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GENERALIZED LYAPUNOV EQUATION FOR THE STABILITY ANALYSIS OF LINEAR TIME – INVARIANT SYSTEMS.

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ABSTRACT
In this paper, a generalized Lyapunov equation is presented to check the behavior of Linear Time-Invariant System, which is used to determine criteria for the stability of the system. The existence of generalized Lyapunov quadratic equation is established and used to study the exponential stability of the system. Illustrative examples have been given to show the effectiveness and the accuracy of the method used.

Key words: Lyapunov Equation, LTI System Analysis, exponential stability.

INTRODUCTION
Linear Time-Invariant System (LTI) analysis plays a central role in control theory, circuits, signal processing, seismology, and spectroscopy etc. LTI system is used to model systems dynamics in science and engineering field and it is usually represented by a set of first order differential equations. Over the years, several analytic method for stability of Linear Time-Invariant System have been proposed by different researchers. These methods includes, Quadratic Inequality and Frequency-domain Techniques (Bugong.2000), Lyapunov Techniques (Weiming et al. 2016, Rifat and Neijat. 2006, Zhang et al. 2009), Polytopic Domain (Dong et al. 2013), Efficient and Exhaustive Frequency-Sweeping Techniques within a Single Loop (Rifat and Neijat 2009), Homogenous Parameter-Dependent Quadratic Lyapunov function HPD-QLFs (Graziano 2010), One-dimensional Routh-Hurwitz Test (De, 2006), Imaginary Spectra (Gingbin and Neijat. 2017, 2016).

In some of these papers, the method used gave a necessary and sufficient conditions for the stability of a class of LTI system. The condition for the asymptotic stability or the stability can be solve by a generalized eigenvalue problem. See (Granziano 2010). Rifat and Neijat (2009) worked on the stability analysis of LTI systems with three independent delays. They introduced practical numerical steps for determining the Robustness behavior of a general class of higher order LTI systems against uncertainties in the delay. They also introduce a new numerical procedures and also applied an efficient and exhaustive frequency-sweeping technique within a single loop. In the study of robust stability analysis of continuous-time uncertain linear system in polytopic domain, a lyapunov functional approach in the analysis and a sufficient linear matrix inequality condition was also presented in (Dong and young, 2013). Robust stability of the general class of vector LTI equation with a single delay was considered in the work of Rifat and Neijat (2006). The cluster treatment of characteristic roots was used and the procedure requires a complete and precise determination of the imaginary spectra of the system. Zhang (2012) worked on the stability condition of a LTI for bounded-input bounded-out system over integral interval (0, 1) was established. The necessary and sufficient conditions of stability system was derived and the numerical examples was presented to illustrate the condition

In this paper, we apply the lyapunov theory to investigate and analyzed the stability of LTI systems,

$$\frac{dy}{dt} = f(t, y) \qquad (1.1)$$

where $y(t)$ is the state which takes values from $X \in R^n$, and $X = R^n$ denotes the n-dimensional Euclidean space with corresponding norm $\|\circ\|$. The function $f(t, y): R^+ \times X \to X$ and $f(t,0) = 0$ for all $t \in R^+$. We shall assume that conditions are imposed on equation (1.1) such that existence of solution is guaranteed.

Preliminaries

In this section, we present some basic definitions of Lyapunov functions and theorems which are useful for our discussion.

Definition 2.1: The point $y_e \in R^n$ is an equilibrium point for the linear time invariant system (1.1) if $f(t, y) = 0$ for all t.

Definition 2.2: A set $M \subset R^n$ is open if for every $m \in M, \exists\, Br(m) \supset M$

Definition 2.3: A system (y_e) is said to be stable if for every $\varepsilon > 0$, $\exists\, \delta > 0$

$$\|y(0) - y_e\| < \delta, \Rightarrow \|y(t) - y_e\| < \varepsilon \ \forall\, t \geq t_0$$

Definition 2.4: The equilibrium point is called unstable equilibrium if the above condition is not satisfied'

Definition 2.5: A function $V(y): D \to R$ is said to be positive definite on the set D if $V(0) = 0$ and $V(y) > 0$, for all $y \neq 0$ and $y \in D$.

