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SOME OBSERVATIONS ON DERIVATIVES OF ORTHOGONAL POLYNOMIALS

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ABSTRACT
In this paper it is shown that if $\{\phi_n(x)\}$ *is a set of orthogonal polynomials with the weight function* $w(x)$ *in the finite interval (a, b), and if we assume that the derivatives* $\{\phi_n'(x)\}$ *also form a set of orthogonal polynomials in a certain interval (c, d) (finite or infinite), with a non-negative weight function* $q(x)$*, where (c,d) ≡ (a,b), then* $\{\phi_n(x)\}$ *is a set of Jacobi polynomials.*

Key Words: *Polynomials, Weight Function, Derivative.*

INTRODUCTION

A set $\phi_0(x) = 1$, $\phi_1(x)$, $\phi_2(x)$, … of polynomials of degrees 0, 1, 2, … is called a set of orthogonal polynomials if they satisfy

$$\int_a^b \phi_m(x)\phi_n(x)d\psi(x) = 0,\quad \int_a^b d\psi(x) > 0 \quad (m \neq n) \tag{1.1}$$

where $\psi(x)$ is a non-decreasing function of bounded variation (Azari and Muller, 1992).

It has been shown in Szego (1975) that if the derivatives also form a set of orthogonal polynomials, then the original set were Jacobi, Hermite, or Laguerre polynomials. His method consisted in showing that the polynomials satisfy a differential equation of the type

$$\left(a + bx + cx^2\right)\phi_n'' + (d + cx)\phi_n' + \lambda_n\phi_n = 0\,. \tag{1.2}$$

From this it follows that the set were Jacobi, Hermite, or Laguerre polynomials.

Here we propose to give a new proof of this result, our point of view being to answer the question: What condition on the weight function results from assuming that both $\{\phi_n(x)\}$ and $\{\phi_n'(x)\}$ are sets of orthogonal polynomials? However, we shall assume that (a, b) is a finite interval and $d\psi(x) = w(x)dx$, where the weight function is L-integrable.

A Relation for the Weight Function Q(X)

Let the set $\{\phi_n'(x)\}$ be orthogonal in the interval (c, d), finite or infinite, with the weight function q(x), that is,

$$\int_c^d q(x)\phi_n'(x)\phi_m'(x)dx = 0, \quad (m \neq n). \tag{2.1}$$

Okolo and Bamiduro (2001 and 2004) documented that the polynomials $\{\phi_n(x)\}$, $\{\phi_n'(x)\}$ satisfy the recurrence relations,

$$\phi_{n+2}(x) = (x - C_{n+2})\phi_{n+1}(x) - \lambda_{n+2}\phi_n(x)$$

, (2.2)

$$\frac{1}{n+2}\phi'_{n+2}(x) = \frac{x - C'_{n+2}}{n+1}\phi'_{n+1}(x) - \lambda'_{n+2}\phi'_n(x) \qquad (2.3)$$

($n \geq 0$; C'_n, λ_n, λ'_n are constants).

Differentiating both sides of (2.2) and eliminating the term containing x, by means of (2.3), we get

$$\phi_{n+1}(x) = \frac{1}{n+2}\phi'_{n+2}(x) + C''_{n+2}\phi'_{n+1}(x) + \lambda''_{n+2}\phi'_n(x), \qquad (2.4)$$

(C''_n, λ''_n are constants).

Remembering that $\phi'_n(x)$ with the weight function $q(x)$, is orthogonal to any polynomial of degree ≤ (n-2), we get

$$\int_c^d q(x)\phi_{n+1}(x)G_{n-2}(x)dx = 0, \qquad (2.5)$$

where $G_n(x)$ is an arbitrary polynomial of degree ≤ n.

Wilf (1962) and Elhay et. al. (1991) showed that if $Q(x)$ be non-negative in (c, d), such that the numbers

$$\beta_k = \int_c^d Q(x)x^k dx, \quad (k=0,1,\ldots..), \qquad (2.6)$$

exist, and for a certain positive integer r

$$\int_c^d Q(x)\phi_n(x)G_{n-r-1}(x)dx = 0, \qquad (2.7)$$

(n = r + 1, r + 2, …).

Then almost everywhere

$$Q(x) = \begin{cases} P_r(x)w(x) & \text{in } (a, b) \\ 0 & \text{elsewhere,} \end{cases} \qquad (2.8)$$

where $P_r(x)$ is a polynomial of degree ≤ r.

Consider the function

$$R(x) = \left(u_0 + u_1x_1^1 + u_2x^2 + \ldots\ldots. + u_ix^r\right)w(x) \qquad (2.9)$$

We determine the $\{u_i\}$ so that

$$\int_a^b R(x)x^i dx = \int_c^d Q(x)x^i dx \qquad (2.10)$$

(i = 0, 1, ….., r);

That is, the u_i satisfy the equations

$$\begin{aligned} &\alpha_0u_0 + \alpha_1u_1 + \ldots\ldots.. + \alpha_ru_r = \beta_0, \\ &\alpha_1u_0 + \alpha_2u_1 + \ldots\ldots.. + \alpha_{r+1}u_r = \beta_1, \qquad (2.11) \\ &\text{- -} \\ &\alpha_r\mu_0 + \alpha_{r+1}\mu_1 + \ldots\ldots.. + \alpha_{2r}\mu_r = \beta_r, \end{aligned}$$

where

$$\alpha_k = \int_a^b x^k w(x)dx\ . \qquad (2.12)$$

Now, as a special case, take in (2.7), n = r + i; that is,

$$\int_c^d Q(x)\phi_{r+i}(x)G_{i-1}(x)dx = 0,$$

then take $G_{i-1}(x) \equiv 1$.

