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POWER SERIES VARIATION ITERATION METHOD FOR SOLVING FOURTH ORDER BOUNDARY VALUE INTEGRO-DIFFERENTIAL EQUATIONS

I. N. Njoseh[1] and E. J. Mamadu[2]

[1]*Department of Mathematics, Delta State University, Abraka, Nigeria*

[2]*Department of Mathematics, University of Ilorin, P.M.B 1515, Ilorin, Nigeria*
Email: njoseh@delsu.edu.ng[1], mamaduebimene@hotmail.com[2]

[1]*Correspondence Author*

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ABSTRACT

In this paper, we implemented the power series variation iteration method for solving fourth order boundary value integro-differential equations. The method is a mixture of power series approximation method (PSAM) and the variation iteration method (VIM). In this method, the approximate solution is given as a power series. The series solution is made to satisfy the given boundary conditions and hence, is the initial approximation. The accuracy of the method was illustrated using two examples with known analytic solution. All numerical evidences were obtained by using the computer application software Maple 18.

Key words: Power series**,** Variation iteration method, Approximate solution, Lagrange multiplier, Integro-differential equations, Boundary value problems,

INTRODUCTION

Integro-differential equation is one active area in the field of pure and applied mathematics. Major physical phenomena such as fluid dynamics, chemical kinetic, potential theory, physics, astronomy, biological models, control theory, etc., are often modelled using integro-differential equations. Many of these equations are difficult to handle analytically. Hence, iterative methods have become more renowned in solving these equations. Over the years, several numerical methods for integro-differential equations have being proposed by various researchers; some of which include, the variation iteration method (Mamadu and Njoseh, 2016a; He, 1999; He, 2000; Njoseh, 2016), the homotopy perturbation method (Yildirim, 2009; Ganji, *et. al*. 2003), the power series approximation method (Njoseh and Mamadu, 2016), the orthogonal collocation method (Mamadu and Njoseh, 2016b), the Tau method (Hosseini, *et. al*., 2003), the Adomian decomposition method (Wazwaz, 1997; Batiha, *et. al.*, 2008), etc.

The power series approximation method (PSAM) was introduced by Njoseh and Mamadu (2016), for solving higher order boundary value problems. The method involves transforming the prescribed boundary conditions to system of ordinary differential equations. The approximate solution is then given as a power series in x, where the coefficients are determined recursively.

In this paper, we are interested in solving fourth order boundary value integro-differential equations by the method of power series variation iteration method. The method is a mixture of PSAM and the variation iteration method (VIM). In this method, the approximate solution is given as a power series in x. The series solution is made to satisfy the given boundary conditions and hence, is the initial approximation. Next, the correction functional is constructed for which the Lagrange multiplier is estimated.

The method offer several advantages, which include, rapid rate of convergence of the approximate solution to the analytic solution, no weak or hidden assumptions, no linearization or perturbation, and computational and truncation errors are optimally minimized.

MATERIALS AND METHODS

Basic ideas of PSAM

In this section, we briefly review the concept of PSAM as presented by Njoseh and Mamadu (2016).

Let us consider the nth order boundary value problem (BVP),

$$y^{(n)}(x) + \alpha(x)y(x) = g(x), 0 < x < a, \quad (1)$$

subject to the boundary conditions

$$y(0) = A_0, y''(0) = A_1, \quad (2)$$

$$y(a) = \beta_0 \text{ and } y''(a) = \beta_1. \quad (3)$$

where $\alpha(x), y(x)$ and $g(x)$are assumed real and differentiable on $0 \leq x \leq a, A_i$, β_i, $i = 0,1$, are real constants in $[0, a]$.

The PSAM requires transforming (1) – (3) into system of ordinary differential equations (ODEs) such that

$$\frac{dy}{dx} = y_1, \frac{dy_1}{dx} = y_2, \frac{dy_2}{dx} = y_3, \cdots, \frac{dy_n}{dx} = g(x) - \alpha(x)y(x), \quad (4)$$

subject to the conditions

$$y^{(2m)}(0) = A_m, m = 0,1. \quad \text{P} \quad (5)$$

$$y^{(2m)}(a) = \beta_m, m = 0,1. \quad (6)$$

In PSAM, we write the approximate solution as a power in x,

$$y(x) = \sum_{i=0}^{n-1} \alpha_i x^i, \quad (7)$$

where a_i, $i = 0(1)(n-1)$, are constants in [0, a] to be determined and n is the order of the derivative.

The error and convergence analysis of the PSAM has been discussed in Njoseh and Mamadu (2016).

Basic ideas of VIM

To illustrate the VIM, we consider the generalized differential equation of the form Mamadu and Njoseh (2016 a & b)

$$Ly(x) + Ny(x) = g(x), \quad (8)$$

with prescribed auxiliary conditions, L is a linear operator of the highest order derivative, N is a nonlinear term, and g(x) is the source term.

