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TRANSVERSE DISPLACEMENT OF CLAMPED-CLAMPED NON-UNIFORM RAYLEIGH BEAMS UNDER MOVING CONCENTRATED MASSES RESTING ON A CONSTANT ELASTIC FOUNDATION.

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ABSTRACT

In this paper, the vibrational motion of a non-uniform beam clamped at both ends carrying moving concentrated loads is investigated. The governing equation of motion of the dynamical system is transformed via Mindlin-Goodman's cum Generalised Galerking's methods as given in Oni and Ajibola (2009) The resulting coupled dynamic equation is simplified via struble's asymptotic techniques Oni,(1991), Oni and Omolafe (2005) , Oni and Omolafe (2005) and Oni (1996) a second order differential equation that ensued is solved using integral transform methods to obtain a closed form solution. From the closed form solution, it is obtained that for the same natural frequency, the critical speed for the non-uniform Rayleigh beams traversed by moving force is greater than that under the influence of a moving mass. Hence, resonance is reached earlier in the moving mass problems. Furthermore, the transverse displacement for the moving force and moving mass models were calculated for various time t and presented in plotted curves and in the clamped-clamped non-uniform boundary conditions. It is found that, the moving force solution is not an upper bound for the accurate solution of the moving mass solution. Analysis further shows that an increase in the values of the structural parameters reduces the response amplitude of non-uniform Rayleigh beams of our dynamical problem.

Key words: Rayleigh beam, non-uniform, axial force, non-classical boundary, rotatory-inertia, Foundation-modulli. Clamped-clamped.

INTRODUCTION

The problem of assessing the dynamical response of an elastic system (beam or plate) which supports moving concentrated masses is fundamental in the analysis and design of highway and railway bridges and as such, this problem continues to attract the attention of research Engineers and Scientists Milomir,eta (1969), Sadiku and Leipholz (1981), Oni (1991), Gbadeyan and Oni (1995), Oni and Omolafe (2005), Oni and Awodola (2003), Omer and Aitung (2006) and Savin (2001). It must be noted that this class of dynamical problems concern results for cases when the elastic system have simple supports at the boundaries and solution techniques are not easily adjusted to the cases in which the supports conditions are not simple ones Jia-Jang (2006). The boundary conditions for structural members under moving loads can be classified Frybal (1972) and Bishop and Johnson (1979) into two viz:

(a) Geometric boundary conditions.

(b) Dynamic / force boundary conditions. Hilderbrand (1977), Jaeger and Starfield (1974), Clough and Penzien (1975) and Craig (1981).

In considering a non-classical end conditions, we discuss the elastically supported end conditions. Suppose a beam is hinged or pinned at one of its ends and supported by an elastic spring, with modulus k at the other end, the magnitude of the shearing force must be k times the displacement Oni and Ajibola (2005).

In this paper, the work Oni and Ajibola (2005) is extended to cover boundary conditions other than simple ones. In particular, at end $x = 0$, the Rayleigh beam is clamped and at end $x = L$, the beam is also clamped; thus the termed Clamped-Clamped Rayleigh beam.

In the mathematical model, the beam properties vary along span L of the beam. The method of Generalised Garlerkin's method already alluded to in Oni and Ajibola (2005) shall be used. This method is employed to simplify the governing fourth order partial differential equation with singular and variable coefficients. The transformed process, in the case of Clamped-Clamped boundary conditions is clearly more cumbersome than we had when working with simply supported boundary conditions. The resulting Garlerkin equations are solved via the modified Struble's asymptotic techniques already alluded to in Oni and Ajibola (2005).

Governing Equation

In this paper, a non-uniform Rayleigh beam resting on a constant elastic foundation where the beams properties such as the moment of inertial I, and the mass per unit length of the beam μ vary along the span L of the beam is considered. R^0 is the Rotatory inertial, K is the elastic foundation Moduli; x is the spatial coordinate and t is the time. The transverse displacement $U(x,t)$ of the beam when it is under the action of a moving load of mass M which is moving with velocity c is governed by the fourth order partial differential equation given by

$$\frac{\partial^2}{\partial x^2}\left[EI(x)\frac{\partial^2}{\partial x^2}U(x,t)\right]+\mu(x)\frac{\partial^2 U(x,t)}{\partial t^2}-\frac{\partial}{\partial x}\left[\mu(x)R^0\frac{\partial^3 U(x,t)}{\partial x\partial t^2}\right]$$

$$+M\delta(x-ct)\left(\frac{\partial^2}{\partial t^2}+\frac{2c\partial^2}{\partial x\partial t}+\frac{c^2\partial^2}{\partial x^2}\right)U(x,t)+KU(x,t)=Mg\delta(x-ct)$$

where

g is the acceleration due to gravity. It is remarked here that, since the Rayleigh beam is non-uniform, I and μ are no longer constants but vary with the spatial coordinate along the span of the beam .in particular, adapting the example in Fryba (1972).Let $\mathrm{I}(x)$ and $\mu(x)$ take the forms .

$$\mathrm{I}(x)=I_0\left(1+\sin\frac{\pi x}{L}\right)^3,\quad \mu(x)=\mu_0\left(1+\sin\frac{\pi x}{L}\right) \quad (2)$$

where I_o and μ_o are constants.

The boundary conditions of the above equation (1) are taken to be time dependent, thus at each of the boundary points, there are two boundary conditions written as

$$D_i\,[U(0,t)]=f_i(t)\quad i=1,2\,,\; D_i\,[U(L,t)]=f_i(t)\quad i=3,4 \quad (3)$$

where D_i are linear homogenous differential operators of order less than or equal to three.

For example, if the Rayleigh beam in question is

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Clamped-clamped ends

$$D_1 = 1,\ D_2 = \frac{\partial}{\partial x},\ D_3 = 1,\ D_4 = \frac{\partial}{\partial x}. \qquad (4)$$

The initial conditions for the motion at time $t = 0$ are specified by two arbitrary functions. Thus

$$U(x,0) = U_0(x) \text{ and } \frac{\partial U(x,0)}{\partial x} = \dot{U}_0(x) \qquad (5)$$

Substituting equation (2) into (1) and after some simplifications and rearrangements yields

$$\frac{EI_0}{4}\Bigg[\left(10 - 6\cos\frac{2\pi x}{L} + 15\sin\frac{\pi x}{L} - \sin\frac{3\pi x}{L}\right)\frac{\partial^4 U(x,t)}{\partial x^4} + \left(\frac{24\pi}{L}\sin\frac{2\pi x}{L} + \frac{30\pi}{L}\cos\frac{\pi x}{L} - \frac{6\pi}{L}\cos\frac{3\pi x}{L}\right)\frac{\partial^3 U(x,t)}{\partial x^3} + \left(\frac{24\pi^2}{L^2}\cos\frac{2\pi x}{L} - \frac{15\pi^2}{L^2}\sin\frac{\pi x}{L} + \frac{9\pi^2}{L^2}\sin\frac{3\pi x}{L}\right)\frac{\partial^2 U(x,t)}{\partial x^2}\Bigg]$$
$$+ \mu_0\left(1 + \sin\frac{\pi x}{L}\right)\frac{\partial^2 U(x,t)}{\partial t^2} - \mu_0 R^0\left[\left(1 + \sin\frac{\pi x}{L}\right)\frac{\partial^4 U(x,t)}{\partial x^2 \partial t^2} + \frac{\pi}{L}\cos\frac{\pi x}{L}\frac{\partial^3 U(x,t)}{\partial x \partial t^2}\right]$$
$$+ M\delta(x - ct)\left[\frac{\partial^2 U(x,t)}{\partial t^2} + \frac{2c\partial^2 U(x,t)}{\partial x \partial t} + \frac{c^2\partial^2 U(x,t)}{\partial x^2}\right] + KU(x,t) = Mg\delta(x - ct).$$

Operational Simplifications of Equation

The initial-boundary value problem (6) consisting of a non-homogeneous partial differential equation with a non-homogeneous boundary conditions is transformed to a non-homogeneous partial differential equation with homogeneous boundary conditions, using the Mindlin-Goodman's method. In order to solve the above initial-boundary value problem, we introduce the auxiliary variable $z(x,t)$ in the form

$$U(x,t) = Z(x,t) + \sum_{i=1}^{4} f_i(t) g_i(x)$$

Substituting equation (7) into the boundary value problem (6), transforms the latter into a boundary value problem in terms of $z(x,t)$. The displacement influence functions $g_i(x)$ are chosen so as to render the boundary conditions for the boundary value problem in $z(x,t)$ homogenous.

Substituting equation (7) into (6) and simplifying yields.

(5)

$$\frac{EI_0}{4}\Bigg[\left(10 - 6\cos\frac{2\pi x}{L} + 15\sin\frac{\pi x}{L} - \sin\frac{3\pi x}{L}\right)\frac{\partial^4}{\partial x^4}Z(x,t) + 6\frac{\pi}{L}\left(4\sin\frac{2\pi x}{L} + 5\cos\frac{\pi x}{L} - \cos\frac{3\pi x}{L}\right)\frac{\partial^3}{\partial x^3}Z(x,t)$$
$$+ 3\frac{\pi^2}{L^2}\left(8\cos\frac{2\pi x}{L} - 5\sin\frac{\pi x}{L} + 3\sin\frac{3\pi x}{L}\right)\frac{\partial^2}{\partial x^2}Z(x,t)\Bigg] + \mu_0\left(1 + \sin\frac{\pi x}{L}\right)Z_{tt}(x,t)$$
$$- \mu_0 R^0\left[\frac{\partial^2}{\partial x^2}Z_{tt}(x,t) + \sin\frac{\pi x}{L}\frac{\partial^2}{\partial x^2}Z_{tt}(x,t) + \frac{\pi}{L}\cos\frac{\pi x}{L}\frac{\partial^2}{\partial x^2}Z_{tt}(x,t)\right]$$
$$+ M\delta(x - ct)\left[Z_{tt}(x,t) + \frac{2c\partial}{\partial x}Z_{tt}(x,t) + \frac{c^2\partial^2}{\partial x^2}Z_{tt}(x,t)\right] + KZ(x,t)$$
$$= Mg\delta(x - ct)$$
$$- \sum_{i=1}^{\Psi}\Bigg[\frac{EI_0}{4}f_i(t)\left(10 - 6\cos\frac{2\pi x}{L} + 15\sin\frac{\pi x}{L} - \sin\frac{3\pi x}{L}\right)g_i^{IV}(x) + 6\frac{\pi}{L}\left(4\sin\frac{2\pi x}{L} + 5\cos\frac{\pi x}{L} - \cos\frac{3\pi x}{L}\right)g_i^{III}(x)$$
$$+ 3\frac{\pi^2}{L^2}\left(8\cos\frac{2\pi x}{L} - 5\sin\frac{\pi x}{L} + 3\sin\frac{3\pi x}{L}\right)g_i^{II}(x)\Bigg] + \mu_0\ddot{f}_i(t)\left(1 + \sin\frac{\pi x}{L}\right)g_i(x)$$
$$- \mu_0 R^0\ddot{f}_i(t)\left(g_i^{II}(x) + \sin\frac{\pi x}{L}g_i^{II}(x) + \frac{\pi}{L}\cos\frac{\pi x}{L}g_i^{I}(x)\right)$$
$$+ M\delta(x - ct)\left[\ddot{f}_i(t)g_i(x) + 2c\dot{f}_i(t)g_i^{I}(x) + c^2 f_i(t)g_i^{II}(x) + Kf_i(t)g_i(x)\right] \qquad (8)$$

Where dot $(\cdot)$ represents the derivative with respect to time, while slash $(')$ represents the derivative with respect to space coordinate.

Now the expression in equation (7) must satisfy the boundary conditions in equation (3).