It follows that

$$0 = \int_a^b R(x)\phi_{r+i}(x)dx = \int_c^d Q(x)\phi_{r+i}(x)dx \qquad (2.13)$$

(i = 1, 2, --------).

And then

$$\int_a^b R(x)x^i dx = \int_c^d Q(x)x^i dx$$

(i = 0, 1, 2, ……..).

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Let

$$f(x) = \begin{cases} Q(x) - R(x) \quad \text{in } E_1\text{, the points where (a, b) and (c, d) overlap,} \\ -R(x) \quad \text{in } E_2\text{, the remainder of (a, b),} \\ Q(x) \quad \text{in } E_3\text{, the remainder of (c, d).} \end{cases} \tag{2.14}$$

Then

$$\int_{E_1+E_2+E_3} f(x)x^i dx = 0, \tag{2.15}$$

(i = 0, 1, 2, ………).

From the above definition of $f(x)$, we conclude that $R(x)$, hence $w(x)$, $\equiv 0$ almost everywhere in E_3, so that both intervals (a, b) and (c, d) may be reduced to their common part E_1; in other words, here we may take (c, d) $\equiv$ (a, b).
It follows from (2.5) that

$$q(x) = (rx^2 + sx + t)w(x) \tag{2.16}$$

and we may take c = a, d = b.

EXISTENCE OF $q'(x)$

Consider the function

$$S(x) = k\int_a^b (x-l)w(x)dx, \tag{3.1}$$

where k and l are such that S(b) = 0, $\int_a^b S(x)dx = \int_a^b q(x)dx$.

An integration by parts applied to $\int_a^b S(x)\phi'_{n+1}(x)dx$ gives

$$\int_a^b S(x)\phi'_{n+1}(x)dx =$$

$$\int_a^b \phi_{n+1}(x)k(x-l)w(x)dx = 0, \quad (n \geq 1), \tag{3.2}$$

since S(a) = S(b) = 0.

But q(x) is the weight function for the orthogonal polynomials $\{\phi'_n(x)\}$, hence

$$\int_a^b S(x)\phi'_{n+1}(x)dx = \int_a^b q(x)\phi'_{n+1}(x)dx = 0,$$

$$(n \geq 1). \tag{3.3}$$

This and the relation $\int_a^b S(x)dx = \int_a^b q(x)dx$ gives

$$\int_a^b S(x)x^n dx = \int_a^b q(x)x^n dx, \quad (n \geq 0), \tag{3.4}$$

and then $q(x) = S(x)$ almost everywhere. Since $S(x)$ has a derivative almost everywhere, $q(x)$ has a derivative almost everywhere and

$$q'(x) = k(x-l)w(x), \quad q(a) = q(b) = 0. \tag{3.5}$$

DISCUSSION OF q(x) and w(x)

Dividing (3.5) by (2.16), we get

$$\frac{q'(x)}{q(x)} = \frac{k(x-l)}{rx^2 + sx + t}, \qquad q(a) = q(b) = 0 \tag{4.1}$$

We proceed to show that (i) $rx^2 + sx + t$ has real zeros, (ii) $r \neq 0$.

(i) Assume $rx^2 + sx + t$ has imaginary zeros. Integrating the differential equation (4.1), we get

$$\log\ q(x) = \int \frac{k(x-l)}{rx^2 + sx + t} dx + c \qquad (4.2)$$

$$q(x) = k(rx^2 + sx + t)^{\alpha} e^{\beta \arctan(\gamma x + \delta)},$$

where α, β, γ, δ, k are constants > 0.

This is incompatible with q(a) = q(b) = 0.

(ii) Assume first r = s = 0. Equation (4.1) becomes

$$q'(x) = (2ax + \beta)q(x),\quad q(x) = Ke^{ax^2 + \beta x}$$

(α, β, k are constants), (4.3)

which is not zero at a and b.

Next, suppose r = 0, s ≠ 0. Equation (4.1) gives

$$\frac{q^1(x)}{q(x)} = \frac{k(x-l)}{sx+t} = \alpha + \frac{\beta s}{sx+t}, \qquad (4.4)$$

$q(x) = k(sx+t)^{\beta} e^{ax}$, (α, β, k are constants).

This cannot vanish at both end points, x = a and x = b.

Having thus proved (i) and (ii), we set

$rx^2 + sx + t = r(x-g)(x-h)$, (r ≠ o, g, h real), (4.5)

and rewrite (4.1) as follows:

$$\frac{q'(x)}{q(x)} = \frac{k(x-l)}{rx^2 + sx + t} = \frac{a}{x-g} + \frac{\beta}{x-h}, \qquad (4.6)$$

hence

$q(x) = k(x-g)^{\alpha}(x-h)^{\beta}$, where k, α, β are constants > 0.

The conditions q(a) = q(b) = 0 demand that g = a, h = b, so that (disregarding constant factors)

$$q(x) = -r(x-a)^{\alpha}(b-x)^{\beta}. \qquad (4.7)$$

And then from (2.16)

$$w(x) = \frac{q(x)}{rx^2 + sx + t} = \frac{r(x-a)^{\alpha}(b-x)^{\beta}}{r(x-a)(b-x)},$$

$$w(x) = (x-a)^{\alpha-1}(b-x)^{\beta-1}, \qquad (4.8)$$

and we can see that α and β are both > 0.

CONCLUSION

The equation in (4.8) is the weight function for Jacobi polynomials. An overriding advantage of our proof over the conventional one is that it is quite explicit. While the conventional proof requires the application of a second order differential equation, see equation (1.2), the one being reported employs a first order as seen in equations (2.2) and (2.3). Under the condition that the weight function is L-integrable, the proof reported should be preferable to the conventional one.

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