Scientia Africana, Vol. 15 (No. 2), December 2016. Pp 204-210
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CRITERIA FOR STABILITY OF LINEAR DYNAMICAL SYSTEMS WITH MULTIPLE DELAYS BY FIXED POINT

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Received: 05-09-16
Accepted: 10-12-16

ABSTRACT

In this study we considered a linear Dynamical system with multiple delays and find suitable conditions on the systems parameters such that for a given initial function, we can define a mapping in a carefully chosen complete metric space on which the mapping has a unique fixed point. An asymptotic stability theory for the zero solution with necessary and sufficient condition is proved by means of the contraction mapping principle.

***Key words**: Fixed points, Stability, Multiple delays*

INTRODUCTION

For the past 100 years, the traditional method used for the study of stability properties of ordinary, functional and partial differential equations has been Lyapunov method and its variants (Razumikhin-type theorem, Lyapunov-Krasovskii functional techniques). The application of this technique to investigate stability properties of systems with delays has encountered obstacle if the delay is unbounded or if the equation has an unbounded terms see (Burton, 2002; Burton, 2003; Seifert, 1973; Hale, 1977). In recent years, several researchers has sought ways of overcoming this obstacle posed by the traditional methods by using new techniques. In particular, (Burton and Furumochi, 2001;Burton, 2002; Becker and Burton, 2006; Zhang, 2005) and others have noticed that some of these difficulties may vanish or overcome by the use of fixed point theorem. The use of fixed points to establish stability properties has several advantages over Lyapunov method. While Lyapunov method usually require pointwise conditions, fixed point methods requires conditions of an averaging nature, hence can handle various delays or unbounded terms more easily (Burton 2006). Also fixed point methods can be used to determine stability properties of delay problems perturbed by stochastic terms (Luo 2010).

In recent years many researcher have applied the fixed point theorem to study the stability of dynamical systems. (Burton, 2003) examined the equation

$$x'(t) = -a(t)x(t-r)$$

where $a:[0,\infty) \to R$ is continuous and r a positive constant, and established the following result.

Theorem (Burton, 2003). Suppose there exists a constant $\alpha < 1$ such that

$$\int_{t-r}^{t} |a(u+r)|du + \int_{0}^{t} |a(s+r)| e^{-\int_{s}^{t} a(u+r)} \int_{s-r}^{s} |a(u+r)duds| < \alpha$$

for all $t \geq 0$ and $\int_0^t a(s+r)ds \to \infty$ as $t \to \infty$. Then for every continuous initial function $\psi : [-r, 0] \to R$, the solution $x(t, 0, \psi)$ is bounded and tends to zero as $t \to \infty$.

In the work of (Zhang 2005), the following equation

$$x'(t) = -b(t)x(t - \tau(t)) \qquad (1.1)$$

is considered where $b \in C(R^+, R)$ and $\tau \in C(R^+, R^+)$ with $t - \tau(t) \to \infty$ as $t \to \infty$ for all $t \geq 0$ and proved the following results.

Theorem (Zhang 2005). Suppose τ is differentiable, the inverse function g of $t - \tau(t)$ exists, and there exists a constant $\alpha \in (0,1)$ such that for all $t \geq 0$

$$\liminf_{t\to\infty} \int_0^t a(g(u))du |a(g(s))| \int_{s-\tau(s)}^s |a(g(v))| dv ds + \theta(s) \leq \alpha$$

Where $\theta(s) = \int_0^t e^{-\int_s^t a(g(u))du} |a(s)| |\tau'(s)| ds$.

Then, the zero solution of (1.1) is asymptotically stable if and only if $\int_s^t a(g(s))ds \to \infty$ as $t \to \infty$.

(Ding et al. 2010), considered two scalar nonlinear equations with variable delays of the form

$$x'(t) = -a(t)x(t - r_1(t)) + b(t)g(x(t - r_2)),$$
$$x'(t) = -a(t)f(x(t - r_1(t))) + b(t)g(x(t - r_2)) \quad (1.2)$$

where $r_1(t), r_2(t) : [0, \infty) \to [0, \infty), r = \max\{r_1(0), r_2(0)\}, a, b : [0\infty) \to R, f, g : R \to R$ are continuous functions and the following result is established,

