Scientia Africana, Vol. 11 (No.2), December 2012. Pp 88-92
 ISSN 1118 – 1931

STEADY-STATE SOLUTIONS AND STABILITY OF A MATHEMATICAL MODEL OF TWO INTERACTING LEGUMES

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Received: 22-11-12
Accepted: 05-12-12

ABSTRACT

In this paper, we studied a deterministic population model with a mass action law simplification. The main aim is to determine the steady-state solutions and their stability. From our previous study, we have used the selection method of penalty functions to select the estimated best-fit values of the model parameters a, b, d, f, while the precise values of the parameters c and e are assumed. The first model equations using the 1-norm selection method have four steady-state solutions (0, 0), (0, 3.3534), (3.2599, 0) and (3.1908, 0.9543). The second model equations using the 2-norm selection method have four steady-state solutions (0, 0), (0, 3.6860), (3.0000, 0) and (1.1341, 3.4985) while the third model equations using the infinity-norm selection method have four steady-state solutions (0, 0), (0, 4.0545), (2.7898, 0) and (0.5871, 3.9478). Irrespective of the type of parameter selection method, we observe that the first three steady-state solutions are unstable while the unique positive co-existence steady-state solutions are said to be stable. Computer simulations were used to illustrate our theory. However, the question of stabilizing the unstable steady-state solutions which we have found in this study remains to be numerically answered. This level of analysis will be attempted in our next simulation study.

Key words: Steady-State Solutions, Stability, Mass Action Law.

INTRODUCTION

The standard mathematical concepts of steady-state solutions or equilibria and their stability characterizations are not new. The detailed mathematical definitions and the analysis of these techniques are clearly laid in the works of [1], [2], [3], [4], [5], [6], [7], [8], [9], 10], [11], [12], [13] and [15]. However, the application of these theories in describing the interaction between two legumes within the Niger Delta Region of Nigeria ecosystem renders interesting insight to researchers in mathematical modelling, crop scientists and biologists working in the ecosystem functioning, stabilization and planning.

From the literatures on population modelling, the trivial steady-state solution indicates that the interacting populations will go into the ecological risk of extinction. For the border steady-state solutions, we learn that while one of the populations will survive at its carrying capacity, the other population will go into extinction. These two ecological laws reinforce the notion of the well established competitive exclusion (12], [11]). The co-existence steady-state solution specifies the required population sizes which will enhance favourable co-existence and survival assuming there are plentiful resources according to competition theory ([12]). While a stable steady-state solution provides better ecosystem predictions than the unstable version, most ecologists and mathematical ecologists are interested in stabilizing the unstable steady-steady solutions for the benefit of effective control of ecosystem functioning and stabilization. In as much as research activity is on-going to quantify the applications of the unstable steady-state solution in the context of ecological modelling and prediction, the qualitative behaviour of an unstable steady-state solution mathematically may throw greater challenges and concern to ecologists. Hence, it is a profitable collaboration for mathematicians and physicists to design a feedback controller which can be used to stabilize such unstable steady-state solutions. This proposal is at the present one of the best numerical simulation techniques of achieving this expectation ([14]).

In most of the cited mathematical ecology papers, it appears at the present observation that there is a **dearth** of the quantification of environmental perturbations on growing ecological populations over a time interval. As a result of this limitation and lack of possible experimental studies which can be used for modelling purposes, it will be a meaningful scientific contribution if we can explore the modelling of environmental perturbations on the biomasses of legumes and its impact on the parametric sensitivity of the mathematical model of interacting legumes [3]. There is incorporation of the role of environmental perturbations in the study of steady-state solutions and their stability. Because of the resilient characteristic of the ecosystems and the instability of the ecosystem, we will think at the present modelling activity that the role of random white noise in quantifying the steady-state solutions and their stability will complement and contribute better comprehensive insights in the full understanding of complex ecosystem population interactions.

From the literature on population modelling ([2]), we know that temperature changes do affect the growth rate parameters than any other model parameters. It is important also to investigate the extent of this indirect impact of temperature changes on the steady-state and stability characteristics of two interacting legumes especially in this era of controlling the inevitable impact of global warming so as to save the livelihoods of the growing human populations in the Niger Delta Region of Nigeria.

It is expected that when two legumes interact, it is possible for these legumes to either survive together or both will go into extinction or while one legume survives the other legume may go into extinction. However, it is an interesting modelling challenge to calculate the extent of all the intraspecific and interspeciic coefficients on the steady-state solutions and their stability on two interacting legumes. If it is possible, it will become an interesting numerical simulation to quantify the extent of steady-state solutions and their stability when the starting biomasses of two interacting legumes are precisely estimated.

Mathematical Formulations

In this section, we will present three distinct model formulations between cowpea and groundnut interaction.

Model Formulation 1

Our first model formulation is based on the application of the 1-norm penalty function selection method. The mathematical structure of

Scientia Africana, Vol. 11 (No.2), December 2012. Pp 88-92
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this nonlinear first order ordinary differential equations is

(2.1.1)

$$\frac{dC(t)}{dt} = C(t)\ (0.0225 - 0.006902C(t)\ - 0.0005G(t))$$

(2.2.1)

$$\frac{dG(t)}{dt} = G(t)\ (0.0446 - 0.0133G(t)\ - 0.01C(t))$$

where the initial biomasses are C(0) = C_0 > 0 and G(0) = G_0 > 0. Here, C(t) and G(t) represent the biomasses of cowpea and groundnut at time *t* in days.

