Scientia Africana, Vol. 16 (No. 2), December 2017. Pp 230-251
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FLEXURAL MOTIONS UNDER MOVING LOADS OF STRUCTURALLY PRESTRESSED NON- UNIFORM SIMPLY SUPPORTED BEAM WITH NON-CLASSICAL BOUNDARY CONDITIONS FOR MOVING MASS CASE.

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Received :27-10-17
Accepted:20-11-17

ABSTRACT

This article is sequel to [18] except that a moving mass case is considered that is the inertial term was not neglected as in [5,6,7,8,9,13,14,18]. The dynamical problem is solved using Mindlin Goodman, Generalized Generalized Galakin's method (GGM), Struble's asymptotic techniques, Laplace Integral transformations and then convolution theory as alluded to in [5,6,7,8,9,13,14,18]. Using numerical example, various plots of the deflections for beams are presented and discussed for different values of axial force N, foundation modulli K and at fixed rotatory Inertial (r) and also for fixed axial force N and foundation moduli K but at various rotatory inertial (r) for moving mass. However, both moving force and moving mass cases were compared and reported as well. Obviously, the results presented in this paper shows good agreement with what is obtainable when compared with that of existing literatures.

Key Words: Non- uniform Rayleigh Beam, Moving mass, Critical Speed, Time-Dependent and Resonance, rotatory inertia,.

INTRODUCTION

As mentioned in the abstract this work is the continuation of [18], recall that it was reported in [18] that the moving force case was obtained while that of moving mass was difficult and even cumbersome which led to the emergence of this work. The same equation procedure from the governing equations (1.0-63.0) were as reported in[18]. Hence the totality of equation (63.0) shall be solved which is called simply supported moving mass case where the inertial term which was set to zero in [5,6,7,8,9,13,18] will be strongly considered here and that is the novelty or the contribution in this paper. The introduction, methodology and the solution procedures are as in [5,6,7,8,9,13,14,]. The interest of this paper is to compare the results in [18] with the result of the moving mass in this paper as shown in the plotted graphs. Effects of some very important beam parameters on the motions of the vibrating systems are also investigated and reported.

THEORETICAL FORMULATION OF THE GOVERNING EQUATIONS

Considered here is a simply supported non-uniform Rayleigh beam resting on elastic foundation where the beams properties such as the moment of inertia I , and the mass per unit length of the beam μ vary along the span L of the beam. The r^o is the Rotatory inertia, K is the elastic foundation Modulli, x is the spatial coordinate. The transverse displacement $U(x,t)$ of the beam when it is under the action of a moving load of mass M which is moving with a non-uniform velocity such that the motion of the contact point of the moving load is described by the function

$$f(t)=\left(x_0+ct+\frac{1}{2}at^2\right) \qquad (1.0)$$

where x_0 is the point of application of force $P = Mg$ at the instance $t = 0$, c is the initial velocity and a is the constant acceleration of motion governed by the fourth order partial differential equation given by

$$\frac{\partial^2}{\partial x^2}\left[EI(x)\frac{\partial^2}{\partial x^2}U(x,t)\right]-N\frac{\partial^2}{\partial x^2}U(x,t)+\mu(x)\frac{\partial^2 U(x,t)}{\partial t^2}-\frac{\partial}{\partial x}\left[\mu(x)r^o\frac{\partial^3 U(x,t)}{\partial x\partial t^2}\right]$$

$$+M\delta\left(x-\left(x_0+ct+\frac{1}{2}at^2\right)\right)\left(\frac{\partial^2}{\partial t^2}+\frac{2c\partial^2}{\partial x\partial t}+\frac{c^2\partial^2}{\partial x^2}\right)U(x,t)+KU(x,t)=Mg\delta\left(x-\left(x_0+ct+\frac{1}{2}at^2\right)\right) \quad (2.0)$$

where g is the acceleration due to gravity, $I(x)$ is the variable moment of inertia and $\mu(x)$ is the variable mass of the Rayleigh beam per unit area. Next, the example in [7] shall be adopted and $I(x)$ and $\mu(x)$ take the forms:

$$I(x)=I_0\left(1+\sin\frac{\pi x}{L}\right)^3 \text{ and } \mu(x)=\mu_0\left(1+\sin\frac{\pi x}{L}\right) \qquad (3.0)$$

where I_o and μ_o are constants. The boundary conditions of the above equation (2.0) are taken to be time dependent, thus at each of the boundary points, there are two boundary conditions written as:

$$D_i\left[U(0,t)\right]=f_i(t) \qquad i=1,2 \text{ and}$$

$$D_i\left[U(L,t)\right]=f_i(t) \qquad i=3,4 \qquad (4.0)$$

where D_i are linear homogenous differential operators of order less than or equal to three. The initial conditions of the motion at time $t=0$ are specified by two arbitrary functions thus:

$$U(x,0)=U_0(x) \text{ and } \frac{\partial U(x,0)}{\partial t}=\dot{U}_0(x) \quad (5.0)$$

But

$$\left(1+\sin\frac{\pi x}{L}\right)^3=\frac{1}{4}\left[10-6\cos\frac{2\pi x}{L}+15\sin\frac{\pi x}{L}-\sin\frac{3\pi x}{L}\right]$$

(6.0)

Substituting equations (3.0) to (6.0) into equation (2.00) on simplifications and rearrangements,gives.

$$\frac{EI_0}{4}\frac{\partial^2}{\partial x^2}\left[\left(10-6\cos\frac{2\pi x}{L}+15\sin\frac{\pi x}{L}-\sin\frac{3\pi x}{L}\right)\frac{\partial^2 U(x,t)}{\partial x^2}\right]-N\frac{\partial^2 U(x,t)}{dx^2}+\mu_o\left(1+\sin\frac{\pi x}{L}\right)\frac{\partial^2 U(x,t)}{dt^2}$$

$$-\mu_o r^o\frac{\partial}{\partial t}\left[(1+\sin\frac{\pi x}{L})\frac{\partial^3 U(x,t)}{\partial x^2\partial t}\right]+M\delta\left(x-\left(x_0+ct+\frac{1}{2}at^2\right)\right)\left(\frac{\partial^2}{\partial t^2}+\frac{2(c+at)\partial^2}{\partial x\partial t}+\frac{(c+at)^2\partial^2}{\partial x^2}\right)U(x,t)$$

Scientia Africana, Vol. 16 (No. 2), December 2017. Pp 230-251
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$$+KU(x,t) = Mg\delta\left(x - \left(x_0 + ct + \frac{1}{2}at^2\right)\right) \quad (7.0)$$

3.0 Operational Simplifications of Equation

In this work, the initial-boundary value problem (7.0) consisting of a non-homogeneous partial differential equation with non-homogeneous boundary conditions is transformed to a non-homogeneous partial differential equation with homogeneous boundary conditions, using the Mindlin-Goodman's method described in [1-5]. In order to solve the above initial-boundary value problem. Thus, we introduce the auxiliary variable $Z(x,t)$ in the form

$$U(x,t) = Z(x,t) + \sum_{i=1}^{4} f_i(t) g_i(x) \quad (8.0)$$

Substituting equation (8.0) into the boundary value problem (7.0) and simplifying, transforms the latter into a boundary value problem in terms of $Z(x,t)$. The displacement influence functions $g_i(x)$ are chosen so as to render the boundary conditions for the boundary value problem in $Z(x,t)$ homogenous. Thus, gives;

$$\frac{EI_0}{4}\left[\left(10 - 6\cos\frac{2\pi x}{L} + 15\sin\frac{\pi x}{L} - \sin\frac{3\pi x}{L}\right)\frac{\partial^4}{\partial x^4}Z(x,t) + 6\frac{\pi}{L}\left(4\sin\frac{2\pi x}{L} + 5\cos\frac{\pi x}{L} - \cos\frac{3\pi x}{L}\right)\frac{\partial^3}{\partial x^3}Z(x,t)\right.$$

$$\left. + 3\frac{\pi^2}{L^2}\left(8\cos\frac{2\pi x}{L} - 5\sin\frac{\pi x}{L} + 3\sin\frac{3\pi x}{L}\right)\frac{\partial^2}{\partial x^2}Z(x,t)\right] + \mu_0\left(1 + \sin\frac{\pi x}{L}\right)Z_{tt}(x,t)$$

$$-\mu_0 r^o\left[\frac{\partial^2}{\partial x^2}Z_{tt}(x,t) + \sin\frac{\pi x}{L}\frac{\partial^2}{\partial x^2}Z_{tt}(x,t) + \frac{\pi}{L}\cos\frac{\pi x}{L}\frac{\partial^2}{\partial x^2}Z_{tt}(x,t)\right]$$

$$+M\delta\left(x - \left(x_0 + ct + \frac{1}{2}at^2\right)\right)\left[Z_{tt}(x,t) + \frac{2c\partial}{\partial x}Z_{tt}(x,t) + \frac{c^2\partial^2}{\partial x^2}Z_{tt}(x,t)\right] + KZ(x,t) - N\frac{\partial^2}{\partial x^2}Z(x,t)$$

$$= Mg\delta\left(x - \left(x_0 + ct + \frac{1}{2}at^2\right)\right) - \sum_{i=1}^{\Psi}\left[\frac{EI_0}{4}f_i(t)\left(10 - 6\cos\frac{2\pi x}{L} + 15\sin\frac{\pi x}{L} - \sin\frac{3\pi x}{L}\right)g_i^{IV}(x)\right.$$

$$\left. + 6\frac{\pi}{L}\left(4\sin\frac{2\pi x}{L} + 5\cos\frac{\pi x}{L} - \cos\frac{3\pi x}{L}\right)g_i^{III}(x) + 3\frac{\pi^2}{L^2}\left(8\cos\frac{2\pi x}{L} - 5\sin\frac{\pi x}{L} + 3\sin\frac{3\pi x}{L}\right)g_i^{II}(x)\right]$$

$$+\mu_0\ddot{f}_i(t)\left(1 + \sin\frac{\pi x}{L}\right)g_i(x) - \mu_0 r^o\ddot{f}_i(t)\left(g_i^{II}(x) + \sin\frac{\pi x}{L}g_i^{II}(x) + \frac{\pi}{L}\cos\frac{\pi x}{L}g_i^{I}(x)\right)$$

$$+M\delta\left(x - \left(x_0 + ct + \frac{1}{2}at^2\right)\right)\left(\ddot{f}_i(t)g_i(x) + 2c\dot{f}_i(t)g_i^{I}(x) + c^2 f_i(t)g_i^{II}(x)\right) + Kf_i(t)g_i(x) + Nf_i(t)g_i(x)\Big] \quad (9.0)$$

3.1 Method of Solution

Evidently, an exact closed form solution of the above partial differential equation does not exist. The method of separation of variables is inapplicable as difficulties arise in getting separate equations whose functions are functions of a single variable. As a result of these difficulties, one resort to an approximate method commonly called Galerkin's method.