Consequently, we have

$$D_i[Z(o,t)]+\sum_{j=1}^{4} f_i(t)D_i[g_i(o)]=f_i(t), \qquad i=1,2.$$

$$D_i[Z(L,t)]+\sum_{j=1}^{4} f_i(t)D_i[g_i(L)]=f_i(t), \qquad i=3,4. \qquad (9)$$

Substituting equation (7) into the initial equation (5) leads to.

$$Z(x,o)=U(x,o)-\sum_{i=1}^{4} f_i(o)g_i(x), \quad \frac{\partial}{\partial t}z(x,o)=\dot{U}_0(x)-\sum_{i=1}^{4}\dot{f}_i(o)g_i(x) \qquad (10)$$

Using the Mindlin – Goodman method [16] the boundary conditions (9) in terms of $Z(x,t)$ can be made homogeneous if the function $g_i(x)$ are chosen such that the sixteen conditions given by

$$D_i[g_i(o)]=\delta_{ij} \quad (i=1,2,\ j=1,2,3,4) \qquad (11a)$$

and

$$D_i[g_i(L)]=\delta_{ij} \qquad (i=3,4,\ j=1,2,3,4) \qquad (11b)$$

Where

$$\delta_{ij}=\begin{cases}0 & for\ \ i\neq j\\ 1 & for\ \ i=j\end{cases} \qquad (12)$$

is the Kronecker delta; are satisfied.

Using equations (11) in the non-homogenous boundary conditions (9) we obtain the homogenous boundary conditions.

$$D_i[z(o,t)]=0 \qquad i=1,2$$

$$D_i[z(L,t)]=0 \qquad i=3,4 \qquad (13)$$

The original problem now reduces to that of solving the non-homogenous partial differential equation (6) subject to the homogenous boundary conditions in (13) with the non-homogenous initial conditions (10).

Solution Procedure

It is observed that the initial – boundary – value problem in equation (8) is a fourth order partial differential equation having some coefficients which are not only variable but are also singular. These coefficients are the Dirac delta functions which multiply each term of the convective acceleration operator associated with the inertia of the mass of the moving load. It is remarked at this juncture that this transformed equation is now amenable to a modification of the approximate method commonly called Galerkin's method

Analytical Approximate Solution

The Galerkin's method requires that the solution of equation (8) takes the form

$$Z_n(x,t)=\sum_{m=1}^{n} Y_m(t)V_m(x)$$

where $V_m(x)$ is chosen such that the desired boundary conditions were satisfied. An appropriate selection of functions for beam problems are beam mode shape. Thus the m^{th} normal mode of vibrations of a uniform beam given by

$$V_m(x)=Sin\frac{\lambda_m x}{L}+A_m Cos\frac{\lambda_m x}{L}+B_m Sinh\frac{\lambda_m x}{L}+C_m Cosh\frac{\lambda_m x}{L} \qquad \textbf{(16)}$$

is chosen as a suitable kernel of the integral (15) where λ_m is the mode frequency, A_m, B_m and C_m are constant. An important feature of the use of this kernel is that it makes the transformation suitable for all variants of the boundary conditions of the dynamical problems. The parameter λ_m, A_m, B_m and C_m are obtained when the equation (16) is substituted into the appropriate boundary conditions.

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By applying the Generalized Galerkin's method (GGM) as in (15), equation (8) takes the form

$$\sum_{m=1}^{\eta}\left[\frac{EI_o}{4}\left(\left(10-6\cos\frac{2\pi x}{L}+15\sin\frac{\pi x}{L}-\sin\frac{3\pi x}{L}\right)V_m^{IV}(x)\right.\right.$$
$$+6\frac{\pi}{L}\left(4\sin\frac{2\pi x}{L}+5\cos\frac{\pi x}{L}-\cos\frac{3\pi x}{L}\right)V_m^{IIIi}(x)$$
$$\left.\left.+3\frac{\pi^2}{L^2}\left(8\cos\frac{2\pi x}{L}-5\sin\frac{\pi x}{L}+3\sin\frac{3\pi x}{L}\right)V_m^{II}(x)\right)Y_m(t)\right.$$
$$+\mu_o\left(V_m(x)+\sin\frac{\pi x}{L}V_m(x)\right)\ddot{Y}_m(t)$$
$$-\mu_o R^0\left(V_m^{II}(x)+\sin\frac{\pi x}{L}V_m^{II}(x)+\frac{\pi}{L}\cos\frac{\pi x}{L}V_m^{I}(x)\right)\ddot{Y}_m(t)$$
$$\left.+M\delta(x-ct)\left(V_m(x)\ddot{Y}_m(t)+2CV_m^{I}(x)\dot{Y}_m(t)+C^2V_m^{II}(x)Y_m(t)\right)+KV_m(x)Y_m(t)\right]$$
$$-Mg\delta(x-ct)$$
$$+\sum_{i=1}^{4}\left[\frac{EI_o}{4}f_i(t)\left(\left(10-6\cos\frac{2\pi x}{L}+15\sin\frac{\pi x}{L}-\sin\frac{3\pi x}{L}\right)g_i^{IV}(x)\right]\right.$$
$$+6\frac{\pi}{L}\left(4\sin\frac{2\pi x}{L}+5\cos\frac{\pi x}{L}-\cos\frac{3\pi x}{L}\right)g_i^{III}(x)$$
$$+3\frac{\pi^2}{L^2}\left(8\cos\frac{2\pi x}{L}-5\sin\frac{\pi x}{L}+3\sin\frac{3\pi x}{L}\right)g_i^{II}(x)+$$
$$\mu_o\ddot{f}_i(t)\left(1+\sin\frac{\pi x}{L}\right)g_i(x)$$
$$-\mu_o R^o\ddot{f}_i(t)\left(g_i^{II}(x)+\sin\frac{\pi x}{L}g_i^{II}(x)+\frac{\pi}{L}\cos\frac{\pi x}{L}g_i^{I}(x)\right)$$
$$+M\delta(x-ct)\left(\ddot{f}_i(t)g_i(x)+2C\dot{f}_i(t)g_i^{I}(x)+C^2f_i(t)g_i^{II}(x)\right)$$
$$+Kf_i(t)g_i(x)\;\Big]=0 \qquad (17)$$

In order to determine $Y_m(t)$, it required that the expression on the left hand side of equation (17) be orthogonal to the function $V_k(x)$.

Thus,

$$\sum_{m=1}^{n}\left[\left[H_i(m,k)+H_2(m,k)-R^0\left(H_3(m,k)+H_4(m,k)+\frac{\pi}{L}H_5(m,k)\right)\right]\ddot{Y}_m(t)\right.$$
$$+\left\{\frac{EI_o}{4}\left([10H_6(m,k)+15H_7(m,k)-6H_8(m,k)-H_9(m,k)]+6\frac{\pi}{L}[4H_{10}(m,k)+15H_{11}(m,k)-H_{12}(m,k)]+\frac{K}{\mu_0}H_i(m,k)\right\}Y_m(t)\right.$$
$$\left.+3\frac{\pi^2}{L^2}[8H_{13}(m,k)+15H_4(m,k)+3H_{14}(m,k)]\;\right)+\frac{M}{\mu_o}\left[H_{15}(m,k)\ddot{Y}_m(t)+2CH_{16}(m,k)\dot{Y}_m(t)+C^2H_{17}(m,k)Y_m(t)\right]\Big]-\frac{Mg}{\mu_0}V_k(ct)$$
$$+[G_a(t)-G_b(t)+G_c(t)-G_d(t)+G_e(t)+G_f(t)-G_g(t)\;-G_d(t)+G_e(t)+G_f(t)-G_g(t)+G_h(t)-G_i(t)$$
$$+G_j(t)+G_k(t)+G_l(t)-G_m(t)-G_N(t)-G_o(t) \qquad (18)$$

where $+G_p(t)+G_q(t)+G_R(t)+G_s(t)\;]=0$

$$H_1(m,k)=\int_0^L V_m(x)V_k(x)dx,\quad H_2(m,k)=\int_0^L\sin\frac{\pi x}{L}V_m(x)V_k(x)dx,\quad H_3(m,k)=\int_o^L V_m^{II}(x)V_k(x)dx$$
$$H_4(m,k)=\int_0^L\sin\frac{\pi x}{L}V_m^{II}(x)V_k(x)dx,\quad H_5(m,k)=\int_0^L\cos\frac{\pi x}{L}V_m^{I}(x)V_k(x)dx,\quad H_6(m,k)=\int_0^L V_m^{IV}(x)V_k(xdx).$$
$$.H_7(m,k)=\int_0^L\sin\frac{\pi x}{L}V_m^{IV}(x)V_k(x)dx,\quad H_8(m,k)=\int_0^L\cos\frac{2\pi x}{L}V_m^{IV}(x)V_k(x)dx,\quad .H_9(m,k)=\int_0^L\sin\frac{3\pi x}{L}V_m^{IV}(x)V_k(x)dx$$
$$H_{10}(m,k)=\int_0^L\sin\frac{2\pi x}{L}V_m^{III}(x)V_k(x)dx,\quad .H_{11}(m,k)=\int_0^L\cos\frac{\pi x}{L}V_m^{III}(x)V_k(x)dx,\quad H_{12}(m,k)=\int_0^L\cos\frac{3\pi x}{L}V_m^{III}(x)V_k(x)dx$$
$$.H_{13}(m,k)=\int_0^L\cos\frac{2\pi x}{L}V_m^{II}(x)V_k(x)dx,\quad H_{14}(m,k)=\int_0^L\sin\frac{3\pi x}{L}V_m^{II}(x)V_k(x)dx\quad .H_{15}(m,k)=\int_0^L\delta(x-ct)V_m(x)V_k(x)dx$$
$$H_{16}(m,k)=\int_0^L\delta(x-ct)V_m^{I}(x)V_k(x)dx \text{ and } H_{17}(m,k)=\int_0^L\delta(x-ct)V_m^{II}(x)V_k(x)dx$$

are the resulting integrals and

$$G_a(t)=10\frac{EI_o}{4\mu_o}\sum_{i=1}^{4} f_i\int_0^L g_i^{IV}(x)V_k(x)dx,\ G_b(t)=\frac{6EI_o}{4\mu_o}\sum_{i=1}^{4} f_i(t)\int_0^L \cos\frac{2\pi x}{L}g_i^{IV}(x)V_k(x)dx\ ,\ G_c(t)=\frac{15EI_o}{4\mu_o}\sum_{i=1}^{4} f_i(t)\int_0^L \sin\frac{\pi x}{L}g_i^{IV}(x)V_k(x)dx\ ,$$

$$G_d(t)=\frac{EI_o}{4\mu_o}\sum_{i=1}^{4} f_i(t)\int_0^L \sin\frac{3\pi x}{L}g_i^{IV}(x)V_k(x)dx\ ,\ G_e(t)=\frac{24EI_o}{4\mu_o}\frac{\pi}{L}\sum_{i=1}^{4} f_i(t)\int_0^L \sin\frac{2\pi x}{L}g_i^{III}(x)V_k(x)dx\ ,\ G_f(t)=\frac{30EI_o}{4\mu_o}\frac{\pi}{L}\sum_{i=1}^{4} f_i(t)\int_0^L \cos\frac{\pi x}{L}g_i^{III}(x)V_k(x)dx$$

$$G_g(t)=\frac{6EI_o}{4\mu_o}\frac{\pi}{L}\sum_{i=1}^{4} f_i(t)\int_0^L \cos\frac{3\pi x}{L}g_i^{III}(x)V_k(x)dx,..G_h(t)=\frac{24EI_o}{4\mu_o}\frac{\pi^2}{L^2}\sum_{i=1}^{4} f_i(t)\int_0^L \cos\frac{2\pi x}{L}g_i^{II}(x)V_k(x)d\ \ G_h(t)=\frac{24EI_o}{4\mu_o}\frac{\pi^2}{L^2}\sum_{i=1}^{4} f_i(t)\int_0^L \cos\frac{2\pi x}{L}g_i^{II}(x)V_k(x)dx\ ,$$

$$G_i(t)=\frac{15EI_o}{4\mu_o}\frac{\pi^2}{L^2}\sum_{i=1}^{4} f_i(t)\int_0^L \sin\frac{\pi x}{L}g_i^{II}(x)V_k(x)dx,\ G_j(t)=\frac{9EI_o}{4\mu_o}\frac{\pi^2}{L^2}\sum_{i=1}^{4} f_i(t)\int_0^L \sin\frac{3\pi x}{L}g_i^{II}(x)V_k(x)dx\ ,\ G_k(t)=\sum_{i=1}^{4}\ddot{f}_i(t)\int_0^L g_i(x)V_k(x)dx$$

$$G_l(t)=\sum_{i=1}^{4}\ddot{f}_i(t)\int_0^L \sin\frac{\pi x}{L}g_i(x)V_k(x)dx\ ,\ G_m(t)=R^o\sum_{i=1}^{4}\ddot{f}_i(t)\int_0^L g_i^{II}(x)V_k(x)dx\qquad G_N(t)=R^o\sum_{i=1}^{4}\ddot{f}_i(t)\int_0^L \sin\frac{\pi x}{L}g_i^{II}(x)V_k(x)dx\ ,$$

$$G_O(t)=R^o\frac{\pi}{L}\sum_{i=1}^{4}\ddot{f}_i(t)\int_0^L \cos\frac{\pi x}{L}g_i^{I}(x)V_k(x)dx\quad G_p(t)=\frac{M}{\mu_o}\sum_{i=1}^{4}\ddot{f}_i(t)\int_0^L \delta(x-ct)g_i(x)V_k(x)dx\ ,$$

$$G_R(t)=\frac{c^2M}{\mu_o}\sum_{i=1}^{4} f_i(x)\int_0^L \delta(x-ct)g_i^{II}(x)V_k(x)dx \text{ and } G_s(t)=\frac{K}{\mu_o}\sum_{i=1}^{4} f_i(t)\int_0^L g_i(x)V_k(x)dx$$

