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IMPROVED ELZAKI TRANSFORM METHOD FOR SINGULAR INITIAL VALUE PROBLEMS IN SECOND ORDER ORDINARY DIFFERENTIAL EQUATIONS

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ABSTRACT
In this paper, we proposed a numerical method called the improved Elzaki transform method for solving singular initial value problems in second order ordinary differential equations. The method is used in handling a generalization of this kind of problem. For this purpose, an Elzaki linear operator is defined and evaluated using certain Elzaki transforms properties. The method was experimented for linear and nonlinear singular initial value problems, and the resulting numerical evidences show that the method converges to the exact solution usually in the first iteration. The method requires less computational rigor such that computational, round-off and truncation errors are eliminated. Similarly, results obtained by the improved Elzaki transform method converge favourably to same results obtained using the modified ADM. Also, the standard Elzaki transform method was treated for the same examples considered, and the results obtained were compared with the improved Elzaki transform method to verify the claim that the improved Elzaki transform method is far superior. All computations were performed with the aid of the computer application programme Maple 18 software.

Key words: Elzaki transform method, Ordinary differential equations, Singular initial value problems, Adomian polynomials, Maple 18 software.

INTRODUCTION
In recent years, there have been many numerical methods for solving singular initial value problems in second order ordinary differential equations. This is largely due to the significance of these problems to the world of mathematical models. Popular numerical methods include, the Adomian decomposition method (ADM) (Adomian, 1990; Rach *et. al.*, 1992; Wazwaz, 1997; Hosseini, 2006), Homotopy analysis method (HAM) (Bataineh *et. al.*, 2009), perturbation method (Bender, 1989), quasilinearization method (Mandelzweig and Tabakin, 2001), linearization method (Ramos, 2003), modified Adomian decomposition method (Hasan and Zhu, 2008; Wazwaz, 1999), etc.

In this paper, we aim at solving these singular initial value problems using the improved version of Elzaki transform method. The proposed method is used in handling a generalization of this kind of problem. For this purpose, an Elzaki linear operator is defined for the singular ordinary differential equations. The proposed method was experimented for linear and nonlinear singular initial value problems. The improved Elzaki transform method converges to the exact solution usually at the first iteration compared to the standard Elzaki method which requires more iteration before converging. Also, the improved Elzaki transform method requires less computational rigor such that computational, round-off and truncation errors are totally eliminated.

MATERIALS AND METHODS

Basic Definitions and Notations

i. Let $f(x)$ for $x \geq 0$, then the Elzaki transform (*see* Elzaki and Ezaki, 2011a; Elzaki and Ezaki, 2011b; Elzaki, 2012; Ziane and Cherif, 2015) of $f(x)$ is a function of s defined by

$$E[f(x)] = s\int_0^\infty f(x)e^{-\frac{s}{x}}dx.$$

ii. The Elzaki transform (Elzaki and Ezaki, 2011a) of DDE derivative is obtained by integration by part. That is,

$$E[u'(x)] = \frac{1}{r}T(r) - ru(0),$$

$$E[u''(x)] = \frac{1}{r^2}T(r) - ru'(0) - u(0),$$

Where

$$T_n(r) = \frac{T(r)}{r^n} - \sum_{m=0}^{n-1} r^{2-n+m}u^{(m)}(0).$$

iii. Some of the Elzaki transform properties can be found in the references Elzaki and Ezaki (2011a), Elzaki and Ezaki (2011b), Elzaki (2012), Ziane and Cherif (2015), and are given as;

a. $E[1] = r^2$

b. $E[x^n] = n!\,r^{n+2}$

c. $E^{-1}[r^{n+2}] = \frac{x^n}{n!}$.

Improved Elzaki Transform Method

Algorithm 1: Consider the generalized initial value problem of the form (Hasan and Zhu, 2008; Wazwaz, 1999)

$$u'' + \frac{2n}{x}u' + \frac{n(n-1)}{x^2}u + Nu(x) = g(x),\ n \geq 0, \qquad (1)$$

with initial conditions

$$u(0) = \alpha, \qquad u'(0) = \beta,$$

where $g(x)$and $Nu(x)$are given real functions and α and β are constants.

The above problem has been solved by Hassan and Zhu (2008) using modified ADM.

Here, we proposes the Elzaki linear operator L, as below

$$L = x^{-n}\frac{d^2}{dx^2}x^n u, \qquad (2)$$

such that problem (1) can be written as

$$Lu + Nu(x) = g(x). \qquad (3)$$

Applying the Elzaki transform method (Elzaki and Ezaki, 2011a; Elzaki and Ezaki, 2011b), equation (3) can be written as

$$E[Lu] = E[g(x)] - E[Nu(x)]. \qquad (4)$$

To aid convergence we rewrite (4) as,

$$E[Lu] = x^{-n}(E[g(x)] - E[Nu(x)]). \qquad (5)$$

where

$$E[Lu] = x^n E[Lu],\ E[g(x)] = E[x^n g(x)]$$

and

$$E[Nu(x)] = E[x^n Nu(x)].$$

Equation (5) is the formation of the improved Elzaki transform method for solving singular initial value problems whose solution is achievable by employing the special properties of the

Elzaki transform method which has been outlined in the Basic Definitions and Notations section of this work.