3.2 Galerkin's Method

The Galerkin's method is used to solve equations of the form

$$\Gamma[Z(x,t)]-P(x,t)=0 \qquad (10)$$

where Γ is the differential operator.

$Z(x,t)$ is the structural displacement and

$P(x,t)$ is the transverse load acting on the structure

A solution of the form

$Z_j(x,t)=q_j(t)\phi_j(x)$ for $j=1,2,3,\ldots,n$.

is sought when j = 1,2,3, ………n. (11)

The function $\phi_j(x)$ are chosen to satisfy the approximate boundary conditions. The Galerkin's method requires that the expression (11) be orthogonal to the function $\phi_i(x)$ for $i=1,2,3,\ldots,n$.

Thus

$$\int_0^l\left[\Gamma\sum_{j=1}^{n}q_j(t)\phi_j(x)-P\right]\phi_i(x)dx=0 \qquad \text{for i}$$

=1,2,………,n (12)

This gives us a set of ordinary differential equations in $q_j(t)$ to be solved. These differential equations are called Galerkin's equations.

3.3 Analytical Approximate Solution.

The Galerkin's method requires that the solution of equation (9.0) takes the form

$$Z_n(x,t)=\sum_{m=1}^{n}Y_m(t)V_m(x) \qquad (13)$$

where $V_m(x)$ is chosen such that the desired boundary conditions are satisfied.

Equation (13) When substituted (11) into equation (9.0) yields

$$\sum_{m=1}^{\eta}\left[\frac{EI_o}{4}\left(\left(10-6\cos\frac{2\pi x}{L}+15\sin\frac{\pi x}{L}-\sin\frac{3\pi x}{L}\right)V_m^{IV}(x)+6\frac{\pi}{L}\left(4\sin\frac{2\pi x}{L}+5\cos\frac{\pi x}{L}-\cos\frac{3\pi x}{L}\right)V_m^{IIIi}(x)\right.\right.$$

$$\left.+3\frac{\pi^2}{L^2}\left(8\cos\frac{2\pi x}{L}-5\sin\frac{\pi x}{L}+3\sin\frac{3\pi x}{L}\right)V_m^{II}(x)\right)Y_m(t)-NV_mY_m(t)+\mu_o\left(V_m(x)+\sin\frac{\pi x}{L}V_m(x)\right)\ddot{Y}_m(t)$$

$$-\mu_o r^o\left(V_m^{II}(x)+\sin\frac{\pi x}{L}V_m^{II}(x)+\frac{\pi}{L}\cos\frac{\pi x}{L}V_m^{I}(x)\right)\ddot{Y}_m(t)$$

$$+M\delta\left(x-\left(x_0+ct+\frac{1}{2}at^2\right)\right)\left(V_m(x)\ddot{Y}_m(t)+2CV_m^{I}(x)\dot{Y}_m(t)+C^2V_m^{II}(x)Y_m(t)\right)$$

$$\left.+KV_m(x)Y_m(t)\ \right]-Mg\delta\left(x-\left(x_0+ct+\frac{1}{2}at^2\right)\right)+\sum_{i=1}^{4}\left[\frac{EI_o}{4}f_i(t)\left(\left(10-6\cos\frac{2\pi x}{L}+15\sin\frac{\pi x}{L}-\sin\frac{3\pi x}{L}\right)g_i^{IV}(x)\right]$$

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$$+6\frac{\pi}{L}\left(4\sin\frac{2\pi x}{L}+5\cos\frac{\pi x}{L}-\cos\frac{3\pi x}{L}\right)g_i^{III}(x)+3\frac{\pi^2}{L^2}\left(8\cos\frac{2\pi x}{L}-5\sin\frac{\pi x}{L}+3\sin\frac{3\pi x}{L}\right)g_i^{II}(x)$$

$$+$$

$$\mu_o\ddot{f}_i(t)\left(1+\sin\frac{\pi x}{L}\right)g_i(x)-\mu_o r^o\ddot{f}_i(t)\left(g_i^{II}(x)+\sin\frac{\pi x}{L}g_i^{II}(x)+\frac{\pi}{L}\cos\frac{\pi x}{L}g_i^{I}(x)\right)$$

$$+M\delta\left(x-\left(x_0+ct+\frac{1}{2}at^2\right)\right)\left(\ddot{f}_i(t)g_i(x)+2C\dot{f}_i(t)g_i^{I}(x)+C^2f_i(t)g_i^{II}(x)\right)-Nf_i(t)g_i+Kf_i(t)g_i(x)\ ]=0 \quad (14)$$

In order to determine $Y_m(t)$, it is required that the expression on the left hand side of equation (13) be orthogonal to the function $V_k(x)$.

Thus,

$$\sum_{m=1}^{n}\left[\left[H_i(m,k)+H_2(m,k)-r^0\left(H_3(m,k)+H_4(m,k)+\frac{\pi}{L}H_5(m,k)\right)\right]\ddot{Y}_m(t)\right.$$

$$+\left\{\frac{EI_o}{4}\left([10H_6(m,k)+15H_7(m,k)-6H_8(m,k)-H_9(m,k)]\right.\right.$$

$$+6\frac{\pi}{L}[4H_{10}(m,k)+15H_{11}(m,k)-H_{12}(m,k)]\ +3\frac{\pi^2}{L^2}[8H_{13}(m,k)+15H_4(m,k)+3H_{14}(m,k)]\ )$$

$$-\frac{N}{\mu_0}H_3(m,k)\ \}Y_m(t)+\frac{K}{\mu_0}H_i(m,k)\ \}Y_m(t)$$

$$\left.+\frac{M}{\mu_o}\left[H_{15}(m,k)\ddot{Y}_m(t)+2cH_{16}(m,k)\dot{Y}_m(t)+c^2H_{17}(m,k)Y_m(t)\right]\right]$$

$$-\frac{Mg}{\mu_0}V_k(ct)+[G_a(t)-G_b(t)+G_c(t)\ -G_d(t)+G_e(t)+G_f(t)-G_g(t)+G_h(t)-G_i(t)$$

$$+G_j(t)+G_k(t)+G_l(t)-G_m(t)-G_N(t)-G_o(t)\ +G_p(t)+G_q(t)+G_R(t)+G_s(t)+G_T(t)]=0 \quad (15)$$

where

$$H_i(m,k)=\int_0^L V_m(x)V_k(x)dx, \qquad H_3(m,k)=\int_o^L V_m^{II}(x)V_k(x)dx$$

$$H_2(m,k)=\int_0^L \sin\frac{\pi x}{L}V_m(x)V_k(x)dx,, \quad H_4(m,k)=\int_0^L\sin\frac{\pi x}{L}V_m^{II}(x)V_k(x)dx$$

$$H_5(m,k)=\int_0^L\cos\frac{\pi x}{L}V_m^{I}(x)V_k(x)dx, \qquad H_6(m,k)=\int_0^L V_m^{IV}(x)V_k(xdx)..$$

$$.H_7(m,k)=\int_0^L\sin\frac{\pi x}{L}V_m^{IV}(x)V_k(x)dx, \qquad H_8(m,k)=\int_0^L\cos\frac{2\pi x}{L}V_m^{IV}(x)V_k(x)dx$$

$$.H_9(m,k)=\int_0^L\sin\frac{3\pi x}{L}V_m^{IV}(x)V_k(x)dx, \qquad H_{10}(m,k)=\int_0^L\sin\frac{2\pi x}{L}V_m^{III}(x)V_k(x)dx$$

$$.H_{11}(m,k)=\int_0^L\cos\frac{\pi x}{L}V_m^{III}(x)V_k(x)dx, \qquad H_{12}(m,k)=\int_0^L\cos\frac{3\pi x}{L}V_m^{III}(x)V_k(x)dx$$

$$.H_{13}(m,k)=\int_0^L \cos\frac{2\pi x}{L}V_m^{II}(x)V_k(x)dx\,,\; H_{14}(m,k)=\int_0^L \sin\frac{3\pi x}{L}V_m^{II}(x)V_k(x)dx$$

$$H_{15}(m,k)=\int_0^L \delta\left(x-\left(x_0+ct+\frac{1}{2}at^2\right)\right)V_m(x)V_k(x)dx$$

$$H_{16}(m,k)=\int_0^L \delta\left(x-\left(x_0+ct+\frac{1}{2}at^2\right)\right)V_m^{I}(x)V_k(x)dx$$

$$H_{17}(m,k)=\int_0^L \delta\left(x-\left(x_0+ct+\frac{1}{2}at^2\right)\right)V_m^{II}(x)V_k(x)dx \qquad (16)$$