Model Formulation 2

Our second model formulation is based on the application of the 2-norm penalty function selection method. The mathematical structure of this nonlinear first order ordinary differential equations is

(2.2.1)

$$\frac{dC(t)}{dt} = C(t)(0.0225\ -\ 0.007500C(t) - 0.004G(t)$$

(2.2.2)

$$\frac{dG(t)}{dt} = G(t)\ (0.0446\ -\ 0.0121G(t)\ -\ 0.002C(t))$$

under the same initial conditions and simplifying assumptions.

Model Formulation 3

In the same manner, our third model formulation based on the infinity-norm penalty function selection method is

(2.3.1)

$$\frac{dC(t)}{dt} = C(t)(0.0225\ -\ 0.008065C(t) - 0.0045G(t))$$

(2.3.2)

$$\frac{dG(t)}{dt} = G(t)(0.0446\ -\ 0.0110G(t) - 0.002G(t)$$

under the same initial conditions and simplifying assumptions.

MATERIALS AND METHODS

In this section, we aim to characterize the steady-state solutions and their stability of the three model equations which we have defined in the previous section.

Characterization of Steady-State Solutions and Stability using the 1-norm model equations

Following [3], and considering some simplifying assumptions such as $C_e \neq 0$, $G_e = 0$; $C_e = 0$, $G_e \neq 0$; $C_e \neq 0$ and $G_e \neq 0$ where
(C_e, G_e) is an arbitrary steady-state solution for this 2-dimensional system of model equations. It is clear that these model equations are characterized by four steady-state solutions which are (0, 0), (0, 3.3534), (3.2599, 0) and (3.1908, 0.9543).

Theoretically, the appropriate steady-state solution except the trivial case are $(\frac{a}{b},\ 0),\ (0,\ \frac{b}{f}),\ (\frac{af - cd}{bf - ce},\ \frac{bd - ac}{bf - ce})$, provided $af > cd,\ bd > ac,\ bf > ce$
where

$a = 0.0225$, b = 0.006902, c = 0.0005, d = 0.0446, e = 0.01, f = 0.0133

Without providing detailed analytical calculations as can be read in the reports of [4], [2] and [14], but using the method of linearization around each chosen steady-state solution and the method of a small perturbation from a steady-state solution, the stability features for this system of model equations are characterized as follows: (0, 0) is unstable because the two calculated eigenvalues are 0.0225 and 0.0446 followed by an unstable steady-state solution (0, 3.3534) having two eigenvalues of -0.0446 and 0.0208. The steady-state solution (3.2599, 0) is similarly unstable because its calculated eigenvalues are -0.0225 and 0.0120. Another steady-state solution is the unique positive point (3.1908, 0.9543) which is characterized as being stable because its calculated eigenvalues are -0.0234 and -0.0113.

Characterization of Steady-State Solutions and Stability using the 2-norm model equations

By applying the assumptions on the arbitrary steady- state solutions as we have done in the previous section, we can observe that the model equations are characterized by four steady-state solutions which are (0, 0), (0, 3.6860), (3.0000, 0) and (1.1341, 3.4985) for the same similar theoretical calculations as mentioned in section 3.1

For this problem, the trivial steady-state solution is unstable having the same eigenvalues as in the case of the 1-norm model equations. The steady-state solution (0, 3.6860) is unstable having the eigenvalues of -0.0446 and 0.0078 while the steady-state solution (3.0000, 0) is unstable having the eigenvalues of -0.0225 and 0.0386. Here, the second unique positive point (1.1341, 3.4985) is characterized as being stable because its calculated eigenvalues are -0.0076 and -0.0432. 5.3.

Characterization of Steady-State Solutions and Stability using the infinity-norm model equations

In this section, the four steady-state solutions are (0, 0), (0, 4.0545), (2.7898, 0) and (0.5871, 3.9478). In the same manner, the trivial steady-state solution is unstable having the same two positive eigenvalues. The steady- state solution (0, 4.0545) is unstable having two eigenvalues of -0.0446 and 0.0043.

Similarly, the steady-state solution (2.7898, 0) is unstable having two eigenvalues of -0.0225 and 0.0340. The third unique positive point (0.5871, 3.9478) is stable having two eigenvalues of -0.0042 and -0.0440.

DISCUSSION

In all our analytical calculations of steady-state solutions and their stability, we have found that qualitatively, the unstable steady-state solutions contribute to the unbounded growth of the solution trajectories over a time interval while the steady-state solution which is stable contributes to the decaying behaviour of the solution trajectories over the same time interval.

For the trivial steady-state solution, the two interacting legumes will go into extinction whereas for the border steady-state solutions, as one of the legumes survives at its carrying capacity, the other legume will be driven into extinction.

This qualitative behaviour of the steady-state solutions is in agreement with the ecological law of competitive exclusion. We also observe that the two interacting legumes also co-exist and survive together.

CONCLUSION

For the first time in the Niger Delta Region of Nigeria ecosystem, we have systematically characterized the steady-state solutions and their stability by using some standard mathematical

Scientia Africana, Vol. 11 (No.2), December 2012. Pp 88-92
 ISSN 1118 – 1931

techniques. The steady-state solutions and their stability characterizations can provide useful ecological insights about the ecosystem stability and planning. Further extensions of this study will involve the sensitivity analysis of the estimated model parameters and the application of stabilizing a mathematical model of two interacting legumes and its significance in ecosystem functioning, ecosystem economics and biodiversity gain.

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