Scientia Africana, Vol. 11 (No.1), June 2012. pp 92-97
 ISSN 1118 – 1931

STABILIZING A MATHEMATICAL MODEL OF STOCK MARKET POPULATION SYSTEM

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Received: 07-02-12
Accepted: 19-04-12

ABSTRACT.

The aim of this study is to stabilize the unstable steady-state solutions of a mathematical model of competition with reference to the stock market dynamics in the Niger Delta.

INTRODUCTION

Our first study in the mathematical modelling of stock market population involves analyzing two populations of investors interacting mutualistically (Nafo and Ekaka-a., 2011a). Our second paper in this context concerns the bifurcation analysis of two populations of investors undergoing competition (Nafo and Ekaka-a., 2011b).

It is very clear from our previous analyses that we have not attempted stabilizing a mathematical model for stock market population system. Following the most current mathematical formulation and stabilization of a Komogrove system of equations (Yan and Ekaka-a, 2011), we propose to extend this important idea to stabilize a mathematical model of stock market population system. We are particularly interested to investigate whether the concept of constructing a feedback control with which to stabilize an unstable steady-state as proposed in the work of Yan and Ekaka-a, (2011) is applicable to stabilize the unstable steady-state solution of stock market population system.

What extra contribution does it make to stabilize a trivial steady-state solution in the context of a stock market population system?

When a trivial steady-state solution is unstable, it means that the two populations of investors will go into extinction. In terms of business and economics implication, it means that these two populations over a trading period will be driven into an unexpected economic risk of extinction. Experts in economics, business and Mathematics are concerned about this unpredictable event in the stock market (Nafo and Ekaka-a, 2011b).Therefore in this study, we will attempt for the first time to construct a feedback controller which if successful will be used to stabilize the trivial steady-state solution.

Next, when the first population of investors goes to extinction and the second population of investors survives at its carrying capacity, this familiar conclusion does not provide adequate insights about the stabilization of this unstable steady-state solution. This outcome of competition only favours the second population of investors. For this unstable steady-state solution to provide further insights into economic and business planning, it is appropriate to investigate the extent and prospects of stabilizing this first boarder steady-state solution.

Our comment for the first boarder steady-state solution are similar to the second boarder steady-state solution where the first population of the investors benefit and the second population of investors does not. Therefore it is our primary aim in this important study to stabilize a mathematical model of a stock market system using our previous estimated stock market population parameters (Ekaka-a and Nafo, 2011).

STOCK MARKET INTERACTION MODEL EQUATIONS

Following (Ekaka-a and Nafo, 2011; Nafo and Ekaka-a, 2011a), we will consider the following systems of first ordinary differential equations.

$$\frac{dy}{dt} = y(t)\,(a_1 - b_1 y(t) - c_1 z(t)),$$

$$\frac{dz}{dt} = z(t)\,(a_2 - b_2 z(t) - c_2 y(t)),$$

where y(0) = y_0 > 0, z(0) = 0. Here a_i, b_i, c_i, i = 1, 2 are positive constants.

The steady states (y_e, z_e) satisfy

$$y_e\,(a_1 - b_1 y_e - c_1 z_e) = 0,$$
$$z_e\,(a_2 - b_2 z_e - c_2 y_e) = 0.$$

The four steady states are

$$y_e = 0, \qquad z_e = 0,$$

$$y_e = 0, \qquad z_e = \frac{a_2}{b_2},$$

$$y_e = \frac{a_1}{b_1}, \qquad z_e = 0,$$

$$y_e = \frac{a_1 b_2 - a_2 c_1}{b_1 b_2 - c_1 c_2}, \quad z_e = \frac{a_2 b_1 - a_1 c_2}{b_1 b_2 - c_1 c_2}$$

How do we stabilize (y_e, z_e) if (y_e, z_e) is unstable?