Furthermore,

$$G_a(t)=10\frac{EI_o}{4\mu_o}\sum_{i=1}^{4} f_i\int_0^L g_i^{IV}(x)V_k(x)dx\; G_b(t)=\frac{6EI_o}{4\mu_o}\sum_{i=1}^{4} f_i(t)\int_0^L \cos\frac{2\pi x}{L}g_i^{IV}(x)V_k(x)dx$$

$$G_c(t)=\frac{15EI_o}{4\mu_o}\sum_{i=1}^{4} f_i(t)\int_0^L \sin\frac{\pi x}{L}g_i^{IV}(x)V_k(x)dx\,, G_d(t)=\frac{EI_o}{4\mu_o}\sum_{i=1}^{4} f_i(t)\int_0^L \sin\frac{3\pi x}{L}g_i^{IV}(x)V_k(x)dx$$

$$,G_e(t)=\frac{24EI_o}{4\mu_o}\frac{\pi}{L}\sum_{i=1}^{4} f_i(t)\int_0^L \sin\frac{2\pi x}{L}g_i^{III}(x)V_k(x)dx$$

$$G_f(t)=\frac{30EI_o}{4\mu_o}\frac{\pi}{L}\sum_{i=1}^{4} f_i(t)\int_0^L \cos\frac{\pi x}{L}g_i^{III}(x)V_k(x)dx$$

$$G_g(t)=\frac{6EI_o}{4\mu_o}\frac{\pi}{L}\sum_{i=1}^{4} f_i(t)\int_0^L \cos\frac{3\pi x}{L}g_i^{III}(x)V_k(x)dx$$

$$G_h(t)=\frac{24EI_o}{4\mu_o}\frac{\pi^2}{L^2}\sum_{i=1}^{4} f_i(t)\int_0^L \cos\frac{2\pi x}{L}g_i^{II}(x)V_k(x)dx$$

$$G_i(t)=\frac{15EI_o}{4\mu_o}\frac{\pi^2}{L^2}\sum_{i=1}^{4} f_i(t)\int_0^L \sin\frac{\pi x}{L}g_i^{II}(x)V_k(x)dx$$

$$G_j(t)=\frac{9EI_o}{4\mu_o}\frac{\pi^2}{L^2}\sum_{i=1}^{4} f_i(t)\int_0^L \sin\frac{3\pi x}{L}g_i^{II}(x)V_k(x)dx\,, G_k(t)=\sum_{i=1}^{4}\ddot{f}_i(t)\int_0^L g_i(x)V_k(x)dx$$

$$G_l(t)=\sum_{i=1}^{4}\ddot{f}_i(t)\int_0^L \sin\frac{\pi x}{L}g_i(x)V_k(x)dx\,,\; G_m(t)=r^o\sum_{i=1}^{4}\ddot{f}_i(t)\int_0^L g_i^{II}(x)V_k(x)dx$$

$$G_N(t)=r^o\sum_{i=1}^{4}\ddot{f}_i(t)\int_0^L \sin\frac{\pi x}{L}g_i^{II}(x)V_k(x)dx\,,$$

$$G_p(t)=\frac{M}{\mu_0}\sum_{i=1}^{4}\ddot{f}_i(t)\int_0^L \delta\left(x-\left(x_0+ct+\frac{1}{2}at^2\right)\right)g_i(x)V_k(x)dx$$

$$G_q(t)=\frac{2(c+at)M}{\mu_0}\sum_{i=1}^{4}\dot{f}_i(t)\int_0^L \delta\left(x-\left(x_0+ct+\frac{1}{2}at^2\right)\right)g^{I}{}_i(x)V_k(x)dx$$

$$G_R(t)=\frac{2(c+at)^2M}{\mu_0}\sum_{i=1}^{4} f_i(t)\int_0^L \delta\left(x-\left(x_0+ct+\frac{1}{2}at^2\right)\right)g_i^{II}(x)V_k(x)dx$$

Scientia Africana, Vol. 16 (No. 2), December 2017. Pp 230-251
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$$G_s(t)=\frac{K}{\mu_o}\sum_{i=1}^{4} f_i(t)\int_0^L g_i(x)V_k(x)dx, \quad G_T(t)=\frac{N}{\mu_o}\sum_{i=1}^{4} f_i(t)\int_0^L g_I(x)V_k(x)dx \tag{17}$$

At this juncture, a solution valid for all cases of classical boundary conditions is sought. Consequently, $V_m(x)$ is chosen as the beam function given as

$$V_m(x)=\sin\frac{\lambda_m x}{L}+A_m\cos\frac{\lambda_m x}{L}+B_m\sinh\frac{\lambda_m x}{L}+C_m\cosh\frac{\lambda_m x}{L}$$

Thus, (18)

$$V_k(x)=\sin\frac{\lambda_k x}{L}+A_k\cos\frac{\lambda_k x}{L}+B_k\sinh\frac{\lambda_k x}{L}+C_k\cosh\frac{\lambda_k x}{L}$$

Consequently,

$$V_k\left(x_0+ct+\frac{1}{2}at^2\right)=\sin\lambda_k\frac{\left(x_0+ct+\frac{1}{2}at^2\right)}{L}+A_k\cos\lambda_k\frac{\left(x_0+ct+\frac{1}{2}at^2\right)}{L}+B_k\sinh\lambda_k\frac{\left(x_0+ct+\frac{1}{2}at^2\right)}{L}+$$

$$+C_k\cosh\lambda_k\frac{\left(x_0+ct+\frac{1}{2}at^2\right)}{L} \tag{19}$$

In order to evaluate the evolving integrals

$$I_1,......I_{144},H_1(m,k),.......H_{18}(m,k),H_{17}(m,n,k)...etc \quad , \tag{20}$$

Use is made of the property of the Dirac Delta function as an even function to express it in Fourier cosine series namely:

$$\delta\left[x-\left(x_0+ct+\frac{1}{2}at^2\right)\right]=\frac{1}{L}+\frac{2}{L}\sum_{n=1}^{\infty}\cos\frac{n\pi}{L}\left(x_0+ct+\frac{1}{2}at^2\right)\cos\frac{n\pi x}{L} \tag{21}$$

Thus, in view of (18), using equation (19) in equation (15), after some simplification and rearrangements yields.

$$\sum_{m=1}^{n}\Big[\alpha_0(m,k)\ddot{Y}_m(t)+\alpha_1(m,k)Y_m(t)$$
$$+\varepsilon\left(\left[H_1(m,k)+2\sum_{n=1}^{\infty}\cos\frac{n\pi}{L}\left(x_0+ct+\frac{1}{2}at^2\right)H_{1A}(m,n,k)\right]\ddot{Y}_m(t)\right.$$
$$+2c\left[H_{18}(m,k)+2\sum_{n=1}^{\infty}\cos\frac{n\pi}{L}\left(x_0+ct+\frac{1}{2}at^2\right)H_{18A}(m,n,k)\right]\dot{Y}_m(t)$$
$$\left.+c^2\left[H_3(m,k)+2\sum_{n=1}^{\infty}\cos\frac{n\pi}{L}\left(x_0+ct+\frac{1}{2}at^2\right)H_{3A}(m,n,k)\right]Y_m(t)\right)\Big]$$

$$= \frac{Mg}{\mu_0}\left[\begin{array}{l} \sin\lambda_k \frac{\left(x_0 + ct + \frac{1}{2}at^2\right)}{L} + A_k \cos\lambda_k \frac{\left(x_0 + ct + \frac{1}{2}at^2\right)}{L} + B_k \sinh \frac{\lambda k\left(x_0 + ct + \frac{1}{2}at^2\right)}{L} + \\ + C_k \cosh\lambda_k \frac{\left(x_0 + ct + \frac{1}{2}at^2\right)}{L} \end{array}\right]$$

$$-\lfloor G_a(t) - G_b(t) + G_c(t) - G_d(t) + G_e(t) + G_f(t) - G_g(t) + G_h(t) - G_i(t) +$$

$$G_j(t) + G_k(t) + G_l(t) - G_m(t) - G_n(t) - G_o(t) + G_p(t) + G_q(t) + G_r(t) + G_s(t) + G_T(t)\rfloor \qquad (22)$$

where $\in = \dfrac{mL}{\mu_0}$ (23)

$$\alpha_0(m,k) = \left[H_1(m,k) + H_2(m,k) - r^0\left(H_3(m,k) + H_4(m,k) + \frac{\pi}{L}H_5(m,k)\right)\right] \text{and}$$

$$\alpha_1(m,k) = \frac{EI_0}{4\mu_0}\left[[10H_6(m,k) + 15H_7(m,k) - 6H_8(m,k) - H_9(m,k)] - \frac{N}{\mu_0}H_3(m,k) + \frac{K}{\mu_0}H_1(m,k)\right.$$

$$\left. + 6\frac{\pi}{L}[4H_{10}(m,k) + 5H_{11}(m,k) - H_{12}(m,k)] + \frac{3\pi^2}{L^2}[8H_{13}(m,k) - 5H_4(m,k) + 3H_{14}(m,k)]\right] \qquad (24)$$

Equation (22) is the transformed equation governing the problem of non-uniform Rayleigh beam resting on a constant elastic foundation and transverse by a moving load. This second order differential equation is valid for all variants of the classical boundary conditions. In what follows, we shall consider boundary conditions such as simply supported boundary conditions as illustrative example.