At this junction, the solution is valid for the case when both ends of the Rayleigh beam are clamped is sought. Consequently, $V_m(x)$ is chosen as in equation (16) which is the beam function suitable for all other boundary conditions other than simple ones. Thus,

$$V_k(x)=\sin\frac{\lambda_k x}{L}+A_k\cos\frac{\lambda_k x}{L}+B_k\sinh\frac{\lambda_k x}{L}+C_k\cosh\frac{\lambda_k x}{L} \text{ and}$$

$$V_k(ct)=\sin\lambda_k\frac{ct}{L}+A_k\cos\lambda_k\frac{ct}{L}+B_k\sinh\frac{\lambda_k ct}{L}+C_k\cosh\frac{\lambda_k ct}{L}\qquad(20)$$

which is the beam function suitable for all other boundary conditions other than simple ones.

$$H_6(m,k)=\frac{\lambda_m^4}{L^4}H_1(m,k)\quad\text{and}\quad H_7(m,k)=\frac{\lambda_m^4}{L^4}H_2(m,k)$$

(21)

Next, use is made of the property of the Dirac-Delta function as an even function to express it in series form, namely

$$\delta(x-ct)=\left(\frac{1}{L}+\frac{2}{L}\sum_{n=1}^{\infty}\cos\frac{n\pi x}{L}\cos\frac{n\pi ct}{L}\right)\qquad(22)$$

In view of equation (22)

$$H_{15}(m,k)=\frac{1}{L}\left[H_1(m,k)+2\sum_{n=1}^{\infty}\cos\frac{n\pi ct}{L}H_{1A}(m,n,k)\right],$$

$$H_{16}(m,k)=\frac{1}{L}\left[H_{18}(m,k)+2\sum_{n=1}^{\infty}\cos\frac{n\pi ct}{L}H_{18a}(m,n,k)\right]$$

$$H_{17}(m,k)=\frac{1}{L}\left[H_3(m,k)+2\sum_{n=1}^{\infty}\cos\frac{n\pi ct}{L}H_{3A}(m,n,k)\right]\quad\text{and}$$

$$H_{18}(m,k)=\int_0^l V_m'(x)V_k(x)dx\qquad(23)$$

where (20)

$$H_{1A}(m,n,k)=\int_o^i\cos\frac{n\pi x}{V_m}L_m(x)V_k(x)dx\ ,$$

$$H_{3A}(m,n,k)=\int_0^l\cos\frac{n\pi x}{L}V_m''(x)V_k(x)dx\quad\text{and}$$

$$H_{18A}(m,n,k)=\int_0^l\cos\frac{n\pi x}{L}V_m'(x)V_k(x)dx\quad(21)$$

Next, we substitute equations (23) and (24) into equation (17) after some simplifications and rearrangements, leads to

$$\sum_{m=1}^{n}\left[\alpha_0(m,k)\ddot{Y}_m(t)+\alpha_1(m,k)Y_m(t)\right.$$

$$+\in\left(\left[H_1(m,k)+2\sum_{n=1}^{\infty}\cos\frac{n\pi ct}{L}H_{1A}(m,n,k)\right]\ddot{Y}_m(t)\right.$$

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$$+2C\left[H_{18}(m,k)+2\sum_{n=1}^{\infty}\cos\frac{n\pi ct}{L}H_{18A}(m,n,k)\right]\dot{Y}_m(t)$$

$$\left.\left.+C^2\left[H_3(m,k)+2\sum_{n=1}^{\infty}\cos\frac{n\pi ct}{L}H_{3A}(m,n,k)\right]Y_m(t)\right)\right]$$

$$=\frac{Mg}{\mu_0}\left[\sin\lambda_k\frac{ct}{L}+A_k\cos\lambda_k\frac{ct}{L}+B_k\sinh\frac{\lambda kct}{L}+C_k\cosh\lambda_k\frac{ct}{L}\right]$$

$$-\lfloor G_a(t)-G_b(t)+G_c(t)-G_d(t)+G_e(t)+G_f(t)-G_g(t)+G_h(t)-G_i(t)+$$

$$G_j(t)+G_k(t)+G_l(t)-G_m(t)-G_n(t)-G_o(t)+G_p(t)+G_q(t)+G_r(t)+G_s(t)\rfloor$$

(25)

Where

$$\in=\frac{mL}{\mu_0}$$

(26)

$$\alpha_0(m,k)=\left[H_1(m,k)+H_2(m,k)-R^0\left(H_3(m,k)+H_4(m,k)+\frac{\pi}{L}H_5(m,k)\right)\right]$$

(27)

and

$$\alpha_1(m,k)=\frac{EI_0}{4\mu_0}\left[\left[10H_6(m,k)+15H_7(m,k)-6H_8(m,k)-H_9(m,k)\right]\right.$$

$$\left.+6\frac{\pi}{L}\left[4H_{10}(m,k)+5H_{11}(m,k)-H_{12}(m,k)\right]+\frac{3\pi^2}{L^2}\left[8H_{13}(m,k)-5H_4(m,k)+3H_{14}(m,k)\right]\right]$$

$$+\frac{K}{\mu_0}H_1(m,k)$$

(28)

Equation (25) is the transformed equation governing the problem of time dependent Clamped-Clamped non-uniform Rayleigh beam resting on a constant Winkler elastic foundation and transverse by a moving load. This second order differential equation is actually valid for all variants of the classical boundary conditions. In what follows, we shall consider Clamped-Clamped boundary conditions as illustrative example.

Clamped-Clamped Boundary Conditions.

In this section, we consider a Rayleigh beam whose ends are clamped at ends x = 0 and x = L, both deflection and slope vanish at these ends. Thus, the conditions are expressed as

$$\bar{Z}(0,t)=0=\bar{Z}(L,t),\ \frac{\partial}{\partial x}\bar{Z}(0,t)=0=\frac{\partial}{\partial x}\bar{Z}(L,t) \quad \textbf{(29)}$$

Thus, for normal modes we have

$$\bar{V}_m(0)=0=\bar{V}_m(L),\quad \frac{\partial\bar{V}_m(0)}{\partial x}=0=\frac{\partial\bar{V}_m(L)}{\partial x} \qquad \textbf{(30)}$$

This implies that

$$\bar{V}_k(0)=0=\bar{V}_k(L),\ \frac{\partial\bar{V}_k(0)}{\partial x}=0=\frac{\partial\bar{V}_k(L)}{\partial x} \qquad \textbf{31)}$$

thus, making use of equations (29)-(31) into equation (16) the beam function, it can be shown that

$$B_m=B_k=-1 \qquad \textbf{(32)}$$

$$A_m=-C_m=-\frac{\sinh\lambda_m-\sin\lambda_m}{\cosh\lambda_m-\cos\lambda_m} \qquad \textbf{(33)}$$

$$A_k=-C_k=-\frac{\sinh\lambda_k-\sin\lambda_k}{\cosh\lambda_k-\cos\lambda_k} \qquad \textbf{(34)}$$

$$V_m(x)=V_{mcc}(x)=\cosh\frac{\lambda_m x}{L}-\cos\frac{\lambda_m x}{L}-\sigma_m\left(\sinh\frac{\lambda_m x}{L}-\sin\frac{\lambda_m x}{L}\right) \qquad \textbf{(35)}$$

The frequency equation is given by

$$\cosh\lambda_m\cos\lambda_m-1=0 \qquad \textbf{(36)}$$

Such that [2]

$\lambda_1=4.73004;\lambda_2=7.85320;\lambda_3=10.99561$ **and so on.**

At this juncture, it is pertinent to obtain the particular function $g_i(x)$ that ensures zeros of the right hand side of the boundary conditions. We now sought the function $g_i(x)$ to be a third degree polynomial.

$$g_i(x)=a_i x^3+b_i x^2+c_i x+d_i,\ i=1,2,3,4. \qquad (37)$$

To obtain $g_1(x)$ explicitly, it is required to satisfying four conditions defined in (3), that is,

$D_1[g_1(0)]=\delta_{11}=1$, $D_2[g_1(0)]=\delta_{12}=0$, $D_3[g_1(0)]=\delta_{13}=0$ and $D4[g_1(0)]=\delta_{14}=0$ (38)

For clamped-clamped end conditions as in equation (4)

$$g_1(x)=a_1(x)+b_1(x)+c_1(x)+d_1$$

$$D_1[g_1(0)]=1[a_1(0)+b_1(0)+c_1(0)+d_1]=1$$

hence, $d_1 = 1$ (39)

$$D_2[g_1(0)] = \frac{\partial}{\partial x}[g_1(x)] = 3a_1x^2 + 2b_1x + c_1$$

$$\frac{\partial}{\partial x}[g_1(0)] = 3a_1 0^2 + 2b_1 0 + c_1 = 0$$

therefore, $c_1 = 0$ (40)

$$D_3[g_1(L)] = 0$$

$$1\left[a_1L^3 + b_1L^2 + c_1L + d_1\right] = 0$$

using equations (39) and (40) , one obtains

$$a_1L^3 + b_1L^2 + 1 = 0 \qquad (41)$$

$$D_4[g_1(L)] = \frac{\partial}{\partial x}[g_1(L)] = 3a_1L^2 + 2b_1L + c_1$$

$$3a_1L^2 + 2b_1L + c_1 = 0$$

using equation (40), then

$$3a_1L^2 + 2b_1L = 0 \qquad (42)$$

Solving equations (41) and (42) simultaneously, it is obvious that

$$a_1 = \frac{2}{L^3} \text{ and } b_1 = -\frac{3}{L^2} \qquad (43)$$

therefore

$$g_1(x) = a_1x^3 + b_1x^2 + c_1x + d_1 = \frac{2}{L^3}x^3 - \frac{3}{L^2}x^2 + 1 = 1 - 3\left(\frac{x}{L}\right)^2 + 2\left(\frac{x}{L}\right)^3 \qquad (44)$$

To obtain $g_2(x)$ explicitly, it requires satisfying four conditions as enumerated above.