Theorem (Ding et al. 2010).Suppose the following conditions are satisfied:

i) $g(0) = 0$, and there exists a constant $L > 0$ so that if $|x|, |y| \leq L$, then $|g(x) - g(y)| \leq |x - y|$

ii) there exist a constant $a \in (0,1)$ and a continuous function $h : [-r, \infty) \to R$ such that

$$\int_{t-r_1(t)}^t |h(s)| ds + \int_0^t e^{-\int_s^t h(u)du} |h(s)| \int_{s-r_1(s)}^s du ds + \int_0^t e^{-\int_s^t h(u)du} \left[|h(s - r_1(s))(1 - r'(s)) - a(s)| \right] ds \leq \alpha$$

iii) $\liminf_{t\to\infty} \int_0^t h(s)ds > -\infty$

then the zero solution of (1.2) is asymptotically stable if and only if

$$\int_0^t h(s)ds \to \infty, \ as\ t \to \infty$$

In this paper we consider the stability of nonlinear systems with time varying delay of the form

$$x'(t) = -a(t)x(t) + \sum_{i=1}^n b_i(t)x(t - h_i(t)) \qquad (1.3)$$

$x(t) = \psi(t)$ for $t \in [m(t_0), t_0]$, $\psi \in C([m(t_0), t_0], R)$

Where $x(t)$ is the state vector, $h_i(t), i = 1,2,\ldots,p$ is time varying delays of the state and $a(t), b(t)$ are continuous functions. In the next section, sufficient conditions for stability by fixed point is

Scientia Africana, Vol. 15 (No. 2), December 2016. Pp 204-210
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presented. This is achieved by fist rewriting equation (1.3) as an integral mapping equation suitable for the contraction mapping principle using the variation of parameter and integration of parts (see Ardjouni and Djoudi, 2014) and references therein.

MAIN RESULT

Let $C(S_1, S_2)$ denote the set of all continuous functions $\varphi : S_1 \to S_2$ with the sup norm, and for each $t \ge 0$, define

$m_i(t_0) = \inf\{t - h_i(t) : t \ge 0\}$, let $m(t_0) = \min\{m_i(t_0), 1 \le i \le p\}$.

and $C(t_0) = C([m(t_0), t_0], R)$ the space all continuous functions with the supremum norm $\|.\|$.

$h_i : R^+ \to R^+$ are all continuous functions such that $t - h_i(t) \ge 0$, $t - h_i(t)$ be strictly increasing and $\lim_{t\to\infty}(t - h_i(t)) = \infty$, and the inverse of $t - h_i(t)$ if it exists be denoted by $g_i(t)$. Let $0 \le b_i(t) \le M_i; i = 1,2,...,n$, and $M = \max\{M_1,...,M_n\}$, hence $0 \le b_i(t) \le M$ and set

$$Q(t) = \sum_{i=1}^{n} b_i(g_i(t)) \qquad (2.1)$$

$$\theta(t) = \sum_{i=1}^{n} \int_0^t e^{-\int_s^t Q(u)du} |b_i| |h_i(s)| ds \qquad (2.2)$$

if $h_i(t)$ is differentiable. For each $(t_0, \psi) \in R^+ \times C(t_0)$ a solution of ( ) through (t_0, ψ) is a continuous function $x : [m(t_0), t_0 + \alpha) \to R^n$ for some positive constant $\alpha > 0$ such that $x(t)$ satisfies ( ) on $[t_0, t_0 + \alpha)$ and $x(s) = \psi(s)$ for some $s \in [m(t_0), t_0]$ and $x(t) = x(t, t_0, \psi)$ denotes the solution. For each $(t_0, \psi) \in R^+ \times C(t_0)$ there exist a unique solution $x(t) = x(t, t_0, \psi)$ of (1.3) defined on $[t_0, \infty)$, for fixed t_0, we defined

$$\|\psi\| = \max\{\psi(s) : m(t_0) \le s \le t_0\}$$

Theorem: Suppose h_i is differentiable, the inverse function $g_i(t)$ of $t - h_i(t)$ exists, and there exists a constants $0 < \alpha < 1$ such that for $t \ge 0$

i. $\lim_{t\to\infty} \inf \int_0^t Q(s)ds > -\infty$

ii.

$$\sum_{i=1}^{n} \int_{t-h_i(t)}^{t} |b_i(g_i(s))| ds + \sum_{i=1}^{n} \int_{t_0}^{t} e^{-\int_s^t Q(u)du} \left[|g_i(s - h_i(s))(1 - h_i'(s)) - b_i(s)| \right] ds$$

$$+ \sum_{i=1}^{n} \int_{t_0}^{t} e^{-\int_s^t Q(u)du} |Q(s)| \left(\int_{s-h_i(s)}^{s} |b_i(g_i(v))| dv \right) ds \le \alpha$$