Definition 2.6: A function $V(y): D \to R$ is said to be positive semi definite in the set D when $V(y)$ has positive sign throughout D except at certain points (including the origin) where it is zero.

Definition 2.7: A function $V(y): D \to R$ is said to be negative definite (negative semi definite) on the set if and only if $-V(y)$ is positive definite (positive semi definite) on D.

Definition 2.8: The real symmetric $n \times n$ matrix B is said to be positive definite if the leading principal minors of B are all positive.

Theorem 2.1: (Stability); Given an equilibrium point $y = 0$ of $\dot{y} = f(y)$, let $V: D \to R$ be a continuously differentiable function such that:

i. $V(0) = 0$

ii. $V(y) > 0, in\ D$

iii. $\dot{V}(y) \leq 0, in\ D$

then $y = 0$ is "stable"

Theorem 2.2: (Asymptotically Stable)

Given equilibrium point $y = 0$ of $\dot{y} = f(y)$, $f: D \to R^n$. Let $V: D \to R$ be a

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continuously differentiable function such that:
i. $V(0) = 0$
ii. $V(y) > 0, in\ D$
iii. $\dot{V}(y) < 0, in\ D$
then $y = 0$ is "asymptotically stable"

Theorem 2.3: (Globally Asymptotically Stable).
Let $y = 0$ be an equilibrium point of $\dot{y} = f(y), \quad f: D \to R^n$. Let $V: D \to R$ be a continuously differentiable function such that:
i. $V(0) = 0$
ii. $V(y) > 0, in\ D$
iii. $\dot{V}(y) \leq 0, in\ D$
iv. $V(y)\ is\ radially\ unbounded.$
then $y = 0$ is " globally asymptotically stable"

Main result

In this section we will present the concept of generalized Lyapunov function. We say that a scalar valued function V is a generalized Lyapunov function for LTI system $\dot{y} = f(t, y)$ if for all nonzero $y \in X$ we have $V(y) > 0$ and $\dot{V}(y) < 0$; V is called a generalized Lyapunov function of if V is a Lyapunov function and V can be written as $V(y) = y^T Py$ for some symmetric matrix P, in this case we say that P is a Lyapunov matrix for the system.

Given a linear time-Invariant system

$$\begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \dot{y}_3 \\ \cdot \\ \cdot \\ \cdot \\ \dot{y}_n \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & \cdot & \cdot & \cdot & C_{1n} \\ C_{12} & C_{22} & C_{23} & \cdot & \cdot & \cdot & C_{2n} \\ C_{13} & C_{23} & C_{33} & \cdot & \cdot & \cdot & C_{3n} \\ \cdot & \cdot & \cdot & & & & \\ \cdot & \cdot & \cdot & & & & \\ \cdot & \cdot & \cdot & & & & \\ C_{1n} & C_{2n} & C_{3n} & \cdot & \cdot & \cdot & C_{nn} \end{bmatrix} \quad (3.1)$$

where $C_{nn} \in R^{nxn}$. Given a continuously differentiable function called Lyapunov function defined as

$$V(y) = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \cdot \\ \cdot \\ \cdot \\ y_n \end{bmatrix}^T \begin{bmatrix} P_{11} & P_{12} & P_{13} & \cdot & \cdot & \cdot & P_{1n} \\ P_{12} & P_{22} & P_{23} & \cdot & \cdot & \cdot & P_{2n} \\ P_{13} & P_{23} & P_{33} & \cdot & \cdot & \cdot & P_{3n} \\ \cdot & \cdot & \cdot & & & & \\ \cdot & \cdot & \cdot & & & & \\ \cdot & \cdot & \cdot & & & & \\ P_{1n} & P_{2n} & P_{3n} & \cdot & \cdot & \cdot & P_{nn} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \cdot \\ \cdot \\ \cdot \\ y_n \end{bmatrix} \quad (3.2)$$

whereP_{nn} is symmetric, positive definite and constant. Taking the derivation of the Lyapunov function