But $g_2(x) = a_2x^3 + b_2x^2 + c_2x + d_2$

$$D_1[g_2(0)] = \delta_{21} = 0$$

$$D_2[g_2(0)] = \delta_{22} = 1$$

$$D_3[g_2(0)] = \delta_{23} = 0$$

$$D4[g_2(0)] = \delta_{24} = 0 \qquad (45)$$

$$g_2(0) = a_2(0) + b_2(0) + c_2(0) + d_2$$

$$D_1[g_2(0)] = 1[a_2(0) + b_2(0) + c_2(0) + d_2] = 0$$

hence, $d_2 = 0$ (46)

$$D_2[g_2(x)] = \frac{\partial}{\partial x}[g_2(x)] = 3a_2x^2 + 2b_2x + c_2$$

$$\frac{\partial}{\partial x}[g_2(0)] = 3a_2 0^2 + 2b_2 0 + c_2 = 1$$

therefore, $c_2 = 1$ (47)

$$D_3[g_2(L)] = 0$$

$$1\left[a_2L^3 + b_2L^2 + c_2L + d_2\right] = 0$$

Using equations (46) and (47) , one obtains

$$a_2L^3 + b_2L^2 + 1 = 0 \qquad (48)$$

$$D_4[g_2(L)] = \frac{\partial}{\partial x}[g_2(L)] = 3a_2L^2 + 2b_2L + c_2$$

$$3a_2L^2 + 2b_2L + c_2 = 0$$

Using equation (47), then

$$3a_2L^2 + 2b_2L + 1 = 0 \qquad (49)$$

Solving equations (48) and (49) simultaneously, it is obvious that

$$a_2 = \frac{1}{L^2} \text{ and } b_2 = -\frac{2}{L}$$

Therefore

$$g_2(x) = a_2x^3 + b_2x^2 + c_2x + d_2 = \frac{1}{L^2}x^3 - \frac{2}{L}x^2 + x = x - 2\frac{x^2}{L} + \frac{x^3}{L^2} \qquad (51)$$

Similarly, when $i = 3,4.$ we have

$$g_3(x) = 3\left(\frac{x}{L}\right)^2 - 2\left(\frac{x}{L}\right)^3 \qquad (52)$$

and

$$g_4(x) = -\left(\frac{x^2}{L}\right) + \left(\frac{x^3}{L^2}\right) \qquad (53)$$

In view of equations (44) and (52). It is straight forward to show that

$$G_a(t) = G_b(t) = G_b(t) = G_b(t) = 0 \qquad (54)$$

$$G_e(t) = -\frac{72EI_o\pi}{\mu_o L^4}[N_1 + A_kN_2 + B_kN_3 + C_kN_4](f_3(t) - f_1(t))$$

$$G_f(t) = -\frac{90EI_o\pi}{\mu_o L^4}[N_5 + A_kN_6 + B_kN_7 + C_kN_8](f_3(t) - f_1(t))$$

$$G_g(t) = -\frac{18EI_o\pi}{\mu_o L^4}[N_9 + A_kN_{10} + B_kN_{11} + C_kN_{12}](f_3(t) - f_1(t))$$

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$$G_h(t)=\frac{36EI_o\pi^2}{\mu_o L^4}\left(f_3(t)-f_1(t)\right)\left[N_{13}+A_kN_{14}+B_kN_{15}+C_kN_{16}-\frac{2}{L}\left(N_{17}+A_kN_{18}+B_kN_{19}+C_kN_{20}\right)\right.$$

$$G_i(t)=\frac{15EI_o\pi^2}{4\mu_o L^2}\left(f_3(t)-f_1(t)\right)\left[\frac{6}{L^2}\left(N_{21}+A_kN_{22}+B_kN_{23}+C_kN_{24}\right)-\frac{12}{L^3}\left(N_{25}+A_kN_{26}+B_kN_{27}+C_kN_{28}\right)\right]$$

$$G_j(t)=\frac{9EI_o\pi^2}{4\mu_o L^2}\left(f_3(t)-f_1(t)\right)\left[\frac{6}{L^2}\left(N_{29}+A_kN_{30}+B_kN_{31}+C_kN_{32}\right)-\frac{12}{L^3}\left(N_{33}+A_kN_{34}+B_kN_{35}+C_kN_{36}\right)\right]$$

$$G_k(t)=\ddot{f}_1(t)\left(N_{37}+A_kN_{38}+B_kN_{39}+C_kN_{40}\right)+\frac{3}{L^2}\left(\ddot{f}_3(t)-\ddot{f}_1(t)\right)\left(N_{41}+A_kN_{42}+B_kN_{43}+C_kN_{44}\right)$$
$$-\frac{2}{L^3}\left(\ddot{f}_3(t)-\ddot{f}_1(t)\right)\left(N_{45}+A_kN46+B_kN_{47}+C_kN_{48}\right)$$

$$G_L(t)=\ddot{f}_1(t)\left(N_{21}+A_kN_{22}+B_kN_{23}+C_kN_{24}\right)+\frac{3}{L^2}\left(\ddot{f}_3(t)-\ddot{f}_1(t)\right)\left(N_{49}+A_kN_{50}+B_kN_{51}+C_kN_{52}\right)$$
$$-\frac{2}{L^3}\left(\ddot{f}_3(t)-\ddot{f}_1(t)\right)\left(N_{53}+A_kN_{54}+B_kN_{55}+C_kN_{56}\right)$$

$$G_m(t)=R^o\left(\ddot{f}_3(t)-\ddot{f}_1(t)\right)\left[\frac{6}{L^2}\left(N_{37}+A_kN_{38}+B_kN_{39}+C_kN_{40}\right)-\frac{12}{L^3}\left(N_{57}+A_kN_{58}+B_kN_{59}+C_kN_{60}\right)\right]$$

$$G_n(t)=R^o\left(\ddot{f}_3(t)-\ddot{f}_1(t)\right)\left[\frac{6}{L^2}\left(N_{21}+A_kN_{22}+B_kN_{23}+C_kN_{24}\right)-\frac{12}{L^3}\left(N_{25}+A_kN_{26}+B_kN_{27}+C_kN_{28}\right)\right]$$

$$G_o(t)=R^o\left(\ddot{f}_3(t)-\ddot{f}_1(t)\right)\left[\frac{6}{L^2}\left(N_{61}+A_kN_{62}+B_kN_{63}+C_kN_{64}\right)-\frac{6}{L^3}\left(N_{65}+A_kN_{266}+B_kN_{67}+C_kN_{68}\right)\right]$$

$$G_p(t)=\frac{m}{\mu_o}\ddot{f}_1(t)\left[\sin\lambda_k\frac{ct}{L}+A_k\cos\lambda_k\frac{ct}{L}+B_k\sinh\lambda_k\frac{ct}{L}+C_k\cosh\lambda_k\frac{ct}{L}\right]$$
$$+\frac{m}{\mu_o}\left(\ddot{f}_3(t)-\ddot{f}_1(t)\right)\left[\frac{3}{L^3}\left(N_{41}+A_kN_{42}+B_kN_{43}+C_kN_{44}\right)-\frac{2}{L^4}\left(N_{45}+A_kN_{46}+B_kN_{47}+C_kN_{48}\right)\right.$$
$$\left.+\sum_{n=1}^{\infty}\cos\frac{n\pi ct}{L}\left\{\frac{6}{L^3}\left(N_{69}+A_kN_{70}+B_kN_{71}+C_kN_{72}\right)-\frac{4}{L^4}\left(N_{75}+A_kN_{74}+B_kN_{75}+C_kN_{76}\right)\right\}\right]\Bigg\}$$

$$G_q(t)=\frac{2cm}{\mu_0}\left(\dot{f}_3(t)-\dot{f}_1(t)\right)\left[\frac{6}{L^3}\left(N_{57}+A_kN_{58}+B_kN_{59}+C_kN_{60}\right)-\frac{6}{L^4}\left[\left(N_{41}+A_kN_{42}+B_kN_{43}+C_kN_{44}\right)\right.\right.$$
$$\left.\left.+\sum_{n=1}^{\infty}\cos\frac{n\pi ct}{L}\left\{\frac{12}{L^3}\left(N_{77}+A_kN_{78}+B_kN_{79}+C_kN_{80}\right)-\frac{12}{L^4}\left(N_{69}+A_kN_{70}+B_kN_{71}+C_kN_{72}\right)\right\}\right]\right]\Bigg\}$$

$$G_r(t)=\frac{C^2m}{\mu_o}\left(f_3(t)-f_1(t)\right)\left[\frac{6}{L^2}\left(\sin\lambda_k\frac{ct}{L}+A_k\cos\lambda_k\frac{ct}{L}+B_k\sinh\lambda_k\frac{ct}{L}+C_k\cosh\lambda_k\frac{ct}{L}\right)\right.$$
$$\left.-\frac{12}{L^4}\left(N_{57}+A_kN_{58}+B_kN_{59}+C_kN_{60}\right)-\frac{24}{L^4}\sum_{n=1}^{\infty}\cos\frac{n\pi ct}{L}\left(N_{77}+A_kN_{78}+B_kN_{79}+C_kN_{80}\right)\right],$$

$$G_s(t)=\frac{k}{\mu_o}f_1(t)+\frac{k}{\mu_o}\left(f_3(t)-f_1(t)\right)\left[\frac{3}{L^2}\left(N_{41}+A_kN_{42}+B_kN_{43}+C_kN_{44}\right)-\frac{2}{L^3}\left(N_{45}+A_kN_{46}+B_kN_{47}+C_kN_{48}\right)\right] \qquad (55)$$

where

$N_i, i=1-100$ are different integrals.

Substituting equations (55) into equation (25), simplifying and rearranging yields.

$$\sum_{m=1}\left[\alpha_o(m,k)\ddot{Y}_m(t)+\alpha_1(m,k)Y_m(t)\right.$$
$$+\varepsilon\left\{\left(H_1(m,k)+2\sum_{n=1}^{\infty}\cos\frac{n\pi ct}{L}H_{1A}(m,n,k)\ddot{Y}_m(t)\right)\right.$$
$$+2C\left(H_{18}(m,k)+2\sum_{n=1}^{\infty}\cos\frac{n\pi ct}{L}H_{18A}(m,n,k)\dot{Y}_m(t)\right)$$

$$= P\left(\sin\lambda_k \frac{ct}{L} + A_k\cos\lambda_k \frac{ct}{L} + B_k\sinh\lambda_k \frac{ct}{L} + C_k\cosh\lambda_k \frac{ct}{L}\right) + \left(\ddot{f}_3(t) - \ddot{f}_1(t)\right)H_{38} - \ddot{f}_1(t)H_{24}$$

$$+ \left(f_3(t) - f_1(t)\right)\frac{k}{\mu_o}H_{36} + \frac{k}{\mu_o}f_1(t)$$

$$-\varepsilon L\left[\ddot{f}_1(t)P\left(\sin\lambda_k \frac{ct}{L} + A_k\cos\lambda_k \frac{ct}{L} + B_k\sinh\lambda_k \frac{ct}{L} + C_k\cosh\lambda_k \frac{ct}{L}\right)\right.$$

$$+\left(\ddot{f}_3(t) - \ddot{f}_1(t)\right)\left(H_{30} + H_{31}\sum_{n=1}^{\infty}\cos\frac{n\pi ct}{L}\right)$$

$$+2C\left(\dot{f}_3(t) - \dot{f}_1(t)\right)\left(H_{32} + H_{33}\sum_{n=1}^{\infty}\cos\frac{n\pi ct}{L}\right)$$

$$+C^2\left(\dot{f}_3(t) - \dot{f}_1(t)\right)\left\{\frac{6}{L^2}\left(\sin\lambda_k \frac{ct}{L} + A_k\cos\lambda_k \frac{ct}{L} + B_k\sinh\lambda_k \frac{ct}{L} + C_k\cosh\lambda_k \frac{ct}{L}\right)\right.$$

$$\left.\left.-\left(H_{34} + H_{35}\sum_{n=1}^{\infty}\cos\frac{n\pi ct}{L}\right)\right\}\right] \qquad (56)$$

It is remarked here that, it is only necessary to compute those $g_i(x)$ for which the corresponding $f_i(t)$ do not vanish. for our analysis, we shall consider a clamped beams whose end x = 0, (say) is subjected to a sine-wave (undamped) transient displacement, starting from rest and end x = L is subjected to a damped sine-wave transient displacement starting from rest. Thus, we can write.

$$f_1(t) = B\sin\Omega t \text{ and } f_3(t) = Ae^{-\beta t}\sin\Omega t \qquad (57)$$

where A,B are amplitudes, β is the parameter and Ω is the frequency.