LINEARISED SYSTEM ABOUT (y_e, z_e)

Following Yan and Ekaka-a, 2011, denote

$$F(y,z) = y(a_1 - b_1 y - c_1 z)$$
$$G(y,z) = z(a_2 - b_2 y - c_2 z)$$

Consider the system

$$\frac{dy}{dt} = F(y,z)$$
$$\frac{dz}{dt} = G(y,z).$$

Taylor expansion about (y_e. z_e),

$$F(y,z) = F(y_e,z_e) + \frac{\partial F(y_e,z_e)}{\partial y}(y - y_e) + \frac{\partial F(y_e,z_e)}{\partial z}(z - z_e)$$

$$G(y,z) = G(y_e,z_e) + \frac{\partial G(y_e,z_e)}{\partial y}(y - y_e) + \frac{\partial G(y_e,z_e)}{\partial y}(z - z_e)$$

The linearized systems about (y_e, z_e) are

$$\frac{dy}{dt} = \frac{\partial F(y_e,z_e)}{\partial y}(y - y_e) + \frac{\partial F(y_e,z_e)}{\partial z}(z - z_e) + \text{higher-order-terms}$$

$$\frac{dz}{dt} = \frac{\partial G(y_e,z_e)}{\partial y}(y - y_e) + \frac{\partial G(y_e,z_e)}{\partial z}(z - z_e) + \text{higher-order-terms}$$

Substituting y – y_e and z – z_e by *Y* and *Z* separately and denoting

$$u = \begin{pmatrix} Y \\ Z \end{pmatrix}, \quad A = \begin{pmatrix} \frac{\partial F(y_e,z_e)}{\partial y} & \frac{\partial F(y_e,z_e)}{\partial z} \\ \frac{\partial G(y_e,z_e)}{\partial y} & \frac{\partial G(y_e,z_e)}{\partial z} \end{pmatrix}.$$

The linearized system about (y_e, z_e) is

$$\frac{du}{dt} = Au, \quad u(0) = u_o,$$

where

$$u_0 = \begin{pmatrix} y_0 - y_e \\ z_0 - z_e \end{pmatrix}.$$

Scientia Africana, Vol. 11 (No.1), June 2012. pp 92-97
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STABILITY OF THE STEADY STATES

Lemma 3.1 (Yan and Ekaka-a, 2011)**.** Assume that all the eigenvalues of A are negative, then the solution of equation (9) tends to the steady

state (y_e ,z_e) as $t \to \infty$ for some suitable initial value $u_0 = (y_0 - ye, z_0 - z_e)$.

- If A has a positive eigenvalue, then the steady state (y_e , z_e) is not stable.
- We will use the feedback control to stabilize the unstable steady state.

STABILIZATION FOR THE LINEARISED SYSTEM

Theorem 4.1. (Yan and Ekaka-a, 2011). Assume that (y_e , z_e) is unstable, then there exists V: $[0, \infty) \to R^2$ such that

$$\frac{du}{dt} = Au + BV, \qquad u(0) = u_0,$$

is exponentially stable at (y_e , z_e), where

$$V = -R^{-1}\, B^*\Pi u.$$

Here II satisfies the Riccati equation

$$A^*\Pi + \Pi A - \Pi BB^*\Pi + Q = 0,$$

where R = I and Q is any positive definite matric and $B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \; or \; \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$

further there exists $\rho > 0$, such that for all u_0 : $\|u_0\| < \rho$, there exist a unique solution $u \in C^1 (0, +\infty, R^2)$ such that, with some $\gamma > 0$, c > 0,

$$\|u(t)\| \le ce^{-\gamma t}\|u_0\|$$

STABILIZATION FOR THE NONLINEAR SYSTEM

Theorem 5.1 (Yan and Ekaka-a, 2011)**.** Assume that $\begin{bmatrix} y_e \\ z_e \end{bmatrix}$ is unstable. Then

$$V = -R^{-1}B^*\Pi \begin{bmatrix} y - y_e \\ z - z_e \end{bmatrix},$$

will stabilize exponentially the nonlinear system

(6.1)

$$\frac{d}{dt}\begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} F(y,z) \\ G(y,z) \end{bmatrix} + BV(t).$$