3.4 Simply-Supported Boundary Conditions.

The deflection and bending moment at $x = 0$ and $x = L$ vanish for a non-uniform Rayleigh beam having simple supports at both ends.

$$Z(0,t) = 0 = Z(L,t), \frac{\partial^2 Z(0,t)}{\partial x^2} = 0 = \frac{\partial^2 Z(L,t)}{\partial x^2} \qquad (25)$$

also, for normal modes $V_m(0) = 0 = V_m(L)$,

$$\frac{d^2V_m(0)}{dx^2} = 0 = \frac{d^2V_m(L)}{dx^2} \qquad (26)$$

Similarly

$V_k(0) = 0 = V_k(L)$,

$$\frac{d^2V_k(0)}{dx^2} = 0 = \frac{d^2V_k(L)}{dx^2} \qquad (27)$$

Thus, it can be shown that

$$A_m = B_m = C_m = A_k = B_k = C_k = 0 \qquad (28)$$

with the frequency equation

$$\sin\lambda_m = \sin\lambda_k = 0 \qquad (29)$$

which implies that

$$\lambda_m = m\pi \text{ and } \lambda_k = k\pi \ . \qquad (30)$$

Substituting, equations (25) to (30) into equation (24), one obtains

$$\sum_{n=1}\left[\left(I_1 + I_{17} + r^0\frac{m^2\pi^2}{L^2}\left(I_1 + I_{17} - \frac{L}{m}I_{33}\right)\right)\ddot{Y}_m(t) + \left\{\frac{EI_0}{4\mu_0}\left(\frac{m^4\pi^4}{L^4}[10I_1 + 5I_{17} - 6I_{49} - I_{65}]\right.\right.\right.$$

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$$+\frac{m^3\pi^4}{L^4}\left[-24I_{81}+30I_{33}+6I_{97}\right]+\frac{m^2\pi^4}{L^4}\left[-24I_{49}+15I_{17}-9I_{65}\right]\Bigg)+\frac{K}{\mu_0}I_1\Bigg\}Y_m(t)$$

$$+\varepsilon\Bigg(\Bigg[I_1+2\sum_{n=1}^{\infty}\cos\frac{n\pi}{L}\left(x_0+ct+\frac{1}{2}at^2\right)I_{113}\Bigg]\ddot{Y}_m(t)+\frac{2(c+at)m\pi}{L}\Bigg[I_5+2\sum_{n=1}^{\infty}\cos\frac{n\pi}{L}\left(x_0+ct+\frac{1}{2}at^2\right)I_{129}\Bigg]\dot{Y}_m(t)$$

$$-\frac{(c+at)^2m^2\pi^2}{L^2}\left[I_1+2\sum_{n=1}^{\infty}\cos\frac{n\pi\left(x_0+ct+\frac{1}{2}at^2\right)}{L}I_{113}\right]Y_m(t)\Bigg)\Bigg]=\frac{Mg}{\mu_0}\sin\frac{k\pi}{L}\left(x_0+ct+\frac{1}{2}at^2\right)-\big[G_a(t)-G_b(t)+$$

$$G_c(t)-G_d(t)+G_e(t)+G_f(t)-G_g(t)G_h(t)-G_i(t)+G_j(t)+G_k(t)+G_l(t)-G_m(t)-G_n(t)-G_o(t)$$

$$G_p(t)+G_q(t)+G_r(t)+G_s(t)+G_T(t)\big]=0 \quad (31)$$

where the integrals ($I_1,........I_{133}$) when solved gives

$$I_1=\begin{cases}0 & \text{for } k\neq m\\ \frac{L}{2} & \text{for } k=m\end{cases} \quad (32)$$

$$I_5=\begin{cases}\frac{-2kL}{\pi\left(m^2-k^2\right)}, & \text{if } k\pm m=\text{odd}\\ 0, & \text{if } k\pm m=\text{even}\end{cases} \quad (33)$$

$$I_{17}=\begin{cases}\frac{-4mLk}{\pi\left[1-(m-k)^2\right]\left[1-(m+k)^2\right]}, & \text{if } k\pm m=\text{odd}\\ 0, & \text{if } k\pm m=\text{even}\end{cases} \quad (34)$$

$$I_{33}=\begin{cases}\frac{-2kL\left[1+m^2-k^2\right]}{\pi\left[1-(m+k)^2\right]\left[1-(m-k)^2\right]}, & \text{if } k\pm m=\text{odd}\\ 0, & \text{if } k\pm m=\text{even}\end{cases} \quad (35)$$

$$I_{49}=\begin{cases}\frac{-mL}{4}, & \text{if } k\pm m=2\\ 0, & \text{if } k\pm m\neq 2\end{cases} \quad (36)$$

$$I_{65} = \begin{cases} \dfrac{-12mkL}{\pi\left[9-(m-k)^2\right]\left[9-(m+k)^2\right]}, & \text{if } k \pm m = \text{odd} \\ 0\,, & \text{if } k \pm m = \text{even} \end{cases} \qquad (37)$$

$$I_{81} = \begin{cases} \dfrac{kL}{4}\,, & \text{if } m \pm k = 2 \\ 0\,, & \text{if } m \pm k \neq 2 \end{cases} \qquad (38)$$

$$I_{97} = \begin{cases} \dfrac{-2kL\left[9+m^2-k^2\right]}{\pi\left[9-(m+k)^2\right]\left[9-(m-k)^2\right]}, & \text{if } k \pm m \text{ odd} \\ 0\,, & \text{if } k \pm m \text{ even} \end{cases} \qquad (39)$$

$$I_{113} = \left.\frac{L}{4}\right|_{n+m=k \text{ or } n=k-m} - \left.\frac{L}{4}\right|_{n-m=k \text{ or } n=k+m} \qquad (40)$$

and

$$I_{129} = \frac{-2kL\left[n^2+m^2-k^2\right]}{\pi\left[n^2-(m+k)^2\right]\left[n^2-(m-k)^2\right]} \qquad (41)$$

Consequently,

$$2\sum_{n=1}^{\infty}\cos\frac{n\pi}{L}\left(x_0+ct+\frac{1}{2}at^2\right)I_{133} = 2L\sin\frac{n\pi}{L}\left(x_0+ct+\frac{1}{2}at^2\right)\sin\frac{m\pi}{L}\left(x_0+ct+\frac{1}{2}at^2\right) \qquad (42)$$

In view of equations (32) to (42)Simplifying and rearranging equation (31) yields

$$\sum_{m=1}^{\infty}\alpha_0^*(m,k)\ddot{Y}_m(t)+\alpha_1^*(m,k)Y_m(t)+\varepsilon\left[Q_1(t)\ddot{Y}_m(t)+Q_2(t)\dot{Y}_m(t)+Q_3(t)Y_m(t)\right]$$

$$=\frac{Mg}{\mu_0}\sin\frac{k\pi}{L}\left(x_0+ct+\frac{1}{2}at^2\right)-\left[G_a(t)-G_b(t)+G_c(t)-G_d(t)+G_e(t)+G_f(t)-G_g(t)+G_h(t)-G_i(t)\right.$$

$$\left.+G_j(t)+G_k(t)+G_l(t)-G_m(t)-G_n(t)-G_o(t)+G_p(t)+G_q(t)+G_r(t)+G_s(t)+G_T(t)\right]=0 \qquad (43)$$

where

$$\alpha_0^*(m,k)=\left(I_1+I_{17}+r^0\frac{m^2\pi^2}{L^2}\left(I_1+I_{17}-\frac{1}{m}I_{33}\right)\right) \qquad (44)$$

$$\alpha_1^*(m,k)=\frac{EI_0}{4\mu_0}\left[\frac{m^4\pi^2}{L^4}\left(5L+\frac{3}{2}mL\frac{-60mL}{\pi\left[1-(m-k)^2\right]\left[1-(m+k)^2\right]}+\frac{12mkL}{\pi\left[9-(m-k)^2\right]\left[9-(m+k)^2\right]}\right)\right.$$

$$\frac{-m^3\pi^4}{L^4}\left(6kL+\frac{12kL\left(9+m^2-l^2\right)}{\pi\left[9-(m-k)^2\right]\left[9-(m+k)^2\right]}+\frac{60kL\left(1+m^2-k^2\right)}{\pi\left[1-(m-k)^2\right]\left[1-(m+k)^2\right]}\right)$$

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$$+\frac{m^2\pi^4}{L^4}\left(6mL-\frac{60mkl}{\pi\left[1-(m-k)^2\right]\left[1-(m+k)^2\right]}+\frac{108mkl}{\pi\left[9-(m-k)^2\right]\left[9-(m+k)^2\right]}\right)$$

$$+\frac{KL}{2\mu_0}+\frac{Nm^2\pi^2}{2\mu_0 L}\;] \tag{45}$$

$$Q_1(k,m,t)=\frac{L}{2}\left(1+4\sum\sin\frac{k\pi}{L}\left(x_0+ct+\frac{1}{2}at^2\right)\sin\frac{m\pi}{L}\left(x_0+ct+\frac{1}{2}at^2\right)\right) \tag{46}$$

$$Q_2(k,m,n,t)=-4(c+at)mk\left(\frac{1}{m^2-k^2}+\frac{2\sum_{n=1}^{\infty}\left(n^2+m^2-k^2\right)\cos\frac{n\pi}{L}\left(x_0+ct+\frac{1}{2}at^2\right)}{\pi\left[n^2-(m-k)^2\right]\left[n^2-(m+k)^2\right]}\right) \tag{47}$$

and $$Q_3(k,m,t)=-\frac{(c+at)^2m^2\pi^2}{L^2}Q_1(k,m,t) \tag{48}$$

At this juncture, it is pertinent to obtain the particular functions $g_i(x)$ that ensure zeros of the right hand sides of the boundary conditions for simply supported beam. thus,

$$g_1(x)=1-\frac{x}{L},\; g_2(x)=-\frac{L}{3}x+\frac{x^2}{L}-\frac{1}{6L}x^3,$$

$$g_3(x)=\frac{x}{L} \text{ and } g_4(x)=-\frac{L}{6}x \tag{49}$$

As stated in the[7-10], it is only necessary to compute those of the $g_i(x)$ for which the corresponding $f_i(t)$ do not vanish. Thus, we need only $g_i(x)$ and $g_3(x)$ for our boundary displacement functions $f_1(t)$ and $f_3(t)$ defined in [7-10]

In view of equations (49).

$$G_a(t)=G_b(t)=G_c(t)=G_d(t)=G_e(t)=G_f(t)=G_g(t)=0$$

and $G_h(t)=G_I(t)=G_j(t)=G_m(t)=G_n(t)=G_r(t)=G_T(t)=0$

(50)

while,

$$G_k(t)=\ddot{f}_1(t)N_1+\left(\ddot{f}_3(t)-\ddot{f}_1(t)\right)\frac{1}{L}N_2 \tag{51}$$

$$G_l(t)=\ddot{f}_1(t)N_3+\left(\ddot{f}_3(t)-\ddot{f}_1(t)\right)\frac{1}{L}N_4 \tag{52}$$

$$G_0(t)=\frac{r^0N_5}{L}\left(\ddot{f}_3(t)-\ddot{f}_1(t)\right) \tag{53}$$

$$G_p(t)=\left[\begin{array}{l}\frac{M}{\mu_0}\ddot{f}_1(t)\sin\frac{k\pi}{L}\left(x_0+ct+\frac{1}{2}at^2\right)+\\ +\frac{M}{L^2\mu_0}\left(N_2+2\sum_{n=1}^{\infty}\cos\frac{n\pi}{L}\left(x_0+ct+\frac{1}{2}at^2\right)N_6\right)\left(\ddot{f}_3(t)-\ddot{f}_1(t)\right)\end{array}\right]$$