Therefore, the required $g_i(x)$ are $g_1(x)$ and $g_3(x)$. The determination of $f_1(t)$, $f_3(t)$, $g_1(x)$ and $g_3(x)$ allows the complete determination of the right hand side of the initial conditions. Thus, setting $U_0(x)$ and $\frac{d}{dx}U_0(x)$ to zero respectively for simplicity and substituting $f_1(t)$, $f_3(t)$, $g_1(x)$ and $g_3(x)$ into the initial condition. One obtains

$$\bar{Z}(x,0) = 0\text{, and } \bar{Z}_t(x,0) = -\Omega \qquad \textbf{(58)}$$

Which when transformed yield

$$\bar{Z}(m,0) = 0 \text{ and } \bar{Z}_t(m,0) = \eta_2 \qquad \textbf{(59)}$$

where

$$\eta_2 = \eta_{or}\left[(1 - \cos\lambda_m) + B_m(\cosh\lambda_m - 1) + A_m\sin\lambda_m + C_m\sinh\lambda_m\right] \qquad \textbf{(60)}$$

and

$$\eta_{or} = -\frac{L\Omega}{\lambda_m} \qquad (61)$$

Using the influence functions equations (57) and their derivatives in equation (56), after some simplifications and rearrangements, equation (56) becomes.

$$\sum_{m=1}^{n}\left[\alpha_o(m,k)\ddot{Y}_m(t) + \alpha_1(m,k)Y_m(t) + \varepsilon\left\{\left(H_1(m,k) + 2\sum_{n=1}^{\infty}\cos\frac{n\pi ct}{L}H_{1A}(m,n,k)\right)\ddot{Y}_m(t)\right.\right.$$

$$\left.\left.+2C\left(H_{18}(m,k) + 2\sum_{n=1}^{\infty}\cos\frac{n\pi ct}{L}H_{18A}(m,n,k)\right)\dot{Y}_m(t) + C^2\left(H_3(m,k) + 2\sum_{n=1}^{\infty}\cos\frac{n\pi ct}{L}H_{3A}(m,n,k)\right)Y_m(t)\right\}\right]$$

$$= PV_k(ct) + H_{39}\sin\Omega t + H_{40}e^{-\beta t}\sin\Omega t - H_{41}e^{-\beta t}\cos\Omega t$$

$$-\varepsilon L\left[H_{42}\sin\Omega t + H_{43}e^{-\beta t}\sin\Omega t - H_{44}\cos\Omega t + H_{45}e^{-\beta t}\cos\Omega t\right.$$

$$+H_{46}\sum_{n=1}^{\infty}\cos\frac{n\pi ct}{L}\sin\Omega t + H_{47}\sum_{n=1}^{\infty}\cos\frac{n\pi ct}{L}e^{-\beta t}\cos\Omega t + H_{48}\sum_{n=1}^{\infty}\cos\frac{n\pi ct}{L}e^{-\beta t}\sin\Omega t$$

$$\left.-\sum_{n=1}^{\infty}H_{49}\sum_{n=1}^{\infty}\cos\frac{n\pi ct}{L}\cos\Omega t - H_{50}V_k(ct) + H_{51}V_k(ct)e^{-\beta t}\sin\Omega t\right] \qquad (62)$$

Where

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$$H_1 = (N_1 - N_3) + A_k(N_2 - N_4), H_2 = (N_5 - N_7) + A_k(N_6 - N_8), \quad H_3 = (N_9 - N_{11}) + A_k(N_{10} - N_{12})$$
$$H_4 = (N_{13} - N_{15}) + A_k(N_{14} - N_{17}), H_5 = (N_{15} - N_{17}) + A_k(N_{18} - N_{20}), \quad H_6 = (N_{21} - N_{23}) + A_k(N_{22} - N_{24})$$
$$H_7 = (N_{25} - N_{27}) + A_k(N_{26} - N_{28}), \ H_8 = (N_{29} - N_{31}) + A_k(N_{30} - N_{32}) H_9 = (N_{33} - N_{35}) + A_k(N_{34} - N_{36}),$$
$$H_{10} = (N_{37} - N_{39}) + A_k(N_{38} - N_{40}), H_{11} = (N_{41} - N_{43}) + A_k(N_{42} - N_{44}), H_{12} = (N_{45} - N_{47}) + A_k(N_{46} - N_{48})$$
$$H_{11} = (N_{41} - N_{43}) + A_k(N_{42} - N_{44}), \ H_{12} = (N_{45} - N_{47}) + A_k(N_{46} - N_{48}), H_{13} = (N_{49} - N_{17}) + A_k(N_{18} - N_{20}),$$
$$H_{14} = (N_{53} - N_{55}) + A_k(N_{54} - N_{56}), H_{15} = (N_{25} - N_{51}) + A_k(N_{50} - N_{52}), \ H_{16} = (N_{61} - N_{63}) + A_k(N_{62} - N_{64})$$
$$H_{17} = (N_{65} - N_{67}) + A_k(N_{66} - N_{68}), \ H_{18} = (N_{69} - N_{74}) + A_k(N_{70} - N_{72}), H_{19} = (N_{73} - N_{75}) + A_k(N_{74} - N_{76}),$$
$$H_{20} = (N_{77} - N_{79}) + A_k(N_{78} - N_{80}), H_{21} = \left(H_4 + \frac{2}{L}H_5\right), \ H_{22} = \left(H_6 - \frac{2}{L}H_7\right), H_{23} = \left(H_8 - \frac{2}{L}H_9\right),$$
$$H_{24} = (H_{10} + H_6), \ H_{25} = \left(\frac{3}{L^2}H_{11} - \frac{2}{L^3}H_{12}\right), \ H_{26} = \left(\frac{3}{L^2}H_{13} - \frac{2}{L^3}H_{14}\right), H_{27} = \left(\frac{6}{L^2}H_{10} - \frac{12}{L^3}H_{15}\right)$$
$$H_{28} = \left(\frac{6}{L^2}H_6 - \frac{12}{L^3}H_7\right), \quad H_{29} = \left(\frac{6}{L^2}H_{16} - \frac{6}{L^3}H_{17}\right), H_{30} = \left(\frac{3}{L^3}H_{11} - \frac{2}{L^4}H_{12}\right), H_{31} = \left(\frac{6}{L^3}H_{18} - \frac{4}{L^4}H_{19}\right),$$
$$H_{32} = 2C\left(\frac{6}{L^3}H_{15} - \frac{6}{L^4}H_{11}\right), H_{33} = \frac{24C}{L^3}\left(H_{20} - \frac{H_{18}}{L}\right), H_{34} = \frac{12}{L^4}H_{15}, H_{35} = \frac{24}{L^4}H_{20}, \ H_{36} = \left(\frac{3}{L^2}H_{11} - \frac{2}{L^3}H_{12}\right)$$
$$H_{37} = \left[\frac{72EI_o\pi H_1}{\mu_o L^4} + \frac{90EI_o\pi H_2}{\mu_o L^4} - \frac{18EI_o\pi H_3}{\mu_o L^4} - \frac{36EI_o\pi H_{21}}{\mu_o L^4} + \frac{90EI_o\pi^2 H_1}{4\mu_o L^4} - \frac{54EI_o\pi^2 H_{23}}{4\mu_o L^4}\right]$$
$$+\frac{18EI_o\pi}{\mu_o L^4}\left(4H_1 + 6H_3 - 2H_{21} + \frac{6}{4}\pi H_{22} - \frac{3}{4}\pi H_{23}\right), \ H_{38} = H_{37} - \left[(H_{25} + H_{26}) - R^o(H_{27} + H_{28} + H_{29})\right]$$
$$H_{39} = \left(\Omega^2 H_{38} + \Omega^2 H_{24} - \frac{k}{\mu_o}H_{36} + \frac{k}{\mu_o}\right) \ H_{40} = \left[(\beta^2 - \Omega^2)H_{38} + \frac{k}{\mu_o}H_{36}\right], \quad H_{41} = 2\beta\Omega H_{38}, \mathrm{H}_{42} = (\Omega^2 H_{30} + C^2 H_{34})$$
$$H_{43} = \left[(\beta^2 - \Omega^2)H_{30} - 2\beta C H_{32} - C^2 H_{34}\right], \ H_{44} = 2C\Omega H_{32}, \ H_{45} = (2C\Omega H_{22} - 2B\Omega H_{30})$$
$$H_{46} = (\Omega^2 H_{31} + C^2 H_{35}), \ H_{47} = 2C\Omega H_{33}, \ H_{48} = \left[(\beta^2 - \Omega^2)H_{31} - 2\beta C H_{33} - C^2 H_{35}\right]$$
$$H_{49} = (2\beta\Omega H_{31} + 2C\Omega H_{33}), \ H_{50} = \left(\Omega^2 P + \frac{6}{L^2}C^2\right) \text{and } H_{51} = \frac{6C^2}{L^2} \qquad (63)$$

Equation (62) represents the transformed equation of the non-uniform Rayleigh beam model Clamped at both ends which undergo displacements which vary with time when it is traveling under the action of concentrated load. In what follows we shall discuss two special cases of the equation

Clamped-Clamped Traversed by Moving Force.

This model neglects the inertial effect of the moving mass M. Thus, in equation (62), ε is set to zero. On this consideration, the transformed equation (62) reduces to

$$\ddot{Y}_m(t) + \alpha_{mf}^2 Y_m t$$
$$= \frac{1}{\alpha_o(m,k)}\left[P\left(\sin\lambda_k \frac{ct}{L} + A_k \cos\lambda_k \frac{ct}{L} + B_k \sinh\lambda_k \frac{ct}{L}\right.\right.$$
$$+ C_k \sinh\lambda_k \frac{ct}{L} + H_{39}\sin\Omega t$$
$$\left. + H_{40}e^{-\beta t}\sin\Omega t - H_{41}e^{-\beta t}\cos\Omega t\right] \qquad (64)$$

This is the classical case of a moving force problem associated with the system. It is an approximate model which assumes the inertia effect of the moving mass as negligible.

To obtain the solution of equation (64), it is subjected to Lap lace transformation defined as

$$(\tilde{\cdot}) = \int_0^\infty (\cdot)e^{-st}dt \qquad (65)$$

where s is the Lap lace parameter in conjunction with the initial conditions

$$\overline{Z}(m,0) = 0 = \overline{Z}_t(m,0)$$

and finally, by the use of convolution theory ,one obtains

$$Y_m(t) = \frac{1}{\alpha_{mf}}\left[P_o \frac{\alpha_{mf}}{\alpha_{mf}^2 - z_3^2}\left(\sin z_3 t - \frac{z_3}{\alpha_{mf}}\sin\alpha_{mf}t\right) + P_o A_k \frac{\alpha_{mf}}{\alpha_{mf}^2 - z_3^2}\left(\cos\alpha_{mf}t + \cos z_3 t\right)\right.$$
$$+ P_o B_k\left(\frac{\alpha_{mf}}{\alpha_{mf}^2 + z_3^2}\left(\sinh z_3 t - \frac{z_3}{\alpha_{mf}}\sin\alpha_{mf}t\right)\right)$$
$$+ P_o C_k\left(\frac{\alpha_{mf}}{\alpha_{mf}^2 - z_3^2}\left(\cosh z_3 t - \cos\alpha_m t\right)\right)$$
$$+ H_{39}^*\left(\frac{\alpha_{mf}}{\alpha_{mf}^2 - \Omega^2}\left(\sin\Omega t - \frac{\Omega}{\alpha_{mf}}\sin\alpha_{mf}t\right)\right)$$
$$+ H_{40}^*\left(\frac{\alpha_{mf}}{Q_4}\left(e^{-\beta t}\sin\Omega t + \sin\alpha_{mf}t\right) + \frac{2\alpha_{mf}\Omega}{Q_4}\left(e^{-\beta t}\cos\Omega t - \cos\alpha_{mf}t\right)\right)$$
$$+ H_{41}^* e^{-\beta t}\left(\frac{1}{Q_4}\left[\beta\left(\beta^2 + \alpha_{mf}^2 + \Omega^2\right)\sin\alpha_{mf}t + + \alpha_{mf}\left(\beta^2 + \alpha_{mf}^2 - \Omega^2\right)\left(e^{-\beta t}\cos\Omega t - \cos\alpha_{mf}t\right) - 2\beta\alpha_m\Omega e^{-\beta t}\sin\Omega t\right]\right)$$
$$\left. + C^o \sin\alpha_{mf}t\right] \qquad (66)$$

Where

$$P_o = \frac{P}{\alpha_o(m,k)}, \quad H_{39}^* = \frac{H_{39}}{\alpha_o(m,k)},$$
$$H_{40}^* = \frac{H_{40}}{\alpha_o(m,k)}, \quad H_{41}^* = \frac{H_{41}}{\alpha_o(m,k)}$$

$$\alpha_{mf}^2 = \frac{\alpha_1(m,k)}{\alpha_o(m,k)} \quad \text{and} \quad z_3 = \frac{\lambda_k c}{L} \qquad (67)$$

Therefore,

$$Z_n(x,t) = \sum_{m=1}^{n} Y_m(t)\left[\sin\frac{\lambda_k x}{L} + A_k\cos\frac{\lambda_k x}{L} + B_k\sinh\frac{\lambda_k x}{L} + C_k\cosh\frac{\lambda_k x}{L}\right] \qquad (68)$$

Consequently, by equation (7)

$$U(x,t) = Z(x,t) + \sin\Omega t\left(1 - 3\left(\frac{x}{L}\right)^2 + 2\left(\frac{x}{L}\right)^3\right) + e^{-\beta t}\sin\Omega t\left(3\left(\frac{x}{L}\right)^2 - 2\left(\frac{x}{L}\right)^3\right) \qquad (69)$$

Equation (69) is the transverse-displacement response to a moving force of a non-uniform Rayleigh beam clamped at both ends which are constrained to undergo displacements which vary with time.