Then the zero solution of (1.3) is asymptotically stable if and only if

iii. $\lim_{t\to\infty} \int_0^t Q(s)ds \to \infty$

Proof: Suppose that (iii) holds. For each $t_0 \ge 0$, we set

$$K = \sup_{t \ge 0} \left\{ e^{-\int_0^t Q(s)ds} \right\} \qquad (2.3)$$

Let $\psi \in ([m(t_0), t_0], R)$ be fixed and define

$$S = \{x \in C([m(t_0), \infty), R) : x(t) \to 0 \text{ as } t \to \infty,$$

$$x(s) = \psi(s) \text{ for } s \in [m(t_0), t_0]\}$$

Thus S is a complete metric space with metric $\rho(x, y) = \sup_{t \ge m(t_0)} \{|x(t) - y(t)|\}$.

By multiplying both sides of (1.3) by $e^{\int_{t_0}^t Q(s)ds}$, integrating from t_0 to t and then performing integration by parts, we obtain

$$x(t)=\left(x(t_0)-\sum_{i=1}^{n}\int_{t_0-h_i(t_0)}^{t_0}b_i(g_i(s))x(s)ds\right)e^{-\int_s^t Q(u)du}+\sum_{i=1}^{n}\int_{t-h_i(t)}^{t}b_i(g_i(s))x(s)ds$$
$$+\sum_{i=1}^{n}e^{-\int_s^t Q(u)du}\left[g_i(s-h_i(s))(1-h_i'(s))-b_i(s)\right]x(s-h_i(s))ds \tag{2.4}$$
$$+\int_{t_0}^{t}e^{-\int_s^t Q(u)du}Q(s)\left(\sum_{i=1}^{n}\int_{s-h_i(s)}^{s}b_i(g_i(v))x(v)dv\right)ds$$

Using (2.4) we define the operator $P:S\to S$ by $(Px)(t)=\psi(t)$ for $t\in[m(t_0),t_0]$ and

$$(Px)(t)=\left(\psi(t_0)-\sum_{i=1}^{n}\int_{t_0-h_i(t_0)}^{t_0}b_i(g_i(s))\psi(s)ds\right)e^{-\int_s^t Q(u)du}+\sum_{i=1}^{n}b_i(g_i(s))\psi(s)ds$$

$$+\sum_{i=1}^{n}\int_{t_0}^{t}e^{-\int_s^t Q(u)du}\left[g_i(s-h_i(s))(1-h_i'(s))-b_i(s)\right]\psi(s-h_i(s))ds \tag{2.5}$$
$$-\int_{t_0}^{t}e^{-\int_s^t Q(u)du}Q(s)\left(\sum_{i=1}^{n}\int_{t-h_i(t)}^{t}b_i(g_i(v))\psi(v)dv\right)ds$$

for $t\geq 0$. It is clear that $(Px)\in C([m(t_0),\infty),R)$. We now show that $(Px)(t)\to 0$ as $t\to\infty$. To this end, denote the four terms of (2.5) by $I_1, I_2, I_3 and I_4$ respectively.

Let $\phi\in S_\psi$ be fixed. For a given $\in>0$, we choose $T_0>0$ large enough such that $t-h_i(t)\geq T_0$, $i=1,2,\ldots,n$ implies $|\phi(s)|<\in$ if $s\geq t-h_i(t)$. Therefore the second term I_2 in (2.5) satisfies

$$|I_2|=\left|\sum_{i=1}^{n}\int_{t-h_i(t)}^{t}b_i(g_i(s))\phi(s)ds\right|$$
$$\leq\sum_{i=1}^{n}\int_{t-h_i(t)}^{t}|b_i(g_i(s))\phi(s)|ds$$
$$\leq\varepsilon\sum_{i=1}^{n}\int_{t-h_i(t)}^{t}|b_i(g_i(s))|ds<\alpha\varepsilon<\varepsilon$$

Thus $I_2\to 0$ as $t\to\infty$.

