

$$\phi_{n-s}^-(a_s,x)=\frac{A^\dagger}{[E_n-E_s]^{1/2}}\phi_{n-(s+1)}^-(a_{s+1},x). \tag{25}$$

RESULTS

In order to solve equation (11), we have to first solve the associated Riccati equation

$$W^2(r)\mp W'(r)=V_{eff}(r)-\tilde{E}_{0,l}, \tag{26}$$

for which we propose a solution of the form

$$W(r)=\frac{pe^{-\frac{2r}{b}}}{\left(1-e^{-\frac{2r}{b}}\right)}+q\qquad. \tag{27}$$

Thus, we can obtain the exact parameter of our study as,

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$$\frac{(p)^2 e^{-\frac{4r}{b}}}{\left(1-e^{-\frac{2r}{b}}\right)^2}+(q)^2+\frac{2pqe^{-\frac{2r}{b}}}{\left(1-e^{-\frac{2r}{b}}\right)}+\frac{\left(\frac{2}{b}\right)pe^{-\frac{2r}{b}}}{\left(1-e^{-\frac{2r}{b}}\right)^2}=\frac{De^{-\frac{4r}{b}}+Qe^{-\frac{2r}{b}}}{\left(1-e^{-\frac{2r}{b}}\right)^2}-\tilde{E}_{0,l} \qquad (28)$$

or more explicitly,

$$\tilde{E}_{0,l}=-(q)^2 \qquad (29)$$

$$p=-\left(\frac{1}{b}\right)\pm\sqrt{\left(\frac{1}{b}\right)^2+(D+Q)}, \qquad (30)$$

$$q=-\left(\frac{(p)^2-D}{2p}\right) \qquad (31)$$

Now based on Eq. 17, we can obtain the supersymmetric partner potentials as,

$$V_+(r)=\frac{p\left(p-\frac{2}{b}\right)e^{-\frac{4r}{b}}}{\left(1-e^{-\frac{2r}{b}}\right)^2}-\frac{De^{-\frac{2r}{b}}}{\left(1-e^{-\frac{2r}{b}}\right)}+\left(\frac{(p)^2-D}{2p}\right)^2$$

$$V_-(r)=\frac{p\left(p+\frac{2}{b}\right)e^{-\frac{4r}{b}}}{\left(1-e^{-\frac{2r}{b}}\right)^2}-\frac{De^{-\frac{2r}{b}}}{\left(1-e^{-\frac{2r}{b}}\right)}+\left(\frac{(p)^2-D}{2p}\right)^2 \qquad (32)$$

Therefore, it is shown that $V_+(r)$ and $V_-(r)$ are shape invariant, satisfying the shape-invariant condition

$$V_+(r,a_0)=V_-(r,a_1)+R(a_1), \qquad (33)$$

with $a_0=p$ and a_i is a function of a_0, i.e $a_1=f(a_0)=a_0-\left(\frac{2}{b}\right)$. Therefore, $a_n=f(a_0)=a_0-\left(\frac{2n}{b}\right)$. Thus, we can see that the shape invariance holds via a mapping of the form $p\rightarrow p-\left(\frac{2}{b}\right)$. From Eq. 20, we have

$$R(a_1)=\left(\frac{(a_0)^2-D}{2a_0}\right)^2-\left(\frac{(a_1)^2-D}{2a_1}\right)^2,$$

$$R(a_2)=\left(\frac{(a_1)^2-D}{2a_1}\right)^2-\left(\frac{(a_2)^2-D}{2a_2}\right)^2, \tag{34}$$

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$$R(a_n)=\left(\frac{(a_{n-1})^2-D}{2a_{n-1}}\right)^2-\left(\frac{(a_n)^2-D}{2a_n}\right)^2,$$

The energy eigenvalues can be obtained as follows

$$\tilde{E}_{nl}=\tilde{E}_{nl}^{-}+\tilde{E}_{0,l} \quad , \tag{35}$$

where,

$$\tilde{E}_{nl}^{-}=\sum_{k=1}^{n}R(a_k)=\left(\frac{(a_0)^2-D^{ps}}{2a_0}\right)^2-\left(\frac{(a_n)^2-D^{ps}}{2a_n}\right)^2, \tag{36}$$

