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THEORETICAL BIFURCATION ANALYSIS OF A BIOLOGICAL INTERSPECIFIC COMPETITION INTERACTION

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ABSTRACT

Bifurcation analysis of a biological interaction model can provide useful information for ecological functioning and stability. In this paper, we will study the fundamental changes in the qualitative behaviour of the solution trajectories due to a variation of an intrinsic growth rate when other model parameters are fixed. Our contribution is as follows: the trivial steady-state solution is a stable node when $p_1 < 0$ and $d < 0$, and a saddle when $p_1 > 0$ and $d > 0$. The transcritical bifurcation points where the trivial steady-state solution changes stability are $p_1 = 0$ and $d = 0$.

INTRODUCTION

The notion of a bifurcation analysis in ecological modelling is an important mathematical technique for understanding the fundamental changes in the qualitative behaviour of solutions which is due to a variation of a model parameter at a time when other model parameters are fixed ([5]), ([9]). Hence bifurcation analysis in ecological research is an active component of research. According to [4] and the several other cited authors by [9], we know that, for a system of nonlinear first order differential equations, a steady-state solution can either be called stable if the signs of the eigenvalues are both negative and unstable if the eigenvalues are of opposite signs. But the bifurcation values where a stable steady-state solution changes to an unstable steady-state solution remains to be an open problem in the context of this interdisplinary research. These bifurcation values can provide some insights to ecologists.

In the sequel, we will present a few key results of our numerical bifurcation analysis which we have not seen elsewhere.

In the discipline of theoretical biology, bifurcation techniques where a system of first order ordinary differential equations changes stability has received extensive active research ([1], [2], [3], [7], [8] [10]). In bifurcation theory, the points where the system changes stability are popularly called transcritical bifurcation points [9].

BIFURCATION ANALYSIS

If a model parameter is varied while other parameters are fixed, we can study the fundamental changes in the qualitative behaviour of steady-state solutions and hence find the bifurcation points where a stable steady-state solution changes to an unstable steady-state solution. For example, after linearizing the interaction continuous and partially differentiable functions in the neighbourhood of an arbitrary steady-state solution, we will aim to characterize the stability and instability behaviour of the steady-state solution qualitatively. In this respect, we can explore the standard mathematical technique of the changes in the signs of the eigenvalues to specify if a steady-states solution is either stable or unstable. In some instances, a steady-state solution can be characterized as sitting on the cusp. A systematic calculation where a steady-state changes from a stable node to a saddle can have interesting application in the study of biological interaction which is both attractive and cost-effective.

This numerical bifurcation analysis can be useful in ecological monitoring and stability. For other sophisticated bifurcation methods, see [5].

MATHEMATICAL FORMULATION

In this paper, our core assumptions will border on the linear Malthusian growth phenomenon, logistic population growth and the law of mass action which are central in the formulation of a system of first order differential equations that describe the interspecific competition between two plant species in a Lotka-Voltera sense([4], [6]).

Recently, [4] introduced a mathematical model of plant species interaction in a harsh climate. The model is constructed based on the notion of a summer season when the plants grow, followed by a winter season when there is no growth but when the plants are subject to the effects of evenets such as winter storms, see also [4, 6].

The model of competition has the following form

$$\frac{dy}{dt} = \alpha_1 y(t)(1-\beta_1 y(t)-\gamma_1 z(t)) \tag{3.1}$$

$$\frac{dz}{dt} = \alpha_2 z(t)(1-\beta_2 z(t)-\gamma_2 y(t)) \tag{3.2}$$

Here y and z denote the population of two plant species at time *t*. Here the non-negative constants α_i, β_i, γ_i, i = 1,2 are given respectively, as the intrinsic growth rate, the intra-species competitive parameter and the inter-species competitive parameter. These model equations have four steady-states.

$$y = 0, \quad z = 0 \tag{3.3}$$

$$y = 0, \ z = \frac{1}{\beta_2}, \tag{3.4}$$

$$y = \frac{1}{\beta_1}, \ z = 0 \tag{3.5}$$

$$y = \frac{\beta_2 - \gamma_1}{\beta_1\beta_2 - \gamma_1\gamma_2}, \quad z = \frac{\beta_1 - \lambda_2}{\beta_1\beta_2 - \gamma_1\gamma_2} \tag{3.6}$$

They discussed how to choose the parameter value α_i, β_i, γ_i, i = 1,2 such that the model is reasonable. They noticed that although the variation in α_i, β_i, γ_i, i = 1,2 between the species is quite small, the behaviour of two such close species are much different over a growing season of several years length. The population of one species may die away and would become extinct over a growing season of several years length. They pointed that small perturbation in the environment could have quite devastating and unexpected results for ecosystems. Some steady-states are stable, while some are unstable. For the purpose of this paper, we will consider a simplified version of the above model equations

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(3.1) and (3.2) such as

(3.7) $$\frac{dN_1}{dt} = N_1(sp_1 - bN_1 - cN_2),$$

(3.8) $$\frac{dN_2}{dt} = N_2(d - eN_1 - fN_2),$$

where the initial conditions are $N_1(0) = N_{10} > 0$ and $N_2(0) = N_{20} > 0$. Here, s is a positive constant.