Now consider I_3. For the given $\varepsilon>0$, there exists $T_1>0$ such that $s\geq T_1$ implies that $|\phi(s-h_i(s))|<\varepsilon$ for $i=1,2,\ldots,n$. Thus, for $t\geq T_1$, the term I_3 in (2.5) satisfies

$$|I_3|=\left|\sum_{i=1}^{n}\int_{t_0}^{t}e^{-\int_s^t Q(u)du}\left[g_i(s-h_i(s))(1-h_i'(s))-b_i(s)\right]\phi(s-h_i(s))ds\right|$$

$$\leq\sum_{i=1}^{n}\int_{t_0}^{T_1}e^{-\int_s^t Q(u)du}\left|g_i(s-h_i(s))(1-h_i'(s))-b_i(s)\right|\left|\phi(s-h_i(s))\right|ds$$
$$+$$
$$\sum_{i=1}^{n}\int_{T_1}^{t}e^{-\int_s^t Q(u)du}\left|g_i(s-h_i(s))(1-h_i'(s))-b_i(s)\right|\left|\phi(s-h_i(s))\right|ds$$

$$\leq\sup_{\sigma\geq m(t_0)}|\phi(\sigma)|\sum_{i=1}^{n}\int_{t_0}^{T_1}e^{-\int_s^t Q(u)du}\left|g_i(s-h_i(s))(1-h_i'(s))-b_i(s)\right|ds$$
$$+$$
$$\in\sum_{i=1}^{n}\int_{T_1}^{t}e^{-\int_s^t Q(u)du}\left|g_i(s-h_i(s))(1-h_i'(s))-b_i(s)\right|ds$$

By (ii) and (iii) there is a $T_2>T_1$ such that $t\geq T_2$ implies

$$\sup_{\sigma\geq m(t_0)}|\phi(\sigma)|\sum_{i=1}^{n}\int_{t_0}^{T_1}e^{-\int_s^t Q(u)du}\left|g_i(s-h_i(s))(1-h_i'(s))-b_i(s)\right|ds$$

$$=\sup_{\sigma\geq m(t_0)}|\phi(\sigma)|e^{-\int_{T_2}^{tt}Q(u)du}\sum_{i=1}^{n}\int_{t_0}^{T_1}e^{-\int_t^{T_2}Q(u)du}\times\left|g_i(s-h_i(s))(1-h_i'(s))-b_i(s)\right|ds<\varepsilon$$

Scientia Africana, Vol. 15 (No. 2), December 2016. Pp 204-210
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Now apply condition (ii) to have $I_3 < \varepsilon + \alpha\varepsilon < 2\varepsilon$. Thus $I_3 \to 0$ as $t \to \infty$.

Since $x(t) \to 0$ and $t - h_i(t) \to \infty$, for each $\in > 0$, there exists $T_1 > t_0$ such that $s \geq T_1$ implies that $|x(s - h_i(s))| < \in$ for $i = 1, 2, \ldots, n.$ Thus, for $t \geq T_1$ the last term I_4 in (2.5) satisfies

$$|I_4| = \left| \int_0^t e^{-\int_s^t Q(u)du} Q(s) \left(\sum_{i=1}^n \int_{s-h_i(s)}^s b_i(g_i(v))\phi(v)dv \right) ds \right|$$

$$\leq \int_{t_0}^{T_1} e^{-\int_s^t Q(u)du} |Q(s)| \left(\sum_{i=1}^n \int_{s-h_i(s)}^s |b_i(g_i(v))||\phi(v)| \right) ds$$

$$+ \int_{T_i}^t e^{-\int_s^t Q(u)du} |Q(s)| \left(\sum_{i=1}^n \int_{s-h_i(s)}^s |b_i(g_i(v))||\phi(v)| dv \right) ds$$

$$\leq \sup_{\sigma \geq m(t_0)} |\phi(\sigma)| \int_{t_0}^{T_1} e^{-\int_s^t Q(u)du} |Q(s)| \left(\sum_{i=1}^n \int_{s-h_i(s)}^s |b_i(g_i(v))| dv \right) ds$$

$$+ \in \int_{T_1}^t e^{-\int_s^t Q(u)du} |Q(s)| \left(\sum_{i=1}^n \int_{s-h_i(s}^s |b_i(g_i(v))| dv \right) ds$$

By (iii), there exists $T_2 > T_1$ such that $t \geq T_2$ implies

$$\sup_{\sigma \geq m(t_0)} |x(\sigma)| \int_{t_0}^{T_1} e^{-\int_s^t Q(u)du} |Q(s| \left(\sum_{i=1}^n \int_{s-h_i(s)}^s |b_i(g_i(v))| dv \right) ds < \in$$