By the variation iteration method (Mamadu and Njoseh, 2016a & b; He, 1999; He, 2000; Njoseh, 2016) requires the construction of a correction functional for equation (8) as

$$y_{n+1}(x) = y_n(x) + \int_0^x \lambda(s)\left(Ly_n(s) + N(\tilde{y}_n(s)) - g(s)\right)ds, n \geq 0, \quad (9)$$

where $\lambda(s)$ is the general Lagrange multiplier, which can obtained optimally via variational theory, $\tilde{y}_n(s)$ is a restricted variable, that is, $\tilde{y}_n(s) = 0$. Abbasbandy and Sivanian (2009) proposed a generalized estimate for obtaining the Lagrange multiplier, $\lambda(s)$, as

$$\lambda_i(s) = (-1)^i \frac{(s-x)^{(i-1)}}{(i-1)!}, \quad (10)$$

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where i is the order of the derivative. The convergence of the variation iteration method has been discussed explicitly by Mamadu and Njoseh (2016b) for nonlinear integro-differential equations.

Power Series Variation Iteration Method (PSVIM)

Consider the fourth order boundary value Volterra integro-differential equation of the form (Mamadu and Njoseh, 2016a & b)

$$y^{(iv)}(x) = g(x) + \alpha(x)y(x) + \int_0^x k(x,s)y(s)ds\,, 0 < x < a, \quad (11)$$

subject to the boundary conditions

$$y(0) = A_0, y''(0) = A_1, \quad (12)$$
$$y(a) = \beta_0 \text{ and } y''(a) = \beta_1. \quad (13)$$

where $\alpha(x), y(x)$ and $g(x)$ are assumed real and differentiable on $0 \le x \le a, A_i$, β_i, $i = 0,1$, are real constants in [0,a].

Let the approximate solution be defined as in equation (7), that is,

$$y(x) = \sum_{i=0}^{3} \alpha_i x^i, n = 4. \quad , (14)$$

Subjecting equation (14) to the boundary conditions (12) and (13), we obtain,

$$a_0 = A_0, a_1 = \beta_0, a_2 = \frac{A_1}{2} \text{ and } a_3 = \frac{\beta_1}{6}. \quad (15)$$

Thus, equation (14) can be written as

$$y(x) = A_0 + \beta_0 x + \frac{A_1}{2}x^2 + \frac{\beta_1}{3!}x^3, \quad (16)$$

where $\beta_0 = y'(0)$ and $\beta_1 = y'''(0)$. Equation (16) in this case, is the initial approximation, that is,

$$y_0(x) = A_0 + \beta_0 x + \frac{A_1}{2}x^2 + \frac{\beta_1}{3!}x^3. \quad (17)$$

By the variation iteration method, we construct a correction functional for equation (11) as,

$$y_{n+1}(x) = y(x) + \int_0^x \lambda(s)\left(\frac{d^4}{ds^4}y_n(s) - g(x) - \alpha(x)y(x) - \int_0^x k(x,s)y(s)ds\right)ds, n \ge 0, \quad (18)$$

where

$$\lambda(s) = \frac{(s-x)^3}{3!}.$$

Therefore, the other components $y_n(x), n > 0$, are computed using (17) and (18).

The constants β_0 and β_1 are estimated at the boundary $x = a$, and substituted back in $y_{n+1}(x), n \ge 0$, to obtain the approximate solution.

Numerical Examples

To illustrate the accuracy of this method, we consider two examples with known analytic solution.

Example 1. (Yildirim, 2008)

Consider the fourth order linear integro-differential equation:

$$y^{(iv)}(x) = x(1+e^x) + 3e^x + y(x) - \int_0^x y(t)dt\,, 0 < x < 1, \quad (19)$$

with boundary conditions

$$y(0) = 1,\; y''(0) = 2,\; y(1) = 1 + e\,,\; y''(1) = 3e. \quad (20)$$

Applying the methodology in PSVIM, we have,

$$y_0(x) = 1 + \beta_0 x + x^2 + \frac{\beta_1}{3!}x^3,$$

$$y_{n+1}(x) = y_n(x) + \int_0^x \frac{(s-x)^3}{3!}\left(\frac{d^4}{ds^4}y_n(s) - \right.$$

s+s+31+s+12s2+13!s3−yns+0xytdtds, (21)

for $n \ge 0$. Now, solving equation (21) for $n \ge 0$, and estimating β_0 and β_1 at the boundary $x = 1$, yields the following results presented in the table below:

Table 1: Estimates of the constants β_0 and β_1 with the maximum errors for each iterate for Example 1.