(54)

$$G_q(t)=\frac{2(c+at)m}{L\mu_0}\sin\frac{k\pi}{L}\left(x_0+ct+\frac{1}{2}at^2\right)\left(\ddot{f}_3(t)-\ddot{f}_1(t)\right) \tag{55}$$

$$G_5(t)=\left[\frac{k}{\mu_0}f_1(t)N_1+\frac{k}{L\mu_0}\left(f_3(t)-f_1(t)\right)\right] \tag{56}$$

Solving the evolving integrals ($N_{1,\ldots\ldots}N_5$) in equations (51) to (56) thus:

$$N_1=\begin{cases}\frac{2L}{k\pi}, & \text{if } (k\pm1) \text{ is even}\\ 0, & \text{if } (k\pm1) \text{ is odd}\end{cases} \tag{57}$$

$$N_2=\frac{L^2}{k\pi}(-1)^{k+1} \tag{58}$$

$$N_3=\begin{cases}0, & \text{if } k\pm1\\ \frac{L}{2}, & \text{if } k=1\end{cases} \tag{59}$$

$$N_4=\begin{cases}0, & \text{if } (1\pm k) \text{ even}\\ \frac{-4kl^2}{\pi^2\left(1-k^2\right)^2}, & \text{if } (1\pm k) \text{ odd}\end{cases} \tag{60}$$

$$N_5 = \begin{cases} 0\,, & \text{if } (1 \pm k) \text{ even} \\ \dfrac{2Lk}{\pi\left(k^2-1\right)}\,, & \text{if } (1 \pm k) \text{ odd} \end{cases} \tag{61}$$

$$N_6 = \begin{cases} \dfrac{-l^2}{2\pi\left(k^2-n^2\right)}\left[(k-n)(-1)^{k+n} + (k+n)(-1)^{k-n}\right], & \text{if } k \neq n \\ 0\,, & \text{if } k = n \end{cases} \tag{62}$$

substituting (49 to 62) into (43), simplified an arranged gives:

$$\sum_{m=1}^{n}\left[\alpha_0^*(m,k)\ddot{Y}_m(t) + \alpha_1^*(m,k)Y_m(t) + \varepsilon\left[Q_1(t)\ddot{Y}_m(t) + Q_2(t)\dot{Y}_m(t)Q_3(t)Y_m(t)\right]\right]$$

$$= \frac{Mg}{\mu_0}\sin\frac{k\pi}{L}\left(x_0 + ct + \frac{1}{2}at^2\right) - \left[F_1(t) + F_2(t) + F_3(t) + F_4(t)\right] - \varepsilon L\left[F_5(t) + F_6(t) + F_7(t) + F_8(t)\right] \tag{63}$$

Where $F_1(t) = \left(\ddot{f}_1(t) + (-1)^{k+1}\,\ddot{f}_3(t)\right)\dfrac{L}{k\pi}$ (64)

$$F_2(t) = \frac{L}{2}\,\ddot{f}_1(t) \tag{65}$$

$$F_3(t) = \left(f_1(t) + (-1)^{k+1}\,f_3(t)\right)\frac{l}{\mu_0\pi} \tag{66}$$

$$F_4(t) = \left(\ddot{f}_3(t) - \ddot{f}_1(t)\right)\left(\frac{2r^0k}{\pi\left(1-k^2\right)} - \frac{4kl}{\pi^2\left(1-k^2\right)^2}\right) \tag{67}$$

$$F_5(t) = \ddot{f}_1(t)\sin\frac{k\pi}{L}\left(x_0 + ct + \frac{1}{2}at^2\right) \tag{68}$$

$$F_6(t) = \left(\ddot{f}_3(t) - \ddot{f}_1(t)\right)\frac{(-1)^{k+1}}{k\pi} \tag{69}$$

$$F_7(t) = \left(\ddot{f}_3(t) - \ddot{f}_1(t)\right)\sum_{n=1}^{\infty}\left\{\frac{(k-n)(-1)^{k+n} + (k+n)(-1)^{k-n}}{\pi\left(k^2-n^2\right)}\right\}\cos\frac{n\pi}{L}\left(x_0 + ct + \frac{1}{2}at^2\right) \tag{70}$$

$$F_8(t) = \left(\dot{f}_3(t) - \dot{f}_1(t)\right)\frac{2(c+at)}{L}\sin\frac{k\pi}{L}\left(x_0 + ct + \frac{1}{2}at^2\right) \tag{71}$$

Equation (63) represents the transformed equation of the non-uniform Rayleigh beam simply-supported at both ends and having boundary and initial conditions which are time dependent.

In order to solve equation (63), two cases are involved, namely: Moving Force [18] and Moving Mass which is being focused in this paper.

SIMPLY SUPPORTED NON-UNIFORM RAYLEIGH BEAM TRAVERSED BY MOVING MASS

In this case, the moving load has mass commensurable with that of the beam. Consequently, $\varepsilon \neq 0$. As mentioned earlier in the previous chapter there is no exact analytical solution to this problem. Thus, we

Scientia Africana, Vol. 16 (No. 2), December 2017. Pp 230-251
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resort to the modified Struble's asymptotic technique already alluded to in this thesis. In order to solve equation (63), it is rearranged to take the form

$$\sum_{m=1}^{n}\left[\ddot{Y}_m(t)+\frac{\in Q_2(t)}{\left[\alpha_o^x(m,k)+\in Q_1(t)\right]}\dot{Y}_m(t)\right]+\frac{\alpha_1^x(m,k)+\in Q_3(t)}{\left[\alpha_o^x(m,k)+\in Q_1(t)\right]}Y_m(t)$$
$$=\frac{\varepsilon L}{\left[\alpha_0^*(m,k)+\in Q_1(t)\right]}\left[g\sin\frac{k\pi}{L}\left(x_0+ct+\frac{1}{2}at^2\right)-F_1^0(t)+F_2^0(t)+F_3^0(t)+F_4^0(t)+F_5(t)+F_6(t)+F_7(t)+F_8(t)\right] \quad (72)$$

Where

$$F_1^0(t)=\frac{L\mu_o}{k\pi m}\left(\overset{\infty}{f}_1(t)+(-1)\right)^{k+1}\overset{\infty}{f}_3(t),\quad F_2^0(t)=\frac{L\mu_o}{2m}\overset{\infty}{f}_1(t),$$

$$F_3^0(t)=\frac{L}{m\pi}\left(f_1(t)+(-1)^{k+1}f_3(t)\right),F_4^0(t)=\frac{\mu_o}{m}\left(\frac{2R^oK}{\pi(1-k^2)}-\frac{4LK}{\pi^2(1-k^2)^2}\right)\left(\overset{\infty}{f}_3(t)-\overset{\infty}{f}_1(t)\right) \quad (73)$$

and $F_5(t),F_6(t),F_7(t),and\,F_8(t)$ are as defined in equations (68-71)
As in the previous section, the homogeneous part of equation (72) is first considered as a modified frequency corresponding to the frequency of the free system due to the presence of the moving mass is sought. An equivalent free system operator defined by the modified frequency then replaces equation (72). To do this, consider a parameter $\lambda<1$ for any arbitrary mass ratio ε defined as.

$$\lambda=\frac{\varepsilon}{1+\varepsilon} \quad (74)$$

It can be shown that

$$\varepsilon_o=\lambda\left[1+0(\lambda)+0(\lambda^2)+.................\right] \quad (75)$$

All the various time dependent coefficients of the differential operator which acts on $Y_m(t)$ in equation (72) can be written in terms of λ when one considers that to $0(\lambda)$.

$$\varepsilon=\lambda \quad (76)$$

Thus, $$\frac{1}{\left[\alpha_0^*(m,k)+\lambda\frac{L}{2}\left(1+4\sum_{m=1}^{\infty}\sin\frac{k\pi}{L}\left(x_0+ct+\frac{1}{2}at^2\right)\sin\frac{m\pi}{L}\left(x_0+ct+\frac{1}{2}at^2\right)\right)\right]}=$$

$$\frac{1}{\alpha_0^*(m,k)}=\left[1-\frac{1}{\alpha_0^*(m,k)}\lambda\frac{L}{2}\left(1+4\sum_{m=1}^{\infty}\sin\frac{k\pi}{L}\left(x_0+ct+\frac{1}{2}at^2\right)\sin\frac{m\pi}{L}\left(x_0+ct+\frac{1}{2}at^2\right)\right)+0(\lambda^2)+..........\right] \quad (77)$$

where $$\left|\frac{1}{\alpha_0^*(m,k)}\lambda\frac{L}{2}\left(1+4\sum_{m=1}^{\infty}\sin\frac{k\pi}{L}\left(x_0+ct+\frac{1}{2}at^2\right)\sin\frac{m\pi}{L}\left(x_0+ct+\frac{1}{2}at^2\right)\right)\right|<1 \quad (78)$$

Now, using (76) and (77) in (72), one obtains.

$$\sum_{m=1}^{n}\left[\ddot{Y}_m(t)-\frac{\lambda\, Q_2(t)}{\alpha_0^*(m,k)}\dot{Y}_m(t)+\gamma_{mf}^2\left(1-\frac{\lambda}{\alpha_0^*(m,k)}\left(Q_1(t)-\frac{Q_3(t)}{\gamma_{mf}^2}\right)\right)Y_m(t)\right]$$

$$=\frac{\lambda L}{[\alpha_0^*(m,k)]}\left[g\sin\frac{k\pi}{L}\left(x_0+ct+\frac{1}{2}at^2\right)-\left(F_1^0(t)+F_2^0(t)+F_3^0(t)+F_4^0(t)+F_5(t)+F_6(t)-F_7(t)-F_8(t)\right)\right] \quad (79)$$