Appling (ii) we obtain $|I_4| \leq \in + \in \alpha < 2 \in$. Thus, $I_4 \to 0$ as $t \to \infty$. In conclusion $(P\phi)(t) \to 0$ as $t \to \infty$ as required. Hence $Px \in S$. Also by (ii) P is a contraction mapping with contraction constant α. Then for $\phi, \varphi \in S$ and $t \geq t_0$

$$|(P\phi)(t) - (P\varphi)(t)| \leq \sum_{i=1}^n \int_{t-h_i(t)}^t |b_i(g_i(s))||\phi(s) - \varphi(s)| ds$$

$$+ \sum_{i=1}^n \int_{t_0}^t e^{-\int_s^t Q(u)du} \left[|g_i(s - h_i(s))(1 - h_i'(s)) - b_i(s)| \right] |\phi(s - h_i(s)) - \varphi(s - h_i(s))| ds$$

$$+ \sum_{i=1}^n \int_{t_0}^t e^{-\int_s^t Q(u)du} |Q(s)| \left(|b_i(g_i(v))||\phi(v) - \varphi(v)| dv \right) ds$$

$$\leq \left(\sum_{i=1}^n \int_{t-h_i(t)}^t |b_i(g_i(s))| + \sum_{i=1}^n \int_{t_0}^t e^{-\int_s^t Q(u)du} \left[|g_i(s - h_i(s))(1 - h_i'(s))| - |b_i(s)| \right] \right.$$

$$\left. + \sum_{i=1}^n \int_{t_0}^t e^{-\int_s^t Q(u)du} |Q(s)| \left(\int_{s-h_i(s)}^s |b_i(g_i(v))| dv \right) ds \right) \|\phi - \varphi\|$$

By condition (ii), P is a contraction mapping with constant α. Hence by the contraction mapping principle (Smart 1980), P has a unique fixed point x in S which is a solution of (1.3) with $x(t) = \psi(t)$ on $[m(t_0), t_0]$ and $x(t) = x(t, t_0, \psi) \to 0$ as $t \to \infty$. To obtain the asymptotic stability of (1.3), we need to show that the zero solution is stable. For any given $\in > 0 (\delta < \in)$ such that $2\delta K e^{\int_0^{t_0} Q(u)du} + \alpha \in < \in$. If $x(t) = x(t, 0, \psi)$ is a solution of (1.3) with $\|\psi\| < \delta$, then $x(t) = (Px)(t)$ defined in (2.5). We claim that $|x(t)| < \in$ for all $t \geq t_0$. Notice that $|x(s)| < \in$ on $[m(t_0), t_0]$. If there exists $t^* > t_0$ such that $|x(t^*)| = \in$ and $|x(s)| < \in$ for $m(t_0) \leq s < t^*$, then it follows from (2.5)

$$|x(t^*)| \leq |\psi| \left(1 + \sum_{i=1}^n \int_{t-h_i(t)}^t |b_i(g_i(s))| \right) e^{-\int_s^t Q(u)du} + \varepsilon \sum_{i=1}^n \int_{t^*-h_i(t^*)}^{t^*} |b_i(g_i(s))| ds$$

that

$$+ \varepsilon \sum_{i=1}^n \int_{t_0}^{t^*} e^{-\int_s^{t^*} Q(u)du} |g_i(s - h_i(s))(1 - h_i(s)) - b_i(s)| ds$$

S

$$+ \varepsilon \int_{t_0}^{t^*} e^{-\int_s^{t^*} Q(u)du} |Q(s)| \left(\sum_{i=1}^n \int_{s-h_i(s)}^s |b_i(g_i(s))| \right) ds$$

$$\leq 2\delta K e^{\int_0^{t_0} Q(u)du} + \alpha\varepsilon < \varepsilon$$

Which contradicts the definition of t^*. Then $|x(t)| < \varepsilon$ for all $t \geq t_0$, and the zero solution of (1.3) is stable. This shows that the zero solution of (1.3) is asymptotically stable if (iii) holds.