Clamped-Clamped Traversed by Moving Mass.

In this section, the solution to the entire equation (62) is sought when no terms of the coupled differential equation is neglected. Evidently, an exact solution to this second order ordinary differential equation (62) is impossible.
Though the equation yields readily to numerical techniques, an analytical approximate method is desirable as the solution so obtained often sheds light on the vital information about the vibrating system. Therefore, we are going to use a modification of the asymptotic methods due to Struble often used in treating weakly homogeneous and non-homogeneous, non-linear oscillatory system discussed in [2]. To this end equation (62) is rearranged to take the form.

$$\ddot{Y}_m(t) + \frac{2\varepsilon C\left[H_{18}(m,k) + 2\sum_{n=1}^{\infty}\cos\frac{n\pi ct}{L}H_{18A}(m,n,k)\right]\dot{Y}_m(t)}{\left[\alpha_o(m,k) + \varepsilon\left(H_1(m,k) + 2\sum_{n=1}^{\infty}\cos\frac{n\pi ct}{L}H_{1A}(m,n,k)\right)\right]} + \frac{\left[\alpha_1(m,k) + \varepsilon C^2\left(H_3(m,k) + 2\sum_{n=1}^{\infty}\cos\frac{n\pi ct}{L}H_{3A}(m,n,k)\right)\right]Y_m(t)}{\left[\alpha_o(m,k) + \varepsilon\left(H_1(m,k) + 2\sum_{n=1}^{\infty}\cos\frac{n\pi ct}{L}H_{1A}(m,n,k)\right)\right]}$$

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$$=\left(\frac{\varepsilon L}{\left[\alpha_o(m,k)+\varepsilon\left(H_1(m,k)+2\sum_{n=1}^{\infty}\cos\frac{n\pi ct}{L}H_{1A}(m,n,k)\right)\right]}\right)*\Big(H_{52}\sin\lambda_k\frac{ct}{L}+H_{52}A_k\cos\lambda_k\frac{ct}{L}+H_{52}B_k\sinh\lambda_k\frac{ct}{L}+H_{52}C_k\cos\lambda_k\frac{ct}{L}$$

$$+H_{53}\sin\Omega t+H_{44}\cos\Omega t-H_{55}e^{-\beta t}\cos\Omega t-H_{46}\sum_{n=1}^{\infty}\cos\frac{n\pi ct}{L}\sin\Omega t-H_{47}\sum_{n=1}^{\infty}\cos\frac{n\pi ct}{L}e^{-\beta t}\cos\Omega t+H_{54}e^{-\beta t}\sin\Omega t$$

$$-H_{46}\sum_{n=1}^{\infty}\cos\frac{n\pi ct}{L}\sin\Omega t-H_{47}\sum_{n=1}^{\infty}\cos\frac{n\pi ct}{L}e^{-\beta t}\cos\Omega t-H_{48}\sum_{n=1}^{\infty}\cos\frac{n\pi ct}{L}e^{-\beta t}\sin\Omega t+H_{49}\sum_{n=1}^{\infty}\cos\frac{n\pi ct}{L}e^{-\beta t}\cos\Omega t\Big] \quad (70)$$

Where

$$P=\frac{mg}{\mu_o},\quad \varepsilon=\frac{m}{L\mu_o},\quad H_{52}=(g+H_{50}),\quad H_{53}=\left(\frac{\mu_o}{m}H_{39}-H_{42}\right),$$

$$H_{54}=\left(\frac{\mu_o}{m}H_{40}-H_{43}\right),\ \text{and}\ H_{55}=\left(\frac{\mu_o}{m}H_{41}+H_{45}\right) \quad (71)$$

Next, we consider the homogenous part of equation (70) and seek a modified frequency corresponding to the frequency of the free system due to the presence of the moving mass. An equivalent free operator defined by the modified frequency then replaces equation (70), using Struble's technique the equation simplifies to

$\ddot{Y}_m(t)+\beta_{mf}Y_m(t)=0$ and the entire equation becomes

$$\ddot{Y}_m(t)+\beta_{mf}Y_m(t)$$
$$=\frac{\lambda L}{\alpha_o(m,k)}\left[H_{52}\left(\sin\lambda_k\frac{ct}{L}+A_k\cos\lambda_k\frac{ct}{L}+B_k\sinh\lambda_k\frac{ct}{L}+C_k\cosh\lambda_k\frac{ct}{L}\right)\right.$$
$$+H_{53}\sin\Omega t+H_{54}e^{-\beta t}\sin\Omega t+H_{44}\cos\Omega t$$
$$-H_{55}e^{-\beta t}\cos\Omega t-H_{46}\sum_{n=1}^{\infty}\cos\frac{n\pi ct}{L}\sin\Omega t$$
$$-H_{47}\sum_{n=1}^{\infty}\cos\frac{n\pi ct}{L}e^{-\beta t}\cos\Omega t-H_{48}\sum_{n=1}^{\infty}\cos\frac{n\pi ct}{L}e^{-\beta t}\sin\Omega t$$
$$\left.+H_{49}\sum_{n=1}^{\infty}\cos\frac{n\pi ct}{L}\cos\Omega t\right] \quad (72)$$

where

$$\beta_{mf}=\alpha_{mf}\left[1-\frac{\lambda}{2}\left(\frac{H_1(m,k)}{\alpha_o(m,k)}-\frac{C^2H_3(m,k)}{\alpha_o(m,k)\alpha_{mf}^2}\right)\right] \quad (73)$$

is the modified natural frequency due to the presence of moving mass.

To obtain the solution of equation (72), it is subjected to a Laplace transform and convolution theory in conjunction with the initial conditions. Thus,

$$Y_m(t)=\frac{1}{\beta_{mf}}\left[H_{56}y_a+H_{57}y_b+H_{58}y_c\right.$$
$$+H_{59}y_d+H_{60}y_e+H_{61}y_f+H_{62}y_g$$
$$-H_{63}y_h+H_{64}y_i-H_{64}y_j-H_{65}y_k$$
$$\left.-H_{65}y_l-H_{66}y_m+H_{66}y_n+H_{67}y_o+H_{67}y_p\right] \quad (74)$$

where

$$H_{56}=\frac{\lambda LH_{52}}{\alpha_o(m,k)},\ H_{57}=H_{56}A_k,\ H_{58}=H_{56}B_k,$$
$$H_{59}=H_{56}C_k,\ H_{60}=\frac{\lambda LH_{53}}{\alpha_o(m,k)},\ H_{61}=\frac{\lambda LH_{54}}{\alpha_o(m,k)}$$
$$H_{62}=\frac{\lambda LH_{44}}{\alpha_o(m,k)},\ H_{63}=\frac{\lambda LH_{55}}{\alpha_o(m,k)},\ H_{64}=\frac{\lambda LH_{46}}{2\alpha_o(m,k)},$$
$$H_{65}=\frac{\lambda LH_{47}}{2\alpha_o(m,k)},\ H_{66}=\frac{\lambda LH_{48}}{\alpha_o(m,k)}\ \text{and}\ H_{47}=\frac{\lambda LH_{49}}{2\alpha_o(m,k)} \quad (75)$$

and

$$y_a = \left(\sin z_3 t \int_o^t \sin \beta_{mf} \tau \cos z_3 \tau d\tau - \cos z_3 t \int_o^t \sin \beta_{mf} \tau \sin z_3 \tau d\tau \right), \; y_b = \left(\cos z_3 t \int_o^t \sin \beta_{mf} \tau \cos z_3 \tau d\tau + \sin z_3 t \int_o^t \sin \beta_{mf} \tau \cos z_3 \tau d\tau \right)$$

$$y_c = \left(\sin \beta_{mf} t \int_o^t \sinh z_3 \tau \cos \beta_{mf} \tau d\tau - \cos \beta_{mf} t \int_o^t \sinh z_3 \tau \sin \beta_{mf} \tau d\tau \right),$$

$$y_d = \left(\sin \beta_{mf} t \int_o^t \cosh z_3 \tau - \cos \beta_{mf} \tau d\tau - \cos \beta_{mf} t \int_o^t \cosh z_3 \tau - \sin \beta_{mf} \tau d\tau \right)$$

$$y_e = \left(\sin \beta_{mf} t \int_o^t \sin \beta_{mf} \tau \sin \Omega \tau d\tau - \cos \beta_{mf} t \int_o^t \sin \beta_{mf} \tau \cos \Omega \tau d\tau \right)$$

$$y_f = \left(\sin \beta_{mf} t \int_o^t e^{-\beta \tau} \cos \beta_{mf} \tau \sin \Omega \tau d\tau - \cos \beta_{mf} t \int_o^t e^{-\beta \tau} \sin \beta_{mf} \tau \sin \Omega \tau d\tau \right)$$

$$y_g = \left(\cos \Omega t \int_o^t \sin \beta_{mf} \tau \cos \Omega \tau d\tau + \sin \Omega t \int_o^t \sin \beta_{mf} \tau \sin \Omega \tau d\tau \right)$$

$$y_h = e^{-\beta t} \left(\cos \Omega t \int_o^t e^{\beta \tau} \sin \beta_{mf} \tau \cos \Omega \tau d\tau + \sin \Omega t \int_o^t e^{\beta \tau} \sin \beta_{mf} \tau \sin \Omega \tau d\tau \right)$$

$$y_i = \sum_{n=1}^{\infty} \left(\sin(z_1 + \Omega)t \int_o^t \sin \beta_{mf} \tau \cos(z_1 + \Omega)\tau d\tau - \cos(z_1 + \Omega)t \int_o^t \sin \beta_{mf} \tau \sin(z_1 + \Omega)\tau d\tau \right)$$

$$y_j = \sum_{n=1}^{\infty} \left(\sin(z_1 - \Omega)t \int_o^t \sin \beta_{mf} \tau \cos(z_1 - \Omega)\tau d\tau - \cos(z_1 - \Omega)t \int_o^t \sin \beta_{mf} \tau \sin(z_1 - \Omega)\tau d\tau \right)$$

$$y_k = \sum_{n=1}^{\infty} e^{-\beta t} \left(\cos(z_1 + \Omega)t \int_o^t e^{\beta \tau} \sin \beta_{mf} \tau \cos(z_1 + \Omega)\tau d\tau + \sin(z_1 + \Omega)t \int_o^t e^{\beta \tau} \sin \beta_{mf} \tau \sin(z_1 + \Omega)\tau d\tau \right)$$

$$y_l = \sum_{n=1}^{\infty} e^{-\beta t} \left(\cos(z_1 - \Omega)t \int_o^t e^{\beta \tau} \sin \beta_{mf} \tau \cos(z_1 - \Omega)\tau d\tau + \sin(z_1 - \Omega)t \int_o^t e^{\beta \tau} \sin \beta_{mf} \tau \sin(z_1 - \Omega)\tau d\tau \right)$$

$$y_m = \sum_{n=1}^{\infty} \left(\sin \beta_{mf} t \int_o^t e^{-\beta \tau} \cos \beta_{mf} \tau \sin(z_1 + \Omega)\tau d\tau - \cos \beta_{mf} t \int_o^t e^{-\beta \tau} \sin \beta_{mf} \tau \sin(z_1 + \Omega)\tau d\tau \right)$$

$$y_n = \sum_{n=1}^{\infty} \left(\sin \beta_{mf} t \int_o^t e^{-\beta \tau} \cos \beta_{mf} \tau \sin(z_1 - \Omega)\tau d\tau - \cos \beta_{mf} t \int_o^t e^{-\beta \tau} \sin \beta_{mf} \tau \sin(z_1 - \Omega)\tau d\tau \right)$$

$$y_o = \sum_{n=1}^{\infty} \left(\cos(z_1 + \Omega)t \int_o^t \sin \beta_{mf} \tau \cos(z_1 + \Omega)\tau d\tau + \sin(z_1 + \Omega)t \int_o^t \sin \beta_{mf} \tau \sin(z_1 + \Omega)\tau d\tau \right)$$

$$y_p = \sum_{n=1}^{\infty} \left(\cos(z_1 - \Omega)t \int_o^t \sin \beta_{mf} \tau \cos(z_1 - \Omega)\tau d\tau + \sin(z_1 - \Omega)t \int_o^t \sin \beta_{mf} \tau \sin(z_1 - \Omega)\tau d\tau \right) \quad (76)$$