$$\dot{V}(y) = \begin{bmatrix} \dot{y_1} \\ \dot{y_2} \\ \dot{y_3} \\ \cdot \\ \cdot \\ \cdot \\ \dot{y_n} \end{bmatrix}^T \begin{bmatrix} P_{11} & P_{12} & P_{13} & \cdot & \cdot & \cdot & P_{1n} \\ P_{12} & P_{22} & P_{23} & \cdot & \cdot & \cdot & P_{2n} \\ P_{13} & P_{23} & P_{33} & \cdot & \cdot & \cdot & P_{3n} \\ \cdot & \cdot & \cdot & & & & \\ \cdot & \cdot & \cdot & & & & \\ \cdot & \cdot & \cdot & & & & \\ P_{1n} & P_{2n} & P_{3n} & \cdot & \cdot & \cdot & P_{nn} \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \cdot \\ \cdot \\ \cdot \\ y_n \end{bmatrix}^T \begin{bmatrix} P_{11} & P_{12} & P_{13} & \cdot & \cdot & \cdot & P_{1n} \\ P_{12} & P_{22} & P_{23} & \cdot & \cdot & \cdot & P_{2n} \\ P_{13} & P_{23} & P_{33} & \cdot & \cdot & \cdot & P_{3n} \\ \cdot & \cdot & \cdot & & & & \\ \cdot & \cdot & \cdot & & & & \\ \cdot & \cdot & \cdot & & & & \\ P_{1n} & P_{2n} & P_{3n} & \cdot & \cdot & \cdot & P_{nn} \end{bmatrix} \begin{bmatrix} \dot{y_1} \\ \dot{y_2} \\ \dot{y_3} \\ \cdot \\ \cdot \\ \cdot \\ \dot{y_n} \end{bmatrix}$$

$$= \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \cdot \\ \cdot \\ \cdot \\ y_n \end{bmatrix}^T \begin{bmatrix} C_{11} & C_{12} & C_{13} & \cdot & \cdot & \cdot & C_{1n} \\ C_{12} & C_{22} & C_{23} & \cdot & \cdot & \cdot & C_{2n} \\ C_{13} & C_{23} & C_{33} & \cdot & \cdot & \cdot & C_{3n} \\ \cdot & \cdot & \cdot & & & & \\ \cdot & \cdot & \cdot & & & & \\ \cdot & \cdot & \cdot & & & & \\ C_{1n} & C_{2n} & C_{3n} & \cdot & \cdot & \cdot & C_{nn} \end{bmatrix}^T \begin{bmatrix} P_{11} & P_{12} & P_{13} & \cdot & \cdot & \cdot & P_{1n} \\ P_{12} & P_{22} & P_{23} & \cdot & \cdot & \cdot & P_{2n} \\ P_{13} & P_{23} & P_{33} & \cdot & \cdot & \cdot & P_{3n} \\ \cdot & \cdot & \cdot & & & & \\ \cdot & \cdot & \cdot & & & & \\ \cdot & \cdot & \cdot & & & & \\ P_{1n} & P_{2n} & P_{3n} & \cdot & \cdot & \cdot & P_{nn} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \cdot \\ \cdot \\ \cdot \\ y_n \end{bmatrix}$$

$$+ \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \cdot \\ \cdot \\ \cdot \\ y_n \end{bmatrix}^T \left[\begin{bmatrix} P_{11} & P_{12} & P_{13} & \cdot & \cdot & \cdot & P_{1n} \\ P_{12} & P_{22} & P_{23} & \cdot & \cdot & \cdot & P_{2n} \\ P_{13} & P_{23} & P_{33} & \cdot & \cdot & \cdot & P_{3n} \\ \cdot & \cdot & \cdot & & & & \\ \cdot & \cdot & \cdot & & & & \\ \cdot & \cdot & \cdot & & & & \\ P_{1n} & P_{2n} & P_{3n} & \cdot & \cdot & \cdot & P_{nn} \end{bmatrix} \right] \begin{bmatrix} C_{11} & C_{12} & C_{13} & \cdot & \cdot & \cdot & C_{1n} \\ C_{12} & C_{22} & C_{23} & \cdot & \cdot & \cdot & C_{2n} \\ C_{13} & C_{23} & C_{33} & \cdot & \cdot & \cdot & C_{3n} \\ \cdot & \cdot & \cdot & & & & \\ \cdot & \cdot & \cdot & & & & \\ \cdot & \cdot & \cdot & & & & \\ C_{1n} & C_{2n} & C_{3n} & \cdot & \cdot & \cdot & C_{nn} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \cdot \\ \cdot \\ \cdot \\ y_n \end{bmatrix}$$