	β_0	β_1	Maximum Error
$y_0(x)$	0.692474	6.154845	9.9898E-02
$y_1(x)$	0.729616	5.021741	9.4468E-02
$y_2(x)$	0.999891	1.000702	3.215E-04

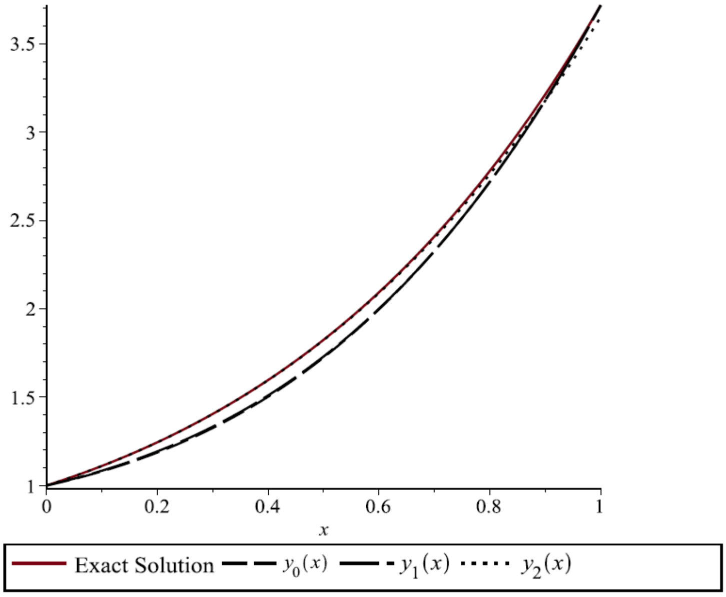

Figure 1: The comparison of each iterate and the exact solution for Example 1.

Example 2. (Yildirim, 2008)
Consider the fourth order nonlinear integro-differential equation:

$$y^{(iv)}(x) = 1 + \int_0^x e^{-t} y^2(t)dt\,, 0 < x < 1, \quad (22)$$

with boundary conditions

$$y(0) = 1, y''(0) = 1, y(1) = e \text{ and } y''(1) = e. \quad (23)$$

Applying the methodology in PSVIM, we have,

$$y_0(x) = 1 + \beta_0 x + x^2 + \frac{\beta_1}{3!}x^3,$$

$$y_{n+1}(x) = y_n(x) + \int_0^x \frac{(s-x)^3}{3!}\left(\frac{d^4}{ds^4} y_n(s) - 1 -\right.$$
0x1−t+12t2−13!t3yn2tdtds, n≥0, (24)

Solving equation (24) for $n \geq 0$, and estimating β_0 and β_1 at the boundary $x = 1$, yield the following results presented in the table below:

Table 2: Estimates of the constants β_0 and β_1 with the maximum errors for each iterate for Example 2.

	β_0	β_1	Maximum Error
$u_0(x)$	1.598568	0.718282	6.4806E-02
$u_1(x)$	0.480829	-14.143697	1.2107E-09
$u_2(x)$	0.460775	-22.285070	4.9433E-09

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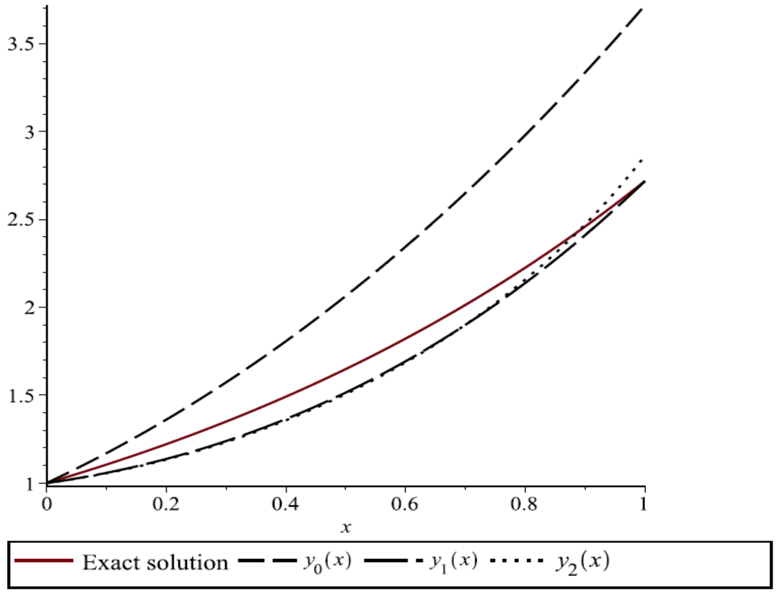

Figure 2: The comparison of each iterate and the exact solution for Example 2.

DISCUSSION

The implementation of the method (PSVIM) on both linear and nonlinear fourth order integro-differential equations shows that for examples 1 and 2, the approximate solution ($y_2(x)$), (represented by the broken line) converges rapidly to the exact solution (solid line) as shown in the Figures 1 and 2, with maximum errors of 3.215E-04 and 4.9433E-09 respectively as shown in the Tables 1 and 2. This indicates that with more iteration, the approximate solution will converge absolutely to the exact solution. All numerical evidences are obtained by using the computer application software, Maple 18.

In this paper, the power series variational iteration method has been executed successfully on both linear and nonlinear fourth order boundary value integro-differential equations. The PSVIM is a reliable and accurate numerical tool for the approximation of integro-differential equations and can be applied to other areas in mathematics such as Ordinary and Partial differential equations, stochastic differential and stochastic partial differential equations, Integral equations, etc.

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