More precisely, there exists $\rho > 0$ such that for all $\begin{bmatrix} y_0 \\ z_0 \end{bmatrix} : \left\| \begin{bmatrix} y_0 \\ z_0 \end{bmatrix} - \begin{bmatrix} y_e \\ z_e \end{bmatrix} \right\| < \rho,$ there exists a unique solution $\begin{bmatrix} y \\ z \end{bmatrix} \in C^1 (0,\infty, R^2)$, such that, with some constant c and $\gamma > 0$,

$$\left\| \begin{bmatrix} y_0 \\ z_0 \end{bmatrix} - \begin{bmatrix} y_e \\ z_e \end{bmatrix} \right\| < ce^{-\gamma t} \left\| \begin{bmatrix} y_0 \\ z_0 \end{bmatrix} \right\|$$

NUMERICAL APPROXIMATION

Consider

$$\frac{d}{dt}\begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} F(y,z) \\ G(y,z) \end{bmatrix} + BV(t).$$

where

$$V = -R^{-1}\, B^*\Pi \begin{bmatrix} y - y_e \\ z - z_e \end{bmatrix}$$

Substituting y - y_e and z - z_e by Y and Z, we got

$$\frac{d}{dt}\begin{bmatrix} Y \\ Z \end{bmatrix} = \begin{bmatrix} F(Y + y_e, Z + z_e \\ G(Y + y_e, Z + z_e \end{bmatrix} + BV(t)$$

where

$$V = -R^{-1}\, B^*\Pi \begin{bmatrix} Y \\ Z \end{bmatrix}$$

Denote

$$u = \begin{bmatrix} Y \\ Z \end{bmatrix}, \quad F(u) = \begin{bmatrix} F(Y + y_e, Z + z_e \\ G(Y + y_e, Z + z_e \end{bmatrix}.$$

The feedback control system is

$$\frac{du}{dt} + A_1 t = F(u)$$

where $A_1 = R^{-1} B B^* \Pi$

Let $0 = t_0 < t_1 < t_2 < \ldots$ be time points, $k = t_j - t_{j-1}$

The backward Euler method, $U^n \approx u(t_n)$:

$$\frac{\left(U^n - U^{n-1}\right)}{k} + A_1 u^n = F(u^{n-1})$$

ERROR ESTIMATES

For the purpose of this study, we will consider the following criteria (Yan and Ekaka-a, 2011):

- Global Lipschitz condition

$$\|F(u) - F(v)\| \le C_1 \|u - v\|, \ \forall u, v \in R^2$$

- Linear growth condition

$$\|F(u)\| \le C_2 \ \|u\|$$

Theorem 7.1 (Yan and Ekaka-a, 2011): Let T > 0. Assume that F satisfies the global Lipschitz condition and growth condition. Then there exists *a* constant C(T) such that, for any $\in > 0$,

$$\|u^n - u(t_n)\| \le C(T) k^{1-\in} \ \|u_0\|$$

Proof: The detail proof of this result can be found in the works of Yan et al. (2008) and Yan and Ekaka-a (2011).

EXAMPLE

In this study, we will consider an example which relates directly with stock market interacting systems (Nafo and Ekaka-a, 2011a) This example concerns the dynamics of two interacting stock market population based on Abia State stock exchange time series data (Okoroafor and Osu, 2009).

$$\frac{dN_1}{dt} = N_1(t)(0.168 - 0.0014 N_1(t) - 0.0005 N_2(t))$$

$$\frac{dN_2}{dt} = N_2(t)(0.03 - 0.0009 N_1(t) - 0.0005 N_2(t))$$

with $N_1(0) = 10$ in thousands of naira, $N_2(0) = 15$ in thousands of naira

This system of model equations is characterized by three realistic steady-state solutions namely (0, 0), (120, 0) and (0, 60).

The trivial steady-state solution is unstable because it has two positive eigenvalues 0.03 and 0.168. The second steady-state solution (120, 0) is stable because it has two negative eigenvalues -0.168 and -0.078. The third stead-state solution (0, 60) is unstable because it has two eigenvalues of opposite signs -0.03 and 0.138.