$$= \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \cdot \\ \cdot \\ \cdot \\ y_n \end{bmatrix}^T \left(\begin{bmatrix} C_{11} & C_{12} & C_{13} & \cdot & \cdot & \cdot & C_{1n} \\ C_{12} & C_{22} & C_{23} & \cdot & \cdot & \cdot & C_{2n} \\ C_{13} & C_{23} & C_{33} & \cdot & \cdot & \cdot & C_{3n} \\ \cdot & \cdot & \cdot & & & & \\ \cdot & \cdot & \cdot & & & & \\ \cdot & \cdot & \cdot & & & & \\ C_{1n} & C_{2n} & C_{3n} & \cdot & \cdot & \cdot & C_{nn} \end{bmatrix}^T \begin{bmatrix} P_{11} & P_{12} & P_{13} & \cdot & \cdot & \cdot & P_{1n} \\ P_{12} & P_{22} & P_{23} & \cdot & \cdot & \cdot & P_{2n} \\ P_{13} & P_{23} & P_{33} & \cdot & \cdot & \cdot & P_{3n} \\ \cdot & \cdot & \cdot & & & & \\ \cdot & \cdot & \cdot & & & & \\ \cdot & \cdot & \cdot & & & & \\ P_{1n} & P_{2n} & P_{3n} & \cdot & \cdot & \cdot & P_{nn} \end{bmatrix} \right.$$

$$\left. + \begin{bmatrix} C_{11} & C_{12} & C_{13} & \cdot & \cdot & \cdot & C_{1n} \\ C_{12} & C_{22} & C_{23} & \cdot & \cdot & \cdot & C_{2n} \\ C_{13} & C_{23} & C_{33} & \cdot & \cdot & \cdot & C_{3n} \\ \cdot & \cdot & \cdot & & & & \\ \cdot & \cdot & \cdot & & & & \\ \cdot & \cdot & \cdot & & & & \\ C_{1n} & C_{2n} & C_{3n} & \cdot & \cdot & \cdot & C_{nn} \end{bmatrix} \begin{bmatrix} P_{11} & P_{12} & P_{13} & \cdot & \cdot & \cdot & P_{1n} \\ P_{12} & P_{22} & P_{23} & \cdot & \cdot & \cdot & P_{2n} \\ P_{13} & P_{23} & P_{33} & \cdot & \cdot & \cdot & P_{3n} \\ \cdot & \cdot & \cdot & & & & \\ \cdot & \cdot & \cdot & & & & \\ \cdot & \cdot & \cdot & & & & \\ P_{1n} & P_{2n} & P_{3n} & \cdot & \cdot & \cdot & P_{nn} \end{bmatrix} \right) \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \cdot \\ \cdot \\ \cdot \\ y_n \end{bmatrix} \quad (3.3)$$

For stability, we assume that

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$$\dot{V} = -\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \cdot \\ \cdot \\ \cdot \\ y_n \end{bmatrix}^T I \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \cdot \\ \cdot \\ \cdot \\ y_n \end{bmatrix} \qquad (3.4)$$

Comparing equation (3.3) and (3.4)