Therefore, when the effect of the mass of the particle is considered, the first approximation to the homogenous system is

$$Y_m(t)=\phi(m,t)\cos[\gamma_m t-\Psi_m] \quad (80)$$

where

$$\gamma_m=\gamma_{mf}\left[1-\lambda\left(\frac{(c+at)^2 m^2\pi^2}{L\gamma_{mf}^2}+L\right)\frac{1}{4\alpha_0^*(m,k)}\right] \quad (81)$$

is called the modified natural frequency representing the frequency of the free system due to the presence of the moving mass. Thus, the homogeneous part of (79) can be written as

$$\ddot{Y}(t)+\gamma_m^2 Y_m(t)=0 \quad (82)$$

Hence, the entire equation (72) taking into account (81) takes the form

$$\ddot{Y}(t)+\gamma_m^2 Y_m(t)$$

$$=\frac{\lambda L}{[\alpha_0^*(m,k)]}\left[g\sin\frac{k\pi}{L}\left(x_0+ct+\frac{1}{2}at^2\right)-\left(F_1^0(t)+F_2^0(t)+F_3^0(t)+F_4^0(t)+F_5(t)+F_6(t)-F_7(t)-F_8(t)\right)\right] \quad (83)$$

In order to solve the non-homogeneous equation (83).it is first simplified and re arranged to take the form.

$$\begin{aligned}
\ddot{Y}(t)+\gamma_m^2 Y_m(t)=\;&T_1^*\sin\frac{k\pi}{L}\left(x_0+ct+\frac{1}{2}at^2\right)+T_2^*\sin\Omega t+T_3^* e^{-\beta t}\sin\Omega t+T_4^* e^{-\beta t}\cos\Omega t\\
&-T_5^*\cos\left(\frac{k\pi}{L}\left(x_0+ct+\frac{1}{2}at^2\right)-\Omega t\right)+T_5^*\cos\left(\frac{k\pi}{L}\left(x_0+ct+\frac{1}{2}at^2\right)-\Omega t\right)\\
&+T_6^* e^{-\beta t}\sin\left(\frac{k\pi}{L}\left(x_0+ct+\frac{1}{2}at^2\right)+\Omega t\right)+T_6^* e^{-\beta t}\sin\left(\frac{k\pi}{L}\left(x_0+ct+\frac{1}{2}at^2\right)-\Omega t\right)\\
&-T_7^* e^{-\beta t}\cos\left(\frac{k\pi}{L}\left(x_0+ct+\frac{1}{2}at^2\right)-\Omega t\right)+T_7^* e^{-\beta t}\cos\left(\frac{k\pi}{L}\left(x_0+ct+\frac{1}{2}at^2\right)+\Omega t\right)\\
&-T_8^* e^{-\beta t}\sin\left(\frac{k\pi}{L}\left(x_0+ct+\frac{1}{2}at^2\right)+\Omega t\right)-T_8^* e^{-\beta t}\sin\left(\frac{k\pi}{L}\left(x_0+ct+\frac{1}{2}at^2\right)-\Omega t\right)\\
&-T_9^* e^{-\beta t}\sin\left(\frac{n\pi}{L}\left(x_0+ct+\frac{1}{2}at^2\right)+\Omega t\right)+T_9^* e^{-\beta t}\sin\left(\frac{n\pi}{L}\left(x_0+ct+\frac{1}{2}at^2\right)-\Omega t\right)\\
&-T_{10}^* e^{-\beta t}\cos\left(\frac{n\pi}{L}\left(x_0+ct+\frac{1}{2}at^2\right)+\Omega t\right)+T_{10}^* e^{-\beta t}\cos\left(\frac{n\pi}{L}\left(x_0+ct+\frac{1}{2}at^2\right)-\Omega t\right)\\
&-T_{11}^* e^{-\beta t}\sin\left(\frac{n\pi}{L}\left(x_0+ct+\frac{1}{2}at^2\right)+\Omega t\right)+T_{11}^* e^{-\beta t}\cos\left(\frac{n\pi}{L}\left(x_0+ct+\frac{1}{2}at^2\right)-\Omega t\right)
\end{aligned} \quad (97)$$

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where $T_1^* = g, z_0 = \frac{k\pi}{L}(c+at).and.z_1 = \frac{n\pi}{L}(c+at)$ (98)

$$T_2^* = \left(\frac{L\mu_0\Omega^2}{k\pi m} - \frac{L\mu_0\Omega^2}{2m} + \frac{L}{\pi m} - \frac{\mu_0\Omega^2}{m}\left(\frac{2R^0k}{\pi(1-k^2)} - \frac{4Lk}{\pi^2(1-k^2)^2} \right) + \frac{\Omega^2(-1)^{k+1}}{k\pi} \right)$$

(99)

$$T_3^* = \left(\frac{-L\mu_0(\beta^2-\Omega^2)(-1)^{k+1}}{k\pi m} + \frac{L(-1)^{k+1}}{m\pi} + \frac{\mu_0(\beta^2-\Omega^2)}{m}\left(\frac{2R^0k}{\pi(1-k^2)} - \frac{4Lk}{\pi^2(1-k^2)^2} \right)(-1)^{k+1} + \frac{\beta^2-\Omega^2(-1)^{k+1}}{k\pi} \right) \quad (100)$$

$$T_4^* = \left(\frac{2L\mu_0\,\beta\Omega(-1)^{k+1}}{k\pi m} - \frac{2\mu_0\beta\Omega(-1)^{k+1}}{m}\left(\frac{2R^0k}{\pi(1-k^2)} - \frac{4Lk}{\pi^2(1-k^2)^2} \right) - \frac{2\beta\Omega}{k\pi}(-1)^{k+1} \right. \quad (101)$$

$$T_5^* = \Omega^2/2, T_6^* = (c+at)\Omega/L, T_7^* = (c+at)\beta/L, \quad (102)$$

$$T_9^* = \frac{(\beta^2-\Omega^2)}{2}\sum_{n=1}^{\infty}\left[\frac{(k-n)(-1)^{k+n} + (k+n)(-1)^{k-n}}{\pi(k^2-n^2)} \right] \quad (103)$$

$$T_{10}^* = \beta\Omega\sum_{n=1}^{\infty}\left[\frac{(k-n)(-1)^{k+n} + (k+n)(-1)^{k-n}}{\pi(k^2-n^2)} \right] \quad 104)$$

$$T_{11}^* = \frac{\Omega^2}{2}\sum_{n=1}^{\infty}\left[\frac{(k-n)(-1)^{k+n} + (k+n)(-1)^{k-n}}{\pi(k^2-n^2)} \right] \quad (105)$$

Equation (97) is solved using Laplace transformation and convolution theory, after simplifications and rearrangements one obtains

$$\begin{aligned} Y_m(t) = \frac{1}{\gamma_m}\Big[& T_1^*I_1^1 - T_1^*I_2^1 + T_2^*I_3^1 - T_2^*I_4^1 + T_3^*I_5^1 - T_3^1I_6^1 + T_4^*I_7^1 - T_4^1I_8^1 - T_5^*I_9^1 + T_5^1I_{10}^1 + T_5^*I_{11}^1 - T_5^1I_{12}^1 \\ & T_6^1I_{13}^1 - T_6^1I_{14}^1 + T_6^1I_{15}^1 - T_6^1I_{16}^1 - T_7^1I_{17}^1 + T_7^1I_{18}^1 + T_7^1I_{19}^1 - T_7^1I_{20}^1 - T_{,6}^1I_{21}^1 + T_6^1I_{22}^1 \\ & - T_6^1I_{23}^1 + T_6^1I_{24}^1 - T_9^1I_{25}^1 + T_9^1I_{26}^1 + T_9^*I_{27}^1 - T_9^1I_{28}^1 + T_{10}^*I_{29}^1 - T_{10}^1I_{30}^1 + T_{10}^*I_{31}^1 - T_{10}^1I_{32}^1 - T_{11}^*I_{33}^1 + T_{11}^1I_{34}^1 \\ & + T_{11}^*I_{35}^1 - T_{11}^1I_{36}^1 + c^0\sin\gamma_m t \Big] \end{aligned} \quad (106)$$

where the evolving integrals $(I_1^1.to.I_{34}^1)$ are evaluated and substituted into (106) simplifying and rearranging, one obtains

$$\begin{aligned} Y_m(t) = {} & A_{10}\sin z_o t + A_{20}\sin\gamma_m t + A_{30}\sin\Omega t + A_{40}\cos\gamma_m t + A_{50}e^{-\beta t}\cos\Omega t \\ & + A_{60}e^{-\beta t}\sin\Omega t + A_{70}\cos(z_o-\Omega)t + A_{80}(z_o+\Omega)t + A_{90}e^{-\beta t}\cos\gamma_m t + A_{91}e^{-\beta t}\sin\gamma_m t \\ & + A_{92}e^{-\beta t}\sin(z_o-\Omega)t + A_{93}e^{-\beta t}\sin(z_o+\Omega)t + A_{94}e^{-\beta t}\cos(z_o+\Omega)t + A_{95}\sin(z_o+\Omega)t \\ & + A_{96}\sin(z_o-\Omega)t + A_{97}e^{-\beta t}\cos(z_o+\Omega)t + A_{98}e^{-\beta t}\sin(z_o+\Omega)t + A_{99}e^{-\beta t}\cos(z_1-\Omega)t \\ & + A_{991}e^{-\beta t}\sin(z_1-\Omega)t + A_{992}\sin(z_1+\Omega)t + A_{993}\sin(z_1-\Omega)t \end{aligned} \quad (107)$$