Hence,

$$z_n(x,t) = \sum_{m=1}^{n} (Y_m(t)) \left[\sin \frac{\lambda_k x}{L} + A_k \cos \frac{\lambda_k x}{L} + B_k \sinh \frac{\lambda_k x}{L} + C_k \cosh \frac{\lambda_k x}{L} \right] \quad (77)$$

Consequently,

$$u(x,t) = z(x,t) + \sin \Omega t \left(1 - 3\left(x/L\right)^2 + 2\left(x/L\right)^3 \right) + e^{-\beta t} \sin \Omega t \left(3\left(x/L\right)^2 - 2\left(x/L\right)^3 \right) \quad (78)$$

Equation (78) is the dynamic response of a Clamped-Clamped non-uniform Rayleigh beam to a moving mass when one end of the beam (x = 0) is subjected to a sine-wave

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transient displacement starting from rest while the other end (x = L) is subjected to a damped sine-wave transient displacement starting from rest.

Discussion of the Analytical Solution

If the undamped system such as this is studied, it is desirable to examine the response amplitude of the dynamical system which may grow without bound. This is termed resonance when it occurs. The Clamped-Clamped elastic Rayleigh beams transverse by a moving force will be in state of resonance whenever $\alpha_{mf}^2 = z_3^2$ which implies that $\alpha_{mf} = z_3$ (79)

and equation (73) shows that the same beam under the action of moving mass experiences resonance effect when

$\beta_{mf}^2 = z_3^2$ which implies that $\beta_{mf} = z_3$ (80)

from equation (73), it implies that

$$\alpha_{mf} = \frac{z_3}{\left[1 - \frac{\lambda}{2}\left(\frac{H_1(m,k)}{\alpha_o(m,k)} - \frac{c^2 H_3(m,k)}{\alpha_o(m,k)\alpha_{mf}^2}\right)\right]} \qquad (81)$$

From equations (80) and (81), we deduced for the same natural frequency, the critical speed for the system of a **Clamped-Clamped** elastic beam on an elastic foundation and traversed by a moving force is grater than that traversed by moving mass. Thus, resonance is reached earlier in the moving mass system than in the moving force system.

NUMERICAL CALCULATION AND DISCUSSION OF THE RESULTS

Illustrating the foregoing analysis, the non-uniform Rayleigh beam of length L=12.192m is considered. Furthermore, the load velocity u = .123, $E = 2.109 \times 10^9 kg/m$, $\frac{EI}{\mu} = 2200 m^4/s^2$

and the ratio of the mass of the load to mass of the beam is 0.25. The traverse deflections of the non-uniform Rayleigh beams are calculated and plotted against time for various values of parameters in the dynamical system. Values of axial force N between 0 and 20000, foundation modulli K were varied between 0N m^2 and 4000000N m^2 .

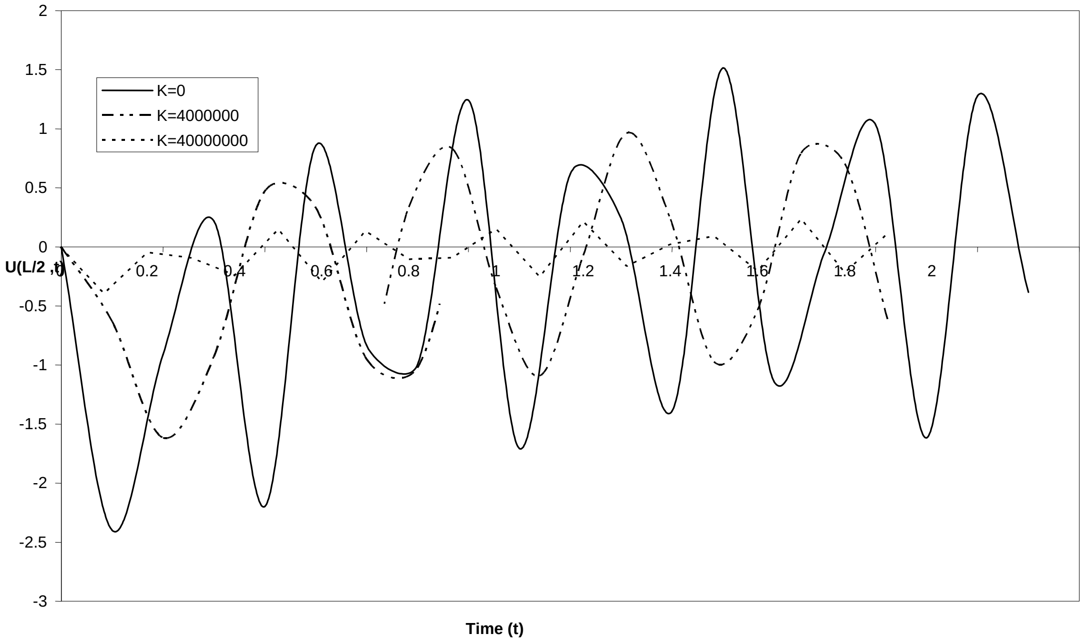

Fig.1: Deflection profile of the Clamped-Clamped Non-Uniform Rayleigh Beam under a moving force for various values of foundation modulli K and for fixed rotatory inertia r(1)

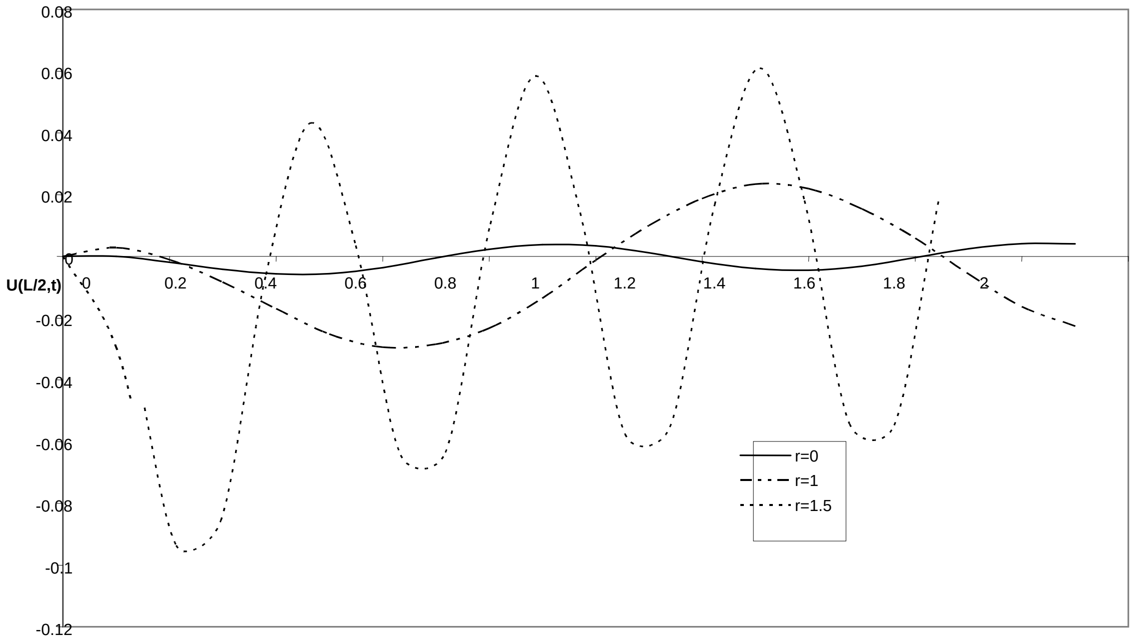

Fig.2: Deflection profile of the Clamped-Clamped Non-Uniform Rayleigh Beam under a moving force for various values of rotatory inertia r and for fixed value of foundation modulus K(40000)

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Fig.1, displays the transverse displacement response of Clamped-Clamped non-uniform Rayleigh beam under the action of a moving force for various values of foundation modulli K and for fixed values of axial force N and rotator inertia r^o.The graph shows that the response amplitude decreases as the values of the foundation modulli K increases. In, fig.2, the deflection profile due to a moving force of Clamped-Clamped non-uniform Rayleigh beam for fixed value of foundation modulli K and axial force N and for various values of rotatory inertia r^o. It is clearly seen that as the rotatory inertia value increases, the response amplitude of the beam reduces. Also, in fig.5, the response amplitude of the Clamped-Clamped non-uniform Rayleigh beam under the action of moving force for various values of axial force N and for fixed values of foundation modulus K and rotatory inertial corrector r^o is displayed. It is observed that as the axial force N increases the response amplitude of the beam decreases. Furthermore, fig.3, depicts the transverse displacement response of Clamped-Clamped non-uniform Rayleigh beam under a moving mass for fixed values of rotatory inertia r^o and axial force N and for various values of foundation modulli K . The response amplitude of the beam was found to decrease as the values of the foundation moduli K increases. In, fig.4, the deflection profile of the Clamped-Clamped non-uniform Rayleigh beam under moving mass for various values of rotatory inertia and for

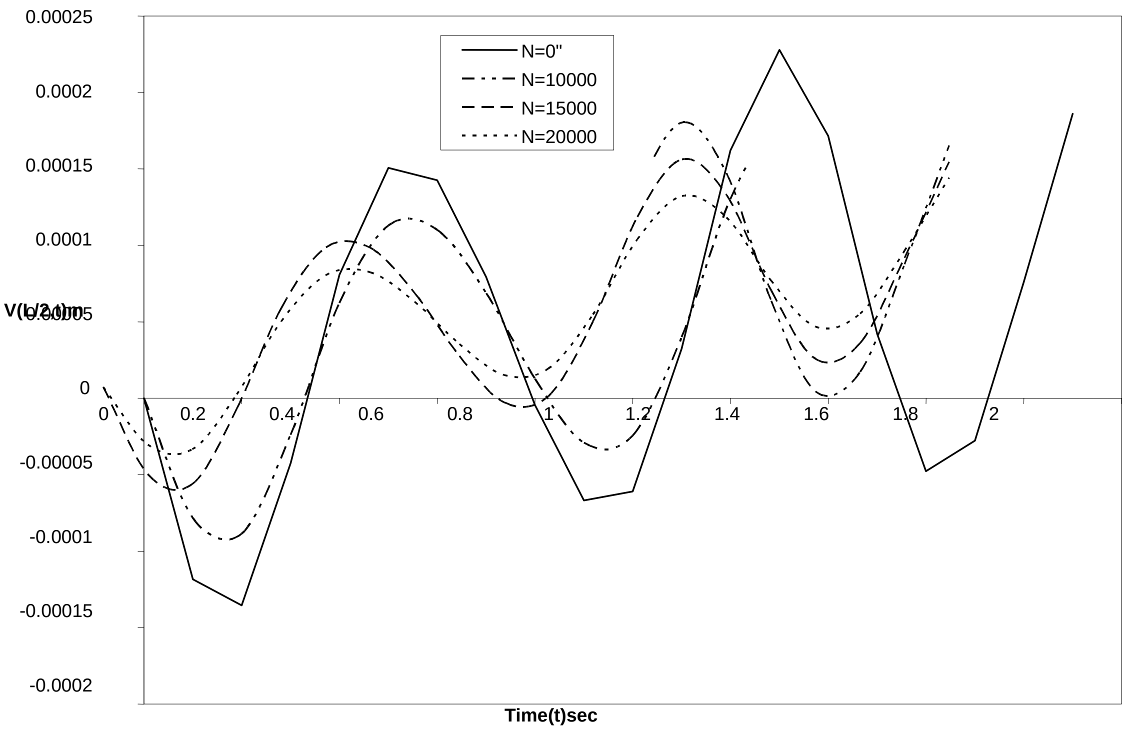

Fig 3. Deflection profile of Clamped-Clamped Non-Uniform Rayleigh beam under the action of moving force for various values of axial force N and fixed value of rotatory inertia r(1) and for fixed value of foundation modulus K(40000)

fixed values of foundation modulli K and axial force N is shown. The graph shows that the response amplitude decreases as the values of rotatory inertia correction factor r^o increases. Also, fig.6, shows the deflection profile of the Clamped-Clamped non-uniform Rayleigh beam under the action of moving mass for various values of axial force N and for fixed values of foundation modulus K and rotatory inertia r^o. From the graph it is shown that as the axial force N increases the response amplitude of the beam decreases. Finally, fig.7 shows the comparison of the transverse displacement for the moving force and moving mass cases of the