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \cdot \\ \cdot \\ \cdot \\ y_n \end{bmatrix}^T \left(\begin{bmatrix} C_{11} & C_{12} & C_{13} & \cdot & \cdot & \cdot & C_{1n} \\ C_{12} & C_{22} & C_{23} & \cdot & \cdot & \cdot & C_{2n} \\ C_{13} & C_{23} & C_{33} & \cdot & \cdot & \cdot & C_{3n} \\ \cdot & \cdot & \cdot & & & & \\ \cdot & \cdot & \cdot & & & & \\ \cdot & \cdot & \cdot & & & & \\ C_{1n} & C_{2n} & C_{3n} & \cdot & \cdot & \cdot & C_{nn} \end{bmatrix}^T \begin{bmatrix} P_{11} & P_{12} & P_{13} & \cdot & \cdot & \cdot & P_{1n} \\ P_{12} & P_{22} & P_{23} & \cdot & \cdot & \cdot & P_{2n} \\ P_{13} & P_{23} & P_{33} & \cdot & \cdot & \cdot & P_{3n} \\ \cdot & \cdot & \cdot & & & & \\ \cdot & \cdot & \cdot & & & & \\ \cdot & \cdot & \cdot & & & & \\ P_{1n} & P_{2n} & P_{3n} & \cdot & \cdot & \cdot & P_{nn} \end{bmatrix} \right.$$

$$\left. + \begin{bmatrix} C_{11} & C_{12} & C_{13} & \cdot & \cdot & \cdot & C_{1n} \\ C_{12} & C_{22} & C_{23} & \cdot & \cdot & \cdot & C_{2n} \\ C_{13} & C_{23} & C_{33} & \cdot & \cdot & \cdot & C_{3n} \\ \cdot & \cdot & \cdot & & & & \\ \cdot & \cdot & \cdot & & & & \\ \cdot & \cdot & \cdot & & & & \\ C_{1n} & C_{2n} & C_{3n} & \cdot & \cdot & \cdot & C_{nn} \end{bmatrix} \begin{bmatrix} P_{11} & P_{12} & P_{13} & \cdot & \cdot & \cdot & P_{1n} \\ P_{12} & P_{22} & P_{23} & \cdot & \cdot & \cdot & P_{2n} \\ P_{13} & P_{23} & P_{33} & \cdot & \cdot & \cdot & P_{3n} \\ \cdot & \cdot & \cdot & & & & \\ \cdot & \cdot & \cdot & & & & \\ \cdot & \cdot & \cdot & & & & \\ P_{1n} & P_{2n} & P_{3n} & \cdot & \cdot & \cdot & P_{nn} \end{bmatrix} \right) \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \cdot \\ \cdot \\ \cdot \\ y_n \end{bmatrix}$$

$$= -\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \cdot \\ \cdot \\ \cdot \\ y_n \end{bmatrix}^T I \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \cdot \\ \cdot \\ \cdot \\ y_n \end{bmatrix}$$

$$\begin{bmatrix} C_{11} & C_{12} & C_{13} & \cdot & \cdot & \cdot & C_{1n} \\ C_{12} & C_{22} & C_{23} & \cdot & \cdot & \cdot & C_{2n} \\ C_{13} & C_{23} & C_{33} & \cdot & \cdot & \cdot & C_{3n} \\ \cdot & \cdot & \cdot & & & & \\ \cdot & \cdot & \cdot & & & & \\ \cdot & \cdot & \cdot & & & & \\ C_{1n} & C_{2n} & C_{3n} & \cdot & \cdot & \cdot & C_{nn} \end{bmatrix}^T \begin{bmatrix} P_{11} & P_{12} & P_{13} & \cdot & \cdot & \cdot & P_{1n} \\ P_{12} & P_{22} & P_{23} & \cdot & \cdot & \cdot & P_{2n} \\ P_{13} & P_{23} & P_{33} & \cdot & \cdot & \cdot & P_{3n} \\ \cdot & \cdot & \cdot & & & & \\ \cdot & \cdot & \cdot & & & & \\ \cdot & \cdot & \cdot & & & & \\ P_{1n} & P_{2n} & P_{3n} & \cdot & \cdot & \cdot & P_{nn} \end{bmatrix}$$