Where

$$A_{10} = \frac{-T_1}{\gamma_m^2 - z_o^2}, A_{20} = \frac{1}{\gamma_m}\left[\frac{-T_1 z_o}{\gamma_m^2 - z_o^2} + \frac{-T_2\Omega}{\gamma_m^2 - \Omega} + \frac{1}{Q_o}\left(T_3\Omega\left(\beta^2 + \Omega^2 - \gamma_m^2\right) + T_4\beta\left(\beta^2 + \gamma_m^2 + \Omega^2\right)\right)\right.$$

$$-T_6\left(\frac{(z_o+\Omega)}{Q_1}\left(\beta^2 + (z_o+\Omega)^2 - \gamma_m^2\right) - \frac{(z_o-\Omega)}{Q_2}\left(\beta^2 + (z_o-\Omega)^2 - \gamma_m^2\right)\right)$$

$$-T_7\beta\left(\frac{\left(\beta^2 + \gamma_m^2 + (z_o+\Omega)^2\right)}{Q_2} - \frac{\left(\beta^2 + \gamma_m^2 (z_o+\Omega)^2\right)}{Q_1}\right) + T_6\left(\frac{z_o+\Omega}{\gamma_m^2 + (z_o+\Omega)^2} + \frac{z_o-\Omega}{\gamma_m^2 - (z_o-\Omega)^2}\right)$$

$$-T_9\left(\frac{(z_1+\Omega)\left(\beta^2 + (z_1+\Omega)^2 - \gamma_m^2\right)}{Q_3} - \frac{(z_1-\Omega)\left(\beta^2 + (z_1-\Omega)^2 - \gamma_m^2\right)}{Q_4}\right)$$

$$+T_{10}\beta\left(\frac{\left(\beta^2 + \gamma_m^2 + (z_1+\Omega)^2\right)}{Q_3} + \frac{\left(\beta^2 + \gamma_m^2 + (z_1-\Omega)^2\right)}{Q_4}\right)$$

$$\left.+T_{11}\left(\frac{z_1+\Omega}{\gamma_m^2 - (z_o+\Omega)^2} - \frac{(z_1-\Omega)}{\gamma_m^2 - (z_1-\Omega)^2} + C^o\right)\right]$$

$$A_{40} = \left[\frac{1}{Q_o}\left(2\beta\Omega T_3 - \left(\beta^2 + \gamma_m^2 - \Omega^2\right)\right) + T_s\left(\frac{1}{\gamma_m^2 - (z_o-\Omega)^2} + \frac{1}{\gamma_m^2 - (z_o+\Omega)^2}\right) + 2\beta T_6\left(\frac{z_o+\Omega}{Q_1} - \frac{z_o-\Omega}{Q_2}\right)\right.$$

$$+T_7\left(\frac{\left(\beta^2 + \gamma_m^2 - (z_o-\Omega)^2\right)}{Q_2^!} - \frac{\left(\beta^2 + \gamma_m^2 - (z_o+\Omega)^2\right)}{Q_1^!}\right) + 2T_9\beta\left(\frac{z_1+\Omega}{Q_3^!} - \frac{z_1-\Omega}{Q_4^!}\right)$$

$$\left.+T_{10}\left(\frac{\left(\beta^2 + \gamma_m^2 - (z_1+\Omega)^2\right)}{Q_3^!} - \frac{\left(\beta^2 + \gamma_m^2 - (z_1-\Omega)^2\right)}{Q_4^!}\right)\right]$$

$$A_{50} = \frac{1}{Q_0^!}\left(T_3 + T_4\left(\beta^2 +^2 - \gamma_m^2 - \Omega^2\right)\right), A_{60} = \frac{1}{Q_0^!}\left(T_3 + \left(\beta^2 +^2 - \gamma_m^2 - \Omega^2\right) - 2\beta\Omega T_4\right),$$

$$A_{70} = \frac{-T_5}{\gamma_m^2 - (z_o-\Omega)^2}, \qquad A_{80} = \frac{-T_5}{\gamma_m^2 - (z_o+\Omega)^2}, \quad A_{30} = \frac{T_2}{\gamma_m^2 - \Omega^2}$$

$$A_{90} = \left[-2\beta T_6\left(\frac{z_o+\Omega}{Q_1^!} + \frac{z_o-\Omega}{Q_2^!}\right) - 2\beta T_9\left(\frac{z_1+\Omega}{Q_3^!} + \frac{z_1-\Omega}{Q_4^!}\right)\right]$$

$$A_{91} = \frac{1}{\alpha_m}\left[T_6\left(\frac{(z_o+\Omega)\left(\beta^2 + (z_o+\Omega)^2 - \gamma_m^2\right)}{Q_1^!} - \frac{(z_o-\Omega)\left(\beta^2 + (z_o+\Omega)^2 - \gamma_m^2\right)}{Q_2^!}\right)\right.$$

$$\left.+T_9\left(\frac{(z_o+\Omega)\left(\beta^2 + (z_1+\Omega)^2 - \gamma_m^2\right)}{Q_3^!} - \frac{(z_o-\Omega)\left(\beta^2 + (z_1+\Omega)^2 - \gamma_m^2\right)}{Q_4^!}\right)\right]$$

$$A_{92} = \frac{2\beta T_7 (z_o-\Omega)}{Q_2^1}, \qquad A_{93} = \frac{-2\beta T_7 (z_o+\Omega)}{Q_1^1},$$

$$A_{95} = \frac{-T_6}{\gamma_m^2 - (z_o+\Omega)^2}, \qquad A_{96} = \frac{-T_6}{\gamma_m^2 - (z_o-\Omega)^2}$$

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$$A_{94} = -T_7\left[\frac{\left(\beta^2+\gamma_m^2-(z_o-\Omega)^2\right)}{Q_2^1}+\frac{\left(\beta^2+\gamma_m^2-(z_o+\Omega)^2\right)}{Q_1^1}\right]$$
$$A_{97} = \frac{T_{10}\left(\beta^2+\gamma_m^2-\left(z_1+\Omega^2\right)\right)}{Q_3}, \qquad \mathrm{A}_{98} = \frac{-2\beta\left(z_1+\Omega^2\right)T_{10}}{Q_3}$$
$$A_{99} = \frac{T_{10}\left(\beta^2+\gamma_m^2-\left(z_1-\Omega^2\right)\right)}{Q_4^!}, \qquad \mathrm{A}_{991} = \frac{-2\beta\left(z_1-\Omega^2\right)T_{10}}{Q_4^!}$$
$$A_{992} = \frac{-T_{11}}{\gamma_m^2-(z_1+\Omega)^2} \text{ and } \quad A_{993} = \frac{T_{11}}{\gamma_m^2-(z_1-\Omega)^2} \qquad (108)$$

Where
$$Q_o^* = \beta^4+\gamma_m^4+\Omega^4+2\left[\beta^2\gamma_m^2+\beta^2\Omega^2-\gamma_m^2\Omega^2\right]$$
$$Q_1^* = \beta^4+\gamma_m^4+(z_o+\Omega)^4+2\left[\beta^2\gamma_m^2+\beta^2(z_o+\Omega)^2-2\gamma_m^2(z_o+\Omega)^2\right]$$
$$Q_2^* = \beta^4+\gamma_m^4+(z_o-\Omega)^4+2\left[\beta^2\gamma_m^2+\beta^2(z_o-\Omega)^2-2\gamma_m^2(z_o-\Omega)^2\right]$$
$$Q_3^* = \beta^4+\gamma_m^4+(z_1+\Omega)^4+2\left[\beta^2\gamma_m^2+\beta^2(z_1+\Omega)^2-2\gamma_m^2(z_1+\Omega)^2\right]$$
$$Q_4^* = \beta^4+\gamma_m^4+(z_1-\Omega)^4+2\left[\beta^2\gamma_m^2+\beta^2(z_1-\Omega)^2-2\gamma_m^2(z_1-\Omega)^2\right] \qquad (109)$$

Therefore,
$$Z_n(x,t) = \sum_{m=1}^{n} Y_m(t)\sin\frac{m\pi x}{L} \qquad (110)$$
Consequently, in view of the inverse of equation (110),the solution becomes

$$U(x,t) = Z(x,t)+\sum_{l=1}^{t} f_i(t)g_i(t)$$

$$= Z(x,t)+\sin\Omega t+\left(e^{-\beta t}-1\right)\left(x/L\right)\sin\Omega t \quad (111)$$
Equation (111) is the dynamic response of a non-uniform Rayleigh beam to Moving Mass whose two simply supported edges undergo displacements which vary with time.

DISCUSSION OF THE ANALYTICAL SOLUTION

If the undamped system such as this is studied, it is desirable to examine the response amplitude of the dynamical system which may grow without bound. This is termed resonance when it occurs. Equation (81) clearly shows that the simply supported elastic Rayleigh beams transverse by a moving force will be in state of resonance whenever
$$\gamma_{mf} = \frac{m\pi u}{L} \qquad (112)$$
While equation (80) shows that the same beam under the action of moving mass experiences resonance effect when
$$\gamma_m = \frac{m\pi u}{L} \qquad (113)$$
From equation (81),
$$\gamma_m = \gamma_{mf}\left[1-\lambda\left(\frac{(c+at)^2m^2\pi^2}{L\gamma_{mf}^2}+L\right)\frac{1}{4\alpha_0^*(m,k)}\right]$$

This implies, $$\gamma_{mf}^2 = \frac{\alpha_1^*(m,k)}{\alpha_0^*(m,k)} \qquad (114)$$

From equations (112) and (113), we deduced for the same natural frequency, the critical speed for the system of a simply supported elastic beam on an elastic foundation and traversed by a moving force is greater than that traversed by moving mass. Thus, resonance is reached earlier in the moving mass system than in the moving force system.