It is interesting to find out how we can stabilize the unstable steady state solutions. Our results which are represented below concern how we have constructed appropriate optimal controller which we have used to stabilize the unstable steady state solutions.

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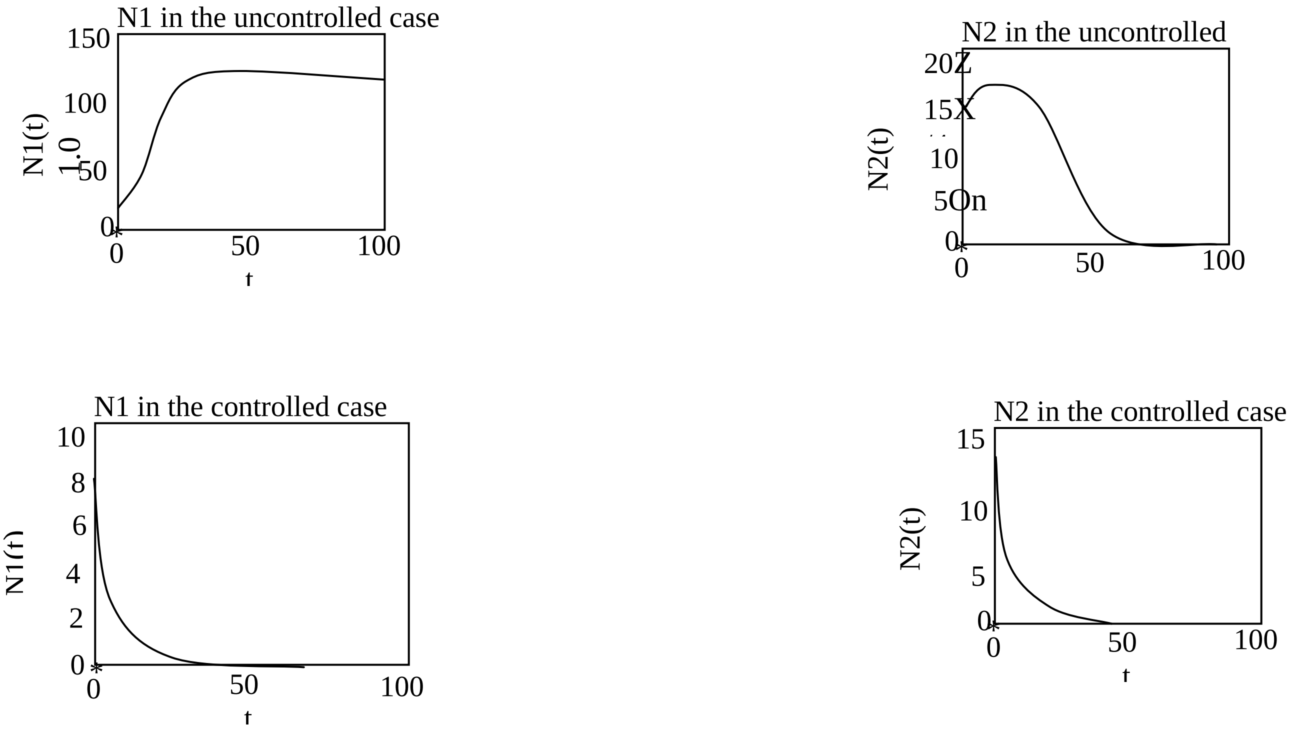

Figure 1. Uncontrolled and Controlled Solution Trajectories of Trivial Steady State (0,0)

Here, we have used final trading period of 100 days and starting trading values of 10,000 naira and 15,000 naira. Other trading periods can similarly be used which provide similar results for the same starting trading values.