$$+ \begin{bmatrix} C_{11} & C_{12} & C_{13} & \cdot & \cdot & \cdot & C_{1n} \\ C_{12} & C_{22} & C_{23} & \cdot & \cdot & \cdot & C_{2n} \\ C_{13} & C_{23} & C_{33} & \cdot & \cdot & \cdot & C_{3n} \\ \cdot & \cdot & \cdot & & & & \\ \cdot & \cdot & \cdot & & & & \\ \cdot & \cdot & \cdot & & & & \\ C_{1n} & C_{2n} & C_{3n} & \cdot & \cdot & \cdot & C_{nn} \end{bmatrix} \begin{bmatrix} P_{11} & P_{12} & P_{13} & \cdot & \cdot & \cdot & P_{1n} \\ P_{12} & P_{22} & P_{23} & \cdot & \cdot & \cdot & P_{2n} \\ P_{13} & P_{23} & P_{33} & \cdot & \cdot & \cdot & P_{3n} \\ \cdot & \cdot & \cdot & & & & \\ \cdot & \cdot & \cdot & & & & \\ \cdot & \cdot & \cdot & & & & \\ P_{1n} & P_{2n} & P_{3n} & \cdot & \cdot & \cdot & P_{nn} \end{bmatrix} = - \qquad (3.5)$$

Equation (3.5) is the generalized Lyapunov equation.

The theorem below guarantees the existence of the above solution

Theorem 3.1

The eigenvalues λ of a matrix $C \in R^{n\,x\,n}$ satisfy $R_e(\lambda) < 0$ if and only if for any symmetric positive definite matrix I, there exists a unique positive definite symmetric matrix P satisfying the Lyapunov equation (3.5)

Proof:

Following the Hokayem and Gallestey (2015) procedure, given $I > 0$, $\exists\ P > 0$ satisfying (3.5). Thus

$$V = y^T P y > 0$$

and

$$\dot{V} = -y^T I y < 0$$

and asymptotic stability follows from theorem 2

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Conversely, assume $R_e(\lambda) < 0$, then P is defined as follows.

$$P = \int_0^\infty e^{C^T t} I e^{Ct} dt$$

P is symmetric and positive definite. To prove that P is positive definite, we assume the opposite. i.e. $\exists y \neq 0$ such that $y^T P y = 0$, substituting P gives

$$\int_0^\infty y^T e^{C^T t} I\, e^{Ct} y dt = 0, \Longrightarrow \int_0^\infty X^T I X dt = 0 \text{ where } X = e^{Ct} y$$

$$\Leftrightarrow X = e^{Ct} y = 0\ \forall t \geq 0$$

$$\Leftrightarrow X = 0$$

since e^{Ct} is nonsingular for all t. This contradicts the assumption. Thus $P > 0$, so that P satisfies Lyapunov equation , we have

$$PC + C^T P = \int_0^\infty e^{C^T t} I\, e^{Ct} C dt + \int_0^\infty C^T e^{C^T t} I e^{Ct} dt$$

$$\int_0^\infty \frac{d}{dt}\left(e^{C^T t} I e^{Ct}\right) dt = e^{C^T t} I e^{Ct}|_0^\infty = -I$$

Lastly, we have to show that P is unique.

Suppose $\tilde{P} \neq P$ which is also a solution. Then we have

$$\left(P - \tilde{P}\right)C + C^T\left(P - \tilde{P}\right) = 0$$

$$\Longrightarrow e^{C^T t}\left[\left(P - \tilde{P}\right)C + C^T\left(P - \tilde{P}\right)\right]e^{Ct} = 0$$

$$\Longrightarrow \frac{d}{dt}\left[e^{C^T t}\left(P - \tilde{P}\right)e^{Ct}\right] = 0$$

$$\Longrightarrow\ e^{C^T t}\left(P - \tilde{P}\right)e^{Ct} \text{is constant } \forall t.$$

This can be stable if and only if $P - \tilde{P} = 0 \Longrightarrow$ P = $\tilde{P}$.This completes the proof'.

Illustrative Example

Example 1

Use Lyapunov equation to check whether the systems is asymptotically stable.

$$\dot{y}_1 = -6y_1 + 2y_2$$

$$\dot{y}_2 = 2y_1 - 6y$$

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Solution

$$C = \begin{bmatrix} -6 & 2 \\ 2 & -6 \end{bmatrix}, \quad P = \begin{bmatrix} P_{11} & P_{12} \\ P_{12} & P_{22} \end{bmatrix} \text{ and } C^T = \begin{bmatrix} -6 & 2 \\ 2 & -6 \end{bmatrix}$$

but

$$C^T P + PC = -I$$

This implies that

$$\begin{bmatrix} -6 & 2 \\ 2 & -6 \end{bmatrix}\begin{bmatrix} P_{11} & P_{12} \\ P_{12} & P_{22} \end{bmatrix} + \begin{bmatrix} P_{11} & P_{12} \\ P_{12} & P_{22} \end{bmatrix}\begin{bmatrix} -6 & 2 \\ 2 & -6 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