By definition ii, equation (5) can be written as

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$E[u(x)] =$
$x^{-n}\left(\sum_{k=0}^{1} r^{2+k}\frac{d^k}{dx^k}u(0) + r^2E[x^n g(x)] -\right.$
r2ExnNu(x), (6)

which implies that,

$E[u(x)] = x^{-n}\left(\alpha r^2 + \beta r^3 +\right.$
r2Exng(x)–ExnNu(x). (7)

Taking Elzaki inverse operator E^{-1} on (7),

$u(x) = E^{-1}\left[x^{-n}\left(\alpha r^2 + \beta r^3 +\right.\right.$
r2Exng(x)–ExnNu(x). (8)

By definition iii c, equation (8) can be written as

$u(x) = \left[x^{-n}\left(\alpha + \beta x +\right.\right.$
E–1r2Exng(x)–ExnNu(x) (9)

The Elzaki transform method introduces the solution $u(x)$ and the nonlinear term $Nu(x)$ as

$$u(x) = \sum_{n=0}^{\infty} u_n(x)\ , \qquad (10)$$

and

$$Nu(x) = \sum_{n=0}^{\infty} A_n\ , \qquad (11)$$

where A_n are the Adomian polynomials which are derived recursively using the relation (*see* Adomian, 1990; Rach *et. al.*, 1992; Wazwaz, 1997; Hosseini, 2006)

$$A_n = \frac{1}{n!}\left[\frac{d^n}{d\lambda^n}\left(\sum_{i=0}^{\infty}\lambda^i y_i\right)\right]_{\lambda=0}. \qquad (12)$$

If the nonlinear term is expressed as $Nu(x) = F(u_0)$, then the first few Adomian polynomials are arranged into the form

$$\left.\begin{array}{c} A_0 = F(u_0), \\ A_1 = u_1' F(u_0), \\ A_2 = u_2F'(u_0) + \frac{u_1^2}{2!}F''(u_0), \\ A_3 = u_3F'(u_0) + u_1u_2F''(u_0) + \frac{u_1^3}{3!}F''(u_0)' \\ \vdots \end{array}\right\} \qquad (13)$$

The components of $u_n, n \geq 0$, are determined recursively.

Substituting (10) and (11) into (9),

$\sum_{n=0}^{\infty} u_n(x) =$
$\left[x^{-n}\left(\alpha + \beta x + E^{-1}\left(r^2(E[x^n g(x)] -\right.\right.\right.$
Exnn=0∞An . (14)

The improved Elzaki transform method requires that the terms

$$x^{-n}\left(\alpha + \beta x + E^{-1}(r^2(E[x^n g(x)])\right)$$

are first evaluated and substituted back in (14). That is, if evaluating

$$x^{-n}\left(\alpha + \beta x + E^{-1}(r^2(E[x^n g(x)])\right)$$

yields $F(x)$ and $G(x)$, then (14) can be written as

$$\sum_{n=0}^{\infty} u_n(x) = F(x) + G(x) - x^{-n}E^{-1}\left[r^2E[x^n \sum_{n=0}^{\infty} A_n]\right]. \qquad (15)$$

To compute the components $u_n, n \geq 0$, recursively we decompose $F(x) + G(x)$ into equal parts such that by comparing both sides of (15) we have,

$$\begin{cases} u_0(x) = F(x), \\ u_{k+1}(x) = G(x) - x^{-n}E^{-1}\left[r^2E[x^n \sum_{n=0}^{\infty} A_n]\right],\ k \geq 0 \end{cases} \qquad (16)$$

which subsequently yield the result $u(x) = \sum_{n=0}^{\infty} u_n(x)$.

Algorithm 2: Going by the Elzaki transform method, we compare both sides of (14) such that the components $u_n, n \geq 0$, are determined recursively,

$$\begin{cases} u_0(x) = x^{-n}\left(\alpha + \beta x + E^{-1}\left(r^2(E[x^n g(x)])\right)\right), \\ u_{k+1}(x) = -x^{-n}E^{-1}\left[r^2E[x^n \sum_{n=0}^{\infty} A_n]\right],\ k \geq 0 \end{cases} \qquad (17)$$

which also yield the result $u(x) = \sum_{n=0}^{\infty} u_n(x)$.

Numerical Examples

In this section, problems considered are solved with the improved Elzaki transform method and the standard Elzaki method to demonstrate their accuracy over one another. Results obtained here are compared with modified ADM found in Hasan and Zhu (2008).

Example 1: (Hasan and Zhu, 2008)
Consider the nonlinear singular initial value problem

$$u'' + \frac{2}{x}u' + u^3 = 6 + 6x^6, \qquad (18)$$

with initial conditions

$$u(0) = 0, u'(0) = 0,$$

Here, $n = 1, \alpha = 0, \beta = 0,\ g(x) = 6 + 6x^6$ and $Nu(x) = u^3$.