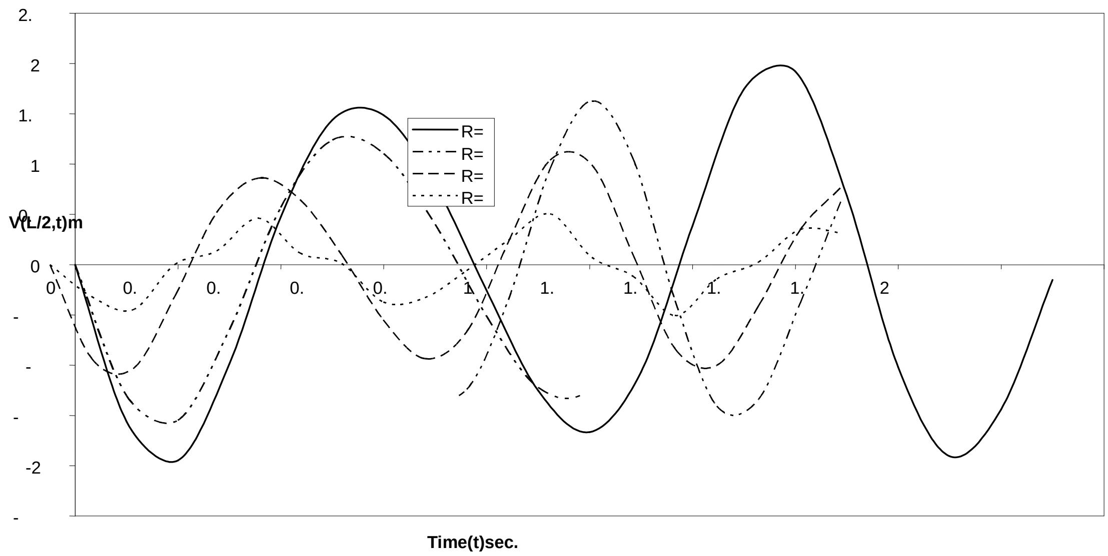

Fig .4: Deflection profile of Clamped-Clamped Non-Uniform Rayleigh beam under the action of moving mass for various values of rotatory inertial and fixed value of axial force N(20000) and foundation modulus K(40000).

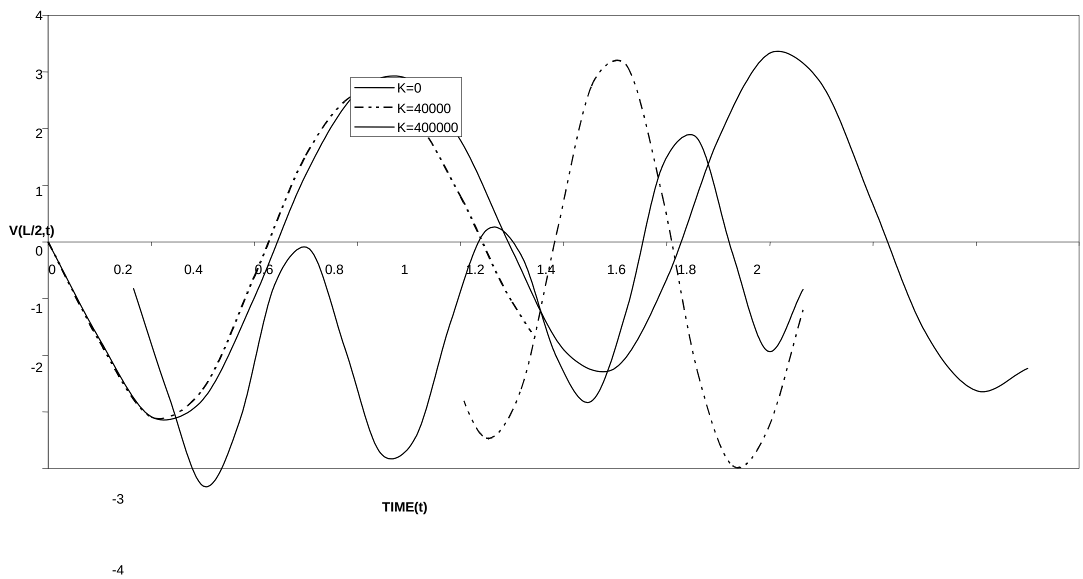

Fig 5: Deflection profile of the Clamped-Clamped Non-Uniform Rayleigh Beam under the action of moving mass for various values of Foundation Modulli K and for fixed value of axial force N and Rotatory inertia r(1)

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Clamped-Clamped non-uniform Rayleigh beams for fixed values of foundation moduli K, axial force N and rotatory inertia r^o.As evident in the figure, the deflection profile for moving mass is higher than that of the moving force confirming also that the moving force solution is not always an upper bound for the accurate solution of the moving mass problem.

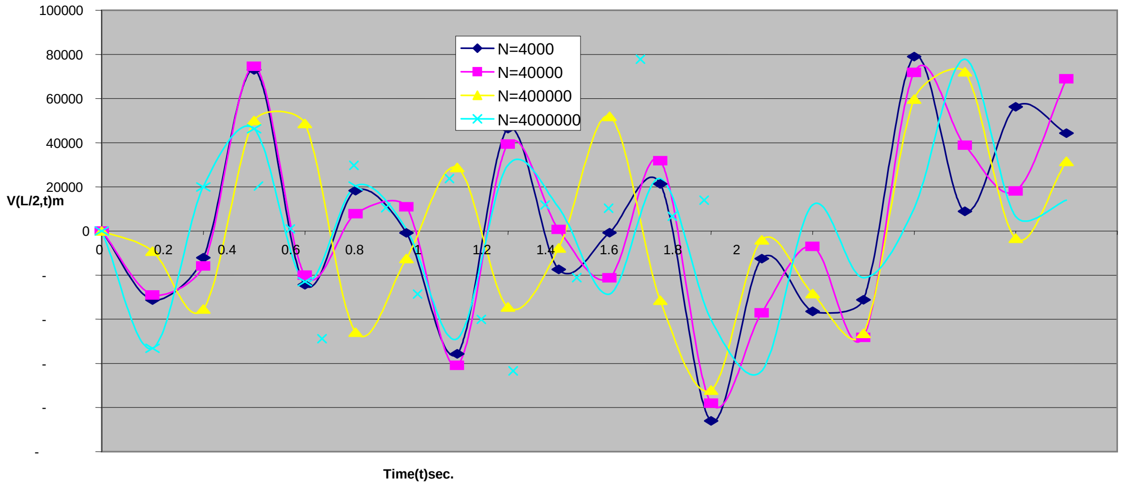

Fig.6 Deflection profile for Non-Uniform Rayleigh beam Clamped-Clamped at both ends under the action of moving mass for various values of axial force N , for fixed value of rotatory innertia r(3) and foundation modulus K(2000000).

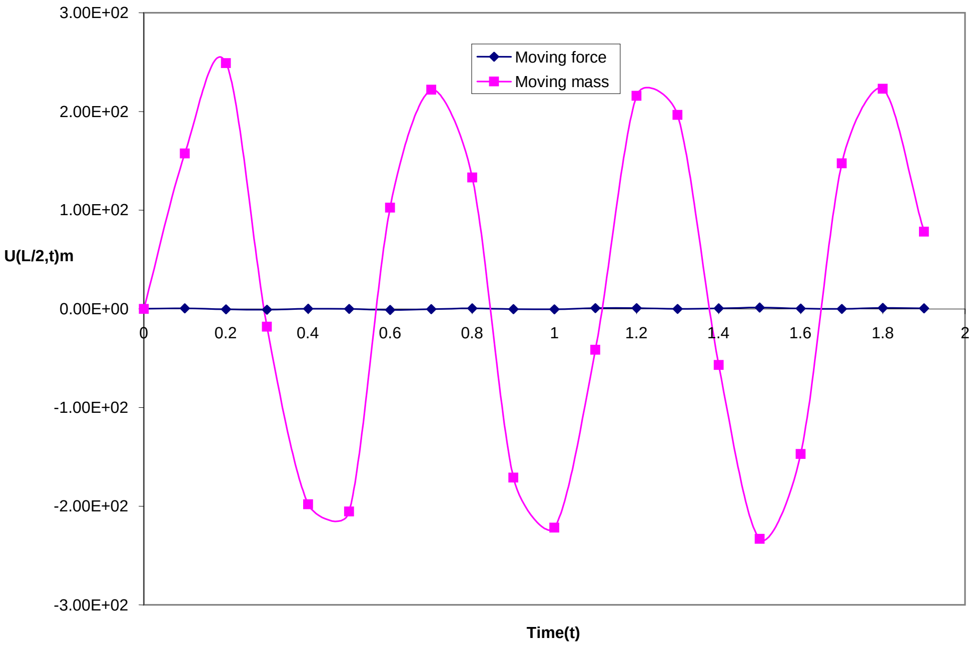

Figure 7: Comparison of the transverse displacement of moving mass cases for Clamped-Clamped Non-uniform Rayleigh beam for fixed values of foundation modulus K(400000) and rotator inertia r(1).

The problem of dynamical analysis of non-uniform Rayleigh beam with time dependent **Clamped-Clamped** boundary conditions when it is under the action of traveling loads is considered in this paper. The main objective is to obtain an approximate analytical solution for the dynamical problem. To this end an approach due to Mindlin and Goodman [16] is extended to transform the governing non-homogeneous partial differential equation with non- homogeneous boundary conditions to a non-homogeneous partial differential equation with homogeneous boundary conditions.

Subsequently, the property of the Dirac-delta function as an even function is used to express it in Fourier cosine series form and the partial differential equation subjected to Generalized Galerkin's method. The Generalized Galerkin's method (GGM) is used to remove the singularity in the Governing equation and to reduce it to a sequence of second order differential equation with variable coefficients. This second order differential equation is then simplified using the modification of the Struble's asymptotic technique. The methods of Integral transformation and the convolution theory are then employed to obtain the analytical solution of the one-dimensional problem.

Analysis of the approximate analytical solutions obtained is carried out and the resonance conditions for the dynamical system are obtained. The influences of the rotatory inertia r^o and foundation moduli K on the dynamic response of the Non-uniform Rayleigh beams having time dependent Clamped-Clamped boundary conditions and under the actions of moving concentrated loads were investigated. The transverse displacements for the moving force and moving mass models are calculated and presented in ploted curves.

As the rotatory inertia r^o and foundation moduli K increases, the displacement response of the Rayliegh beam having time dependent Clamped-Clamped boundary conditions and under the actions of moving concentrated loads for both moving force and moving mass models reduces. We also observed that in Clamped-Clamped non-uniform Rayleigh beams, the moving force solution is not an upper bound for the accurate solution of the moving mass solution. Hence, the non-reliability of moving force solution as a safe approximation to the moving mass solution is confirmed. Furthermore for fixed rotatory inertia and foundation modulus, the response amplitude for the moving mass problem is greater than that of the moving force. However for the same natural frequency the critical speed for moving mass problem is smaller than that of the moving force problem. Hence, resonance is reached earlier in moving mass problem. Finally, higher values of Rotatory inertia and Foundation moduli are required for a more noticeable effect in the case of moving mass than moving force non-uniform Clamped-Clamped boundary conditions.

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