Multiplying we obtain

$$\begin{bmatrix} -12P_{11} + 4P_{12} & 2P_{11} + 2P_{22} - 12P_{12} \\ 2P_{11} + 2P_{22} - 12P_{12} & 4P_{12} - 12P_{22} \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

Which gives the following three equations in three unknown.

$$-12P_{11} + 4P_{12} = -1$$
$$2P_{11} + 2P_{22} - 12P_{12} = 0$$
$$4P_{12} - 12P_{22} = -1$$

hence

$$P_{11} = 0.09375, \; P_{22} = 0.09375 \text{ and } P_{12} = 0.03125$$

Therefore

$$P = \begin{bmatrix} 0.09375 & 0.3125 \\ 0.3125 & 0.09375 \end{bmatrix}$$

The leading principal minors are

$$D_1 = 0.09375 \text{ and } D_2 = 0.12500$$

Which are positive definite, therefore we conclude that the system is asymptotically stable

Example 2

Use lyapunov equation to determine whether the following system is stable.

$$\dot{y}_1 = y_3$$
$$\dot{y}_2 = 2y_2 - y_3$$
$$\dot{y}_3 = -3y_1 + 2y_2 + y_3$$

Solution

$$C = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 2 & -1 \\ -3 & 2 & 1 \end{bmatrix}, \; P = \begin{bmatrix} P_{11} & P_{12} & P_{13} \\ P_{12} & P_{22} & P_{23} \\ P_{13} & P_{23} & P_{33} \end{bmatrix} \text{ and } I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Applying the lyapunov equation, we have that $C^T P + PC = -I$

$$\begin{bmatrix} 0 & 0 & -3 \\ 0 & 2 & 2 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} P_{11} & P_{12} & P_{13} \\ P_{12} & P_{22} & P_{23} \\ P_{13} & P_{23} & P_{33} \end{bmatrix} + \begin{bmatrix} 0 & 0 & 1 \\ 0 & 2 & -1 \\ -3 & 2 & 1 \end{bmatrix} \begin{bmatrix} P_{11} & P_{12} & P_{13} \\ P_{12} & P_{22} & P_{23} \\ P_{13} & P_{23} & P_{33} \end{bmatrix} = -\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} -2P_{13} & -2P_{23} & -2P_{33} \\ 4P_{12} + P_{13} & 4P_{22} + P_{23} & 4P_{23} + P_{33} \\ -2P_{11} + P_{12} + 2P_{23} & -2P_{12} + 2P_{22} + P_{23} & -2P_{13} + P_{23} + 2P_{33} \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Solving we have that

$$P_{11} = -\frac{1}{16}, P_{12} = -\frac{1}{8}, P_{13} = \frac{1}{2},$$
$$P_{21} = -\frac{1}{8}, P_{22} = -\frac{1}{4}, \;\; P_{23} = 0,$$
$$P_{31} = \frac{1}{2}, \; P_{32} = 0 \, P_{33} = 0$$

Thus,

$$P = \begin{bmatrix} -\frac{1}{16} & -\frac{1}{8} & \frac{1}{2} \\ -\frac{1}{8} & -\frac{1}{4} & 0 \\ \frac{1}{2} & 0 & 0 \end{bmatrix}$$

The leading principal minors are

$$D_1 = -\frac{1}{16}, \; D_2 = 0, \; D_3 = \frac{1}{16}$$

This system is not positive definite. Hence, the system is not stable.

The stability analysis of the equilibrium state of LTI system using lyapunov method requires a Lyapunov function having the following properties: (1) $V(y)$ is positive definite and (2) $\dot{V}(y)$ is negative definite. Lyapunov method still serves as the primary tool for establishing stability of the nonlinear system.

Lyapunov method can be applied to analyze the stability of nonlinear system without explicitly solving for the solutions of the differential equation

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