Substituting the above parameters in (14) , we have,

$$\sum_{n=0}^{\infty} u_n(x) = \left[x^{-1}\left(E^{-1}\left(r^2(E[6x + 6x^7] - E[xu^3])\right)\right)\right].$$

By definition iiib, we have

$$\sum_{n=0}^{\infty} u_n(x) = x^{-1}E^{-1}[6r^5 + 7!\,r^{11}] - x^{-1}E^{-1}\left[r^2E[xu^3]\right].$$

By definition iiic, we have,

$$\sum_{n=0}^{\infty} u_n(x) = x^2 + \frac{x^8}{72} - x^{-1}E^{-1}\left[r^2E[xu^3]\right].$$

By the improved Elzaki transform method, we have

$$u_0(x) = x^2,$$
$$u_{k+1}(x) = \frac{x^8}{72} - x^{-1}E^{-1}\left[r^2E\left[x\sum_{n=0}^{\infty} A_n\right]\right], \quad k \geq 0. \qquad (19)$$

For $k = 0$, we have,

$$u_1(x) = \frac{x^8}{72} - x^{-1}E^{-1}\left[r^2E[xA_0]\right],$$

where $A_0 = x^6$.
Hence,

$$u_1(x) = \frac{x^8}{72} - x^{-1}E^{-1}[7!\,r^{11}] = \frac{x^8}{72} - x^{-1}\left(\frac{7!}{9!}x^9\right) = 0.$$

This implies that

$$u_{k+1}(x) = 0, k \geq 0.$$

Hence,

$$u(x) = x^2$$

is the solution of (18) which is the exact solution. The same solution was obtained in Hasan and Zhu (2008) using modified ADM.
Alternatively, if we have used the Elzaki transform method (ETM), the approximations would be

$$u_0(x) = x^2 + \frac{x^8}{72},$$
$$u_1(x) = -\left[\frac{1}{333135504}x^{24} + \frac{1}{851760}x^{18} + \frac{1}{5460}x^{12} + \frac{1}{72}x^6\right],$$

and so on.

Example 2: (Hasan and Zhu, 2008)
Consider the nonlinear singular initial value problem

$$u'' + \frac{2}{x}u' + u = 6 + 12x + x^2 + x^3, \quad (20)$$

with initial conditions

$$u(0) = 0, u'(0) = 0,$$

Here, $n = 1, \alpha = 0, \beta = 0,\ g(x) = 6 + 12x + x^2 + x^3$ and $u(x) = u$.

Substituting the above parameters in (14) , we have,

$$\sum_{n=0}^{\infty} u_n(x) = \left[x^{-1}\left(E^{-1}\left(r^2(E[6x + 12x^2 + x^3 + x^4] - E[xu])\right)\right)\right].$$

By definition iiib, we have

$$\sum_{n=0}^{\infty} u_n(x) = x^{-1}E^{-1}[6r^5 + 4!\,r^6 + 3!\,r^7 + 4!\,r^8] - x^{-1}E^{-1}\left[r^2E[xu^3]\right].$$

By definition iiic, we have,

$$\sum_{n=0}^{\infty} u_n(x) = x^2 + x^3 + \frac{x^5}{20} + \frac{x^6}{30} - x^{-1}E^{-1}\left[r^2E[xu^3]\right].$$

By the improved Elzaki transform method, we have

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$$u_0(x) = x^2 + x^3,$$
$$u_{k+1}(x) = \frac{x^5}{20} + \frac{x^6}{30} - x^{-1}E^{-1}[r^2E[xu^3]], \quad k \geq 0. \quad (21)$$
For $k \geq 0$, we have,
$$u_{k+1}(x) = 0, k \geq 0.$$
Hence,
$$u(x) = x^2 + x^3$$

is the solution of (20) which is the exact solution. The same solution was obtained in Hasan and Zhu (2008) using modified ADM.

Alternatively, if we have used the Elzaki transform method (ETM), the approximations would be
$$u_0(x) = x^2 + x^3 + \frac{x^5}{20} + \frac{x^6}{30},$$

$$u_{k+1}(x) = -x^{-1}E^{-1}\left[r^2E[xu^3]\right], \quad k \geq 0,$$
and so on.

DISCUSSION
From the implementation of the improved ETM on the numerical examples considered, the results obtained by the improved ETM converge favourably to the exact solution usually at the first iteration. This agrees with the similar result obtained in Hasan and Zhu (2008) using modified ADM. In contrast, the standard ETM requires more iteration to converge.

In this paper, we have improved on the standard Elzaki transform method for solving singular initial value problems for linear and nonlinear cases. The standard Elzaki transform method was employed to solve same problems for effective comparison with the improved method. The improved Elzaki transform method requires less computational rigor such that computational, round-off and truncation errors are totally eliminated.

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