4.1.1 Numerical Calculation and Discussion of the Results for Non-Uniform Simply Supported Beam.

In order to illustrate the analytical results in dynamics of structures and Engineering designs for example considered, the non-uniform Rayleigh beam is taken to be of length L=12.192m, the load velocity u = 8.123 and $E = 2109 * 10^9 kg / m$ .The values of the foundation moduli K varied between 0 and 400000 ,axial force NA varied between 0 and 40000 and for fixed values of rotatory inertia R=1.The traverse deflections of the non-uniform Rayleigh beam are calculated and plotted against time for values of rotatory inertia, axial force and foundation stiffness K. Fig. 1 shows response of simply supported moving mass of a non-uniform Rayleigh beam for fixed value of rotatory inertia , fixed value of axial force NA=400000 and various values of foundation moduli K = 0 to K = 4000000. From the graph it shows that the response

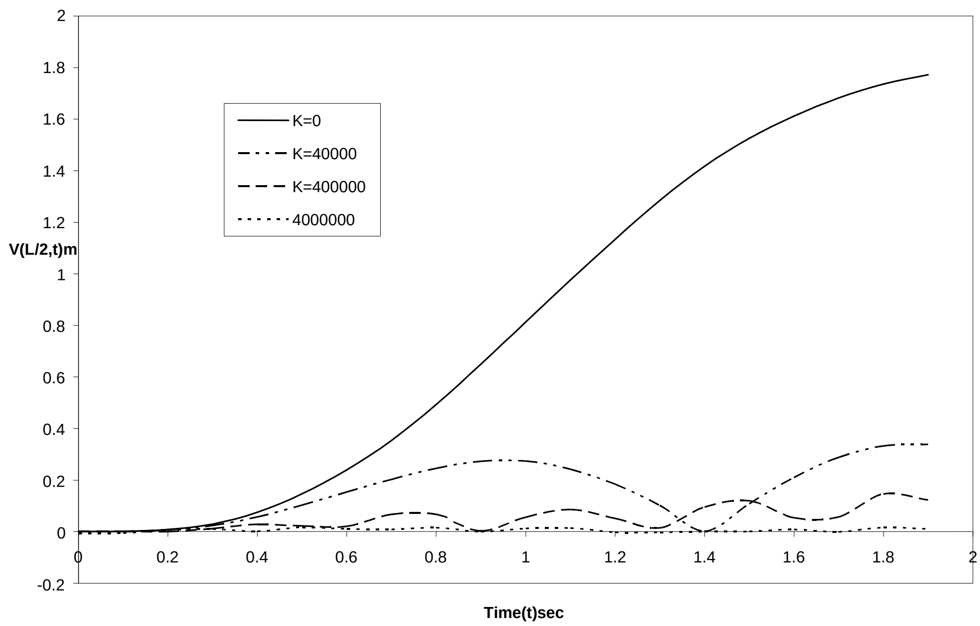

Figure 1: Deflection profile of a simply surported non-uniform Rayleigh beam under the action of moving mass for various values of foundation modulus K and for fixed values of axial force N=20000 and rotatory inertia R=1

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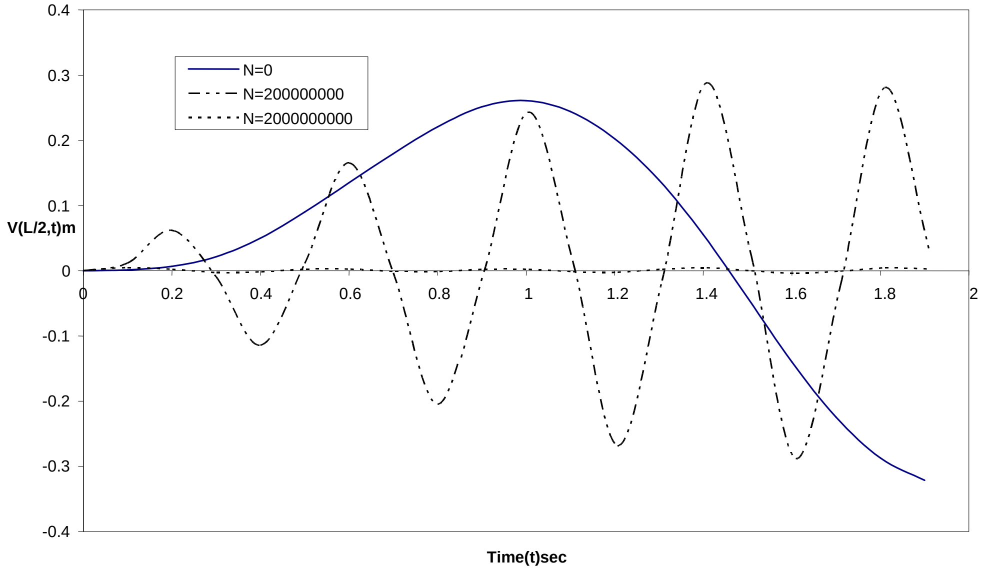

Figure 2: Deflection Profile of a simply surported Rayleigh beam under the actions of moving mass for various values of axial force N and for fixed values of Rotatory inertia R=1 and foundation modulus K=40000

amplitude decreases as the values of the foundation moduli K increases. While, fig. 2,shows the deflection profile of simply supported Non-Uniform Rayleigh beam under the action of moving mass for various values of axial force N=0 to 2000000 and fixed value of Rotatory inertia R=3 and fixed value of foundation modulus K= 20000. The graph reads that as the axial force increases the deflection profile decreases. However, fig.3, exhibits deflection profile of simply supported moving mass of Non-Uniform Rayleigh beam for various values of rotatory inertial R=0 to 3 and for fixed value of axial force N=20000 and for fixed value of foundation modulus K=40000. From the graph it shows

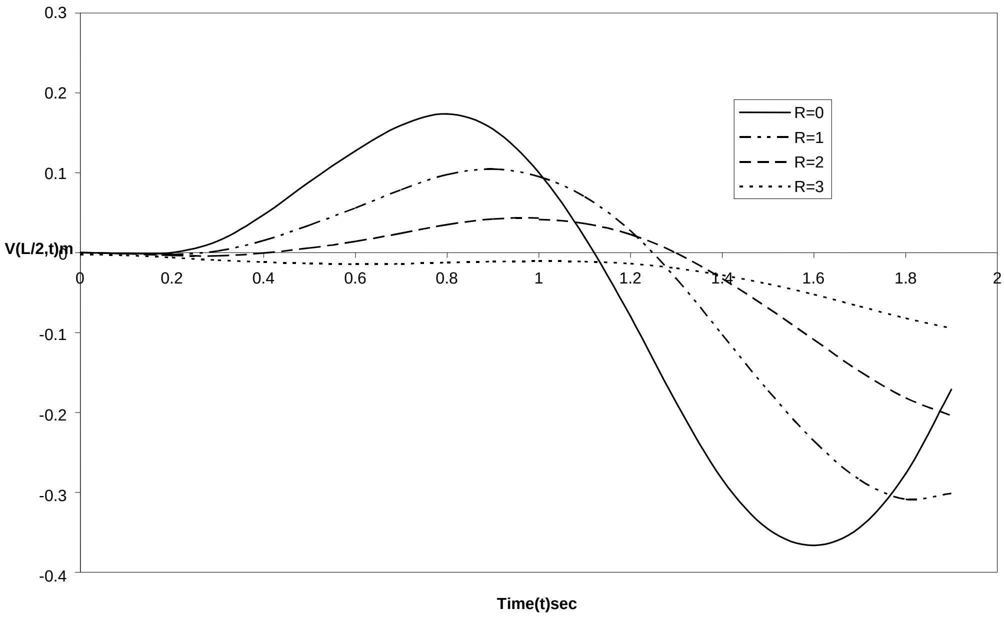

Figure 3: Deflection profile for simply supported non-uniform Raylegih beam under the action of moving mass for various values of rotatory inertia and for fixed values of axial force N=20000 and foundation modulus K=40000

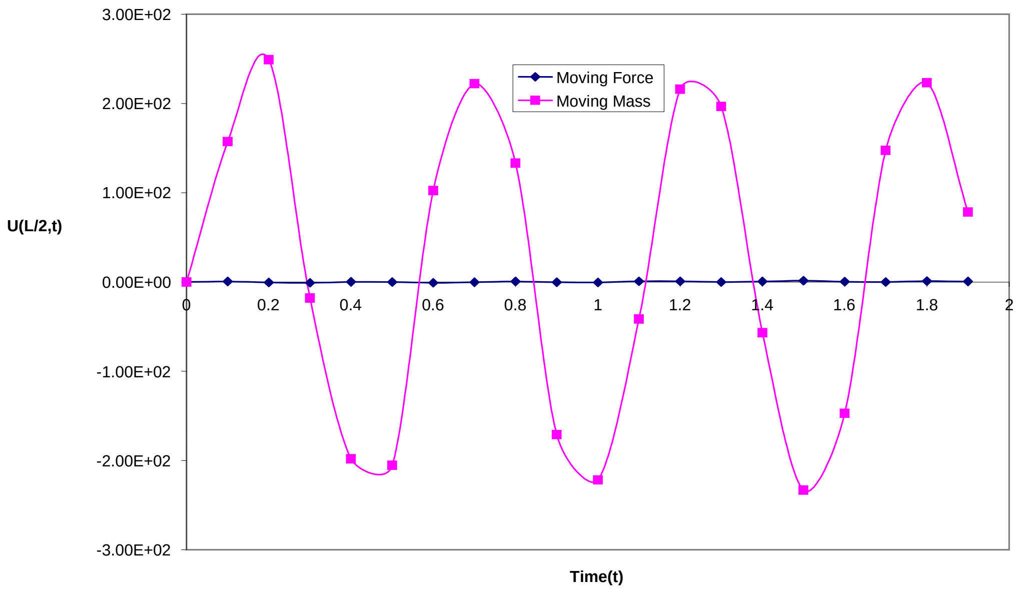

Figure 4: Comparison of the Moving force and Moving mass cases for simply supported non-uniform Rayleigh beam for fixed value of foundation modulli K=40000 and rotatory inertia R=1

that the response amplitude decreases as the values of Rotatory inertial R increases .However, fig.4: shows the comparison of the moving force and moving mass simply supported Non-uniform Rayleigh beams for fixed value of foundation moduli K= 40000, fixed value of axial force NA=40000 and rotatory inertia R=1, respectively.

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