By substituting Eqs.(31) and (36) into Eq.(35), we get

$$\tilde{E}_{nl}=-\left(\frac{(a_n)^2-D}{2a_n}\right)^2 \tag{37}$$

Substituting Eqs. (13-15), (26-27) and Eqs.(29-31) into Eq.(37), we obtain the energy equation for the shifted Hulthén potential as

$$\frac{\mu E_{nl}^2}{\tilde{m}}+2\mu E_{nl}+\frac{1}{b^2}\left[(n+\sigma)-\left(\frac{b}{2}\right)^2\left\{\frac{2\mu V_0+\frac{8\mu}{b^2}-\frac{\mu V_0^2}{\tilde{m}}-\frac{8\mu V_0}{\tilde{m}b^2}-\frac{16\mu}{\tilde{m}b^4}-\frac{2\mu V_0 E_{nl}}{\tilde{m}}-\frac{8\mu E_{nl}}{\tilde{m}b^2}}{n+\sigma}\right\}\right]^2=0, \tag{38}$$

Where

$$a_n=-\left(\frac{2}{b}\right)[n+\sigma], \tag{39}$$

$$\sigma=\frac{1}{2}\left(1\pm\sqrt{1+b^2\left(\begin{array}{l}2\mu V_0+\frac{8\mu}{b^2}-\frac{\mu V_0^2}{\tilde{m}}-\frac{8\mu V_0}{\tilde{m}b^2}-\frac{16\mu}{\tilde{m}b^4}\\-\frac{2\mu V_0 E_{nl}}{\tilde{m}}-\frac{8\mu E_{nl}}{\tilde{m}b^2}+\left(\frac{2}{b}\right)^2 l(l+1)-2\mu V_0-\frac{8\mu}{b^2}+\\\frac{2\mu V_0 E_{nl}}{\tilde{m}}+\frac{8\mu E_{nl}}{\tilde{m}b^2}\end{array}\right)}\right) \tag{40}$$

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The corresponding wave function becomes,

$$\psi_{nl}(r) = N_{nl}\left(e^{-\frac{2r}{b}}\right)^{\sqrt{w_3^{ps}}}(1-e^{-\frac{2r}{b}})^{1/2+\sqrt{w_1^{ps}-w_2^{ps}+w_3^{ps}+1/4}}$$

$$\times {}_2F_1\left(-n, n+2\sqrt{w_3^{ps}}+2\sqrt{\frac{1}{4}+w_1^{ps}+w_3^{ps}-w_2^{ps}}+1; 2\sqrt{w_3^{ps}}+1; e^{-\frac{2r}{b}}\right) \quad (41)$$

where

$$w_1 = \frac{b^2}{4}\left(2\mu V_0 + \frac{8\mu}{b^2} - \frac{\mu V_0^2}{\tilde{m}} - \frac{8\mu V_0}{\tilde{m}b^2} - \frac{16\mu}{\tilde{m}b^4} - \frac{2\mu V_0 E_{nl}}{\tilde{m}} - \frac{8\mu E_{nl}}{\tilde{m}b^2} - \frac{\mu E_{nl}^2}{\tilde{m}} - 2\mu E_{nl}\right), \quad (42)$$

$$w_2 = -\frac{b^2}{4}\left(\left(\frac{2}{b}\right)^2 l(l+1) - 2\mu V_0 - \frac{8\mu}{b^2} + \frac{2\mu V_0 E_{nl}}{\tilde{m}} + \frac{8\mu E_{nl}}{\tilde{m}b^2} + 2\left(\frac{\mu E_{nl}^2}{\tilde{m}} + 2\mu E_{nl}\right)\right), \quad (43)$$

$$w_3 = -\frac{b^2\left(\frac{\mu E_{nl}^2}{\tilde{m}} + 2\mu E_{nl}\right)}{4}. \quad (44)$$

where N_{nl} is the normalization constant.

In this work, we solve the two body spinless Salpeter equation with shifted Hulthén potential with proper approximation to the centrifugal term using the powerful SUSQM technique. We obtain explicitly, the bound state energy eigenvalues and the corresponding wave function in a closed form.

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