Characterization of Steady-State Solutions

If the rates of change are equated to zero and the interactions functions are solved analytically, we will obtain the four steady-state solutions namely

(3.9) $$N_{1e} = 0,\ N_{2e} = 0$$

(3.10) $$N_{1e} = \frac{sp_1}{b},\ N_{2e} = 0,$$

(3.11) $$N_{1e} = 0,\ N_{2e} = \frac{d}{f},$$ and

(3.12) $$N_{1e} = \frac{p_1 f - cd}{bf - ce},\quad N_{2e} = \frac{bd - csp_1}{bf - ce}$$

provided

(3.13) $$P_1 > \frac{cd}{sf}$$

(3.14) $$p_1 < \frac{bd}{se}$$

(3.15) $$bf > ce.$$

By using a standard mathematical technique of linearization at each steady-state solution, we will consider two interaction functions $F(N_{1e}, N_{2e})$ and $G(N_{1e},N_{2e})$ which are assumed to be partially differentiable and continuous at each steady-state solution. In our context, the mathematical structures of these two functions are

(4.1) $$F(N_{1e}, N_{2e}) = sp_1N_{1e} - bN_{1e}^2 - cN_{1e}N_{2e},$$

(4.2) $$G(N_{1e}, N_{2e}) = dN_{2e} - eN_{1e}N_{2e} - fN_{2e}^2$$

To determine the stability property of each steady-state solution, we differentiated these two functions partially with respect to N_{1e} and N_{2e} and obtain the following Jacobian coefficients such as

(4.3) $$J_{11} = \frac{\partial F}{dN_{1e}} = sp_1 - 2bN_{1e} - cN_{2e},$$

(4.4) $$J_{12} = \frac{\partial F}{dN_{2e}} = -cN_{1e},$$

(4.5) $$J_{21} = \frac{\partial G}{dN_{1e}} = -eN_{2e},$$

(4.6) $$J_{22} = \frac{\partial G}{dN_{2e}} = d - eN_{1e} - 2fN_{2e}$$

Upon evaluating these values of partial derivatives at each steady-state solution, we can set up a Jacobian matrix from which two eigenvalues can be calculated. For example, at the steady-state (0,0), the two eigenvalues which are unique to these model parameters are $\lambda_1 = sp_1$ and $\lambda_2 = d$. where $s > 0$. Provided that these eigenvalues have positive signs, the trivial steady-state is said to be unstable without bifurcation analysis. A similar procedure can be used to characterize the other three steady-state solutions.

Steady-State Bifurcation Analysis

The core aspect of this paper is to conduct a theoretical bifurcation analysis of the trivial steady-state solution which we have determined in the previous section. To study the fundamental changes in the qualitative behaviour of the trivial steady-state solution which is due to a variation to a model parameter at a time where other parameters are fixed, we will tackle this problem analytically using nine different scenarios considering

(1) If $\lambda_1 < 0$ and $\lambda_2 < 0$, then $p_1 < 0$ and $d < 0$ implying that the steady-state solution (0,0) is stable.

(2) If $\lambda_1 < 0$ and $\lambda_2 = 0$, then $p_1 < 0$ and $d = 0$ implying that the steady-state solution (0,0) is neither stable nor unstable, hence it is sitting on the cusp

(3) If $\lambda_1 = 0$ and $\lambda_2 < 0$, then $p_1 = 0$ and $d < 0$ implying that the steady-state solution (0,0) is neither stable nor unstable, hence it is sitting on the cusp

(4) If $\lambda_1 = 0$ and $\lambda_2 = 0$, then $p_1 = 0$ and $d = 0$ implying that the stability of the steady-state solution (0,0) is fully lost

(5) If $\lambda_1 = 0$ and $\lambda_2 > 0$, then $p_1 = 0$ and $d > 0$ implying that the steady-state solution (0,0) is neither sable nor unstable, hence it is sitting on the cusp

(6) If $\lambda_1 > 0$ and $\lambda_2 = 0$, then $p_1 > 0$ and $d = 0$ implying that the steady-state solution (0,0) is neither stable or unstable, hence it is sitting on the cusp.

(7) If $\lambda_1 < 0$ and $\lambda_2 > 0$, then $p_1 < 0$ and $d > 0$ implying that the steady-state solution (0,0) is neither stable nor sitting on the cusp but it is said to be unstable.

(8) If $\lambda_1 > 0$ and $\lambda_2 < 0$, then $p_1 > 0$ and $d < 0$ implying that the steady-state solution (0,0) is neither stable nor sitting on the cusp but it is said to be unstable.

(9) If $\lambda_1 > 0$ and $\lambda_2 > 0$, then $p_1 > 0$ and $d > 0$ implying that the steady-state solution (0,0) is fully unstable. These inequalities are the opposite of those in scenario (1).

DISCUSSION OF RESULTS

In this paper, our calculations show that the steady-state (0,0) is a stable node for $p_1 < 0$ and $d < 0$ and a saddle for $p_1 > 0$ and $d > 0$. On the basis of these results, we have found that as the steady-state persists through the bifurcation points $p_1 = 0$ and $d = 0$, (0,0) will change from a stable node to a saddle.

Our first key result shows that when the daily intrinsic growth rate has a negative value, in this scenario the birth rate of each plant species population will be smaller than the death rate of each plant species population.

Our second key result shows that when the daily intrinsic growth rate has a zero value, there is no difference between the birth rate and the death rate of each plant species population.

Our third key result shows that when the daily intrinsic growth rate has a positive value, in this scenario the birth rate of each plant species population will be bigger than the death rate of each plant species population.

In this study, we have achieved a few novel theoretical bifurcation results which extend a dominant ecological theory. We have used the mathematical concept of bifurcation to find the changes in the qualitative behaviour of the trivial steady-state solution due to a variation of daily intrinsic growth rate of the first plant species when other parameters are fixed. The criteria that the model parameter p_1 must satisfy to enable the stable steady-state solution (0,0) to change to an unstable steady-state have been quantitatively determined and discussed. These results which we have not seen elsewhere are expected to provide insights for ecological functioning, ecological stabilization, and ecological monitoring.

A further numerical bifurcation analysis is proposed to tackle the other problems of analyzing the fundamental changes in the qualitative behavior of the other steady-state solutions when the same intrinsic growth rate is varied.

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