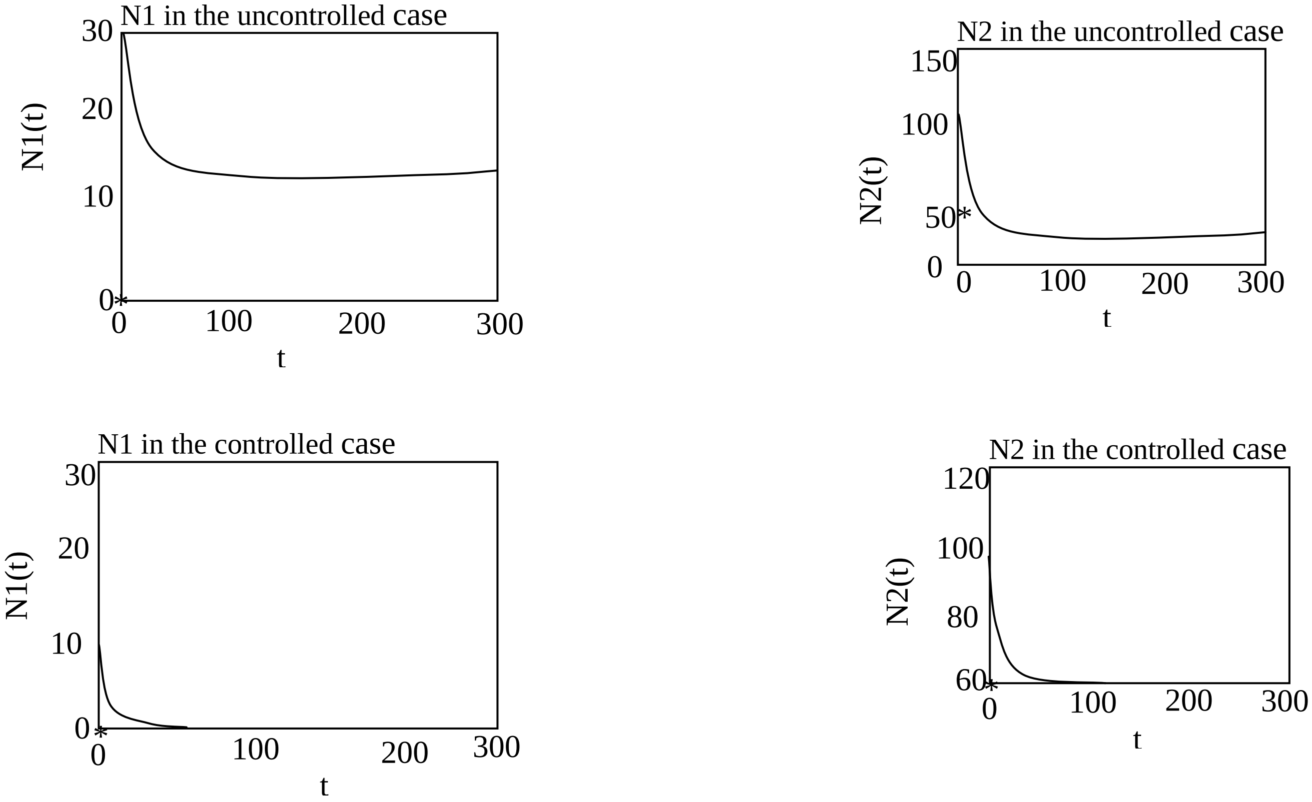

Figure 2. Uncontrolled and Controlled Solution Trajectories of Trivial Steady State (0,60)

Here, we have used final trading period of 400 days and starting trading values of 30,000 naira and 12,000 naira. This unstable steady-state can similarly be stabilized when the trading periods are 200 days, 300 days, and 365 days, at the same starting trading values. However, for shorter trading periods such as 20 days, 50 days, 60 days, 80 days, and 100 days, this unstable steady-state solution is indicating slow convergence. The business and economic implications of these observations are yet to be clearly deduced.

DISCUSSION

In this study, we have used feedback control to construct a controller with which the two unstable steady-states of two interacting stock market populations have been stabilized. These results are short-term numerical simulations which are capable of providing some insights about the profit behaviour of Nigeria stock market systems.

The technique of implementing a feedback controller to stabilize unstable steady-state solutions can be extended to tackle other stock market interacting populations such as mutualism, commensalism, and predation.

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