Conversely, suppose (iii) fails. Then by (i) there exists a sequence $\{t_n\}$, $t_n \to \infty$ as $n \to \infty$ such that $\lim_{n\to\infty} \int_0^t Q(u)du = \ell$ for some $\ell \in R$. We may also choose a positive constant J satisfying

$$-J \le \int_0^{t_n} Q(u)du \le J$$

for all $n \ge 1$. To simplify the expression, we define

$$\omega(s) = \sum_{i=1}^{n} \left[\left| g_i(s - h_i(s))(1 - h_i'(s)) \right| - \left| b_i(s) \right| \right] + Q(s) \left(\sum_{i=1}^{n} \int_{s-h_i(s)}^{s} \left| b_i(g_i(v)) \right| dv \right)$$

for all $s \ge 0$. By (ii), we have

$$\int_{t_0}^{t_n} e^{-\int_s^t Q(u)du} \omega(s) ds \le \alpha$$

This gives

$$\int_{t_0}^{t_n} e^{\int_0^s Q(u)du} \omega(s) ds \le \alpha e^{\int_0^{t_n} Q(u)du} \le e^J$$

The sequence $\left\{ \int_0^{t_n} e^{\int_0^s Q(u)du} \omega(s) ds \right\}$ is bounded, so there exists a convergent subsequence. For brevity in notation, we assume that

$$\lim_{n\to\infty} \int_0^{t_n} e^{\int_0^{t_n} Q(u)du} \omega(s) ds = \gamma$$

For some $\gamma \in R^+$ and choose appositive integer m so large that

$$\int_{t_m}^{t} e^{\int_0^s Q(u)du} \omega(s) ds < \frac{\delta_0}{4K}$$

For all $n \ge m$ n, where $\delta_0 > 0$ satisfies $4\delta_0 K e^J + \alpha < 1$. By (i), K in (2.3) is well defined. We now consider the solution $x(t) = (t, t_m, \psi)$ of (1.3) with $\psi(t_m) = \delta_0$ and $\left| \psi(s) \le \delta_0 \right|$ for all $s \le t_m$. We may choose ψ so that $|x(t)| \le 1$ for all $t \ge t_m$ and

$$\psi(t_m) - \sum_{i=1}^{n} \int_{t_m - h_i(t_m)}^{t_m} b_i g_i(s) \psi(s) ds \ge \frac{1}{2} \delta_0$$

It follows that (2.5) with $x(t) = (Px)(t)$ that for $n \ge m$

$$\left| x(t_n) - \sum_{i=1}^{n} \int_{t_n - h_i(t_n)}^{t_n} b_i(g_i(s)) x(s) ds \right| \ge \frac{1}{2} \delta_0 e^{-\int_{t_m}^{t_n} Q(u)du} - \int_{t_m}^{t_n} e^{-\int_s^{t_n} Q(u)du} \omega(s) ds$$

$$= \frac{1}{2} \delta_0 e^{-\int_{t_m}^{t_n} Q(u)du} - e^{-\int_0^{t_n} Q(u)du} \int_{t_m}^{t_n} e^{\int_0^s Q(u)du} \omega(s) ds$$

$$= e^{-\int_{t_m}^{t_n} Q(u)du} \left[\frac{1}{2} \delta_0 - e^{-\int_0^{t_m} Q(u)du} \int_{t_m}^{t_n} e^{\int_0^s Q(u)du} \omega(s) ds \right]$$

$$\ge e^{-\int_{t_m}^{t_n} Q(u)du} \left[\frac{1}{2} \delta_0 - K \int_{t_m}^{t_n} e^{\int_0^s Q(u)du} \omega(s) ds \right]$$

$$\ge \frac{1}{4} \delta_0 e^{-\int_{t_m}^{t_n} Q(u)du} \ge \frac{1}{4} \delta_0 e^{-2J} > 0$$

On the other hand, if the zero solution of (1.3) is asymptotically stable, then $x(t) = x(t, t_m, \psi) \to 0$ as $t \to \infty$. Since $t_n - h_i(t_n) \to \infty$ as $n \to \infty$ and (ii) holds, we have

$$x(t_n) - \sum_{i=1}^{n} \int_{t_n - h_i(t_n)}^{t_n} b_i(g_i(s)) x(s) ds \to 0$$

as $t \to \infty$

Which contradict (2.7). Hence, condition (iii) is a necessary condition for the asymptotic stability of the zero solution of (1.3).

Lyapunov method has been the traditional method used for the study of stability properties of dynamical systems for the past 100 years. The application of this technique to investigate stability properties of systems with delays has encountered obstacle if the delay is unbounded or if the equation has an unbounded terms.To overcome this obstacle posed by the traditional methods, several researchers has explored new techniques. In this study the contraction mapping principle is used to establish necessary and sufficient criteria for the asymptotic stability of linear dynamical system with multiple delays.

Scientia Africana, Vol. 15 (No. 2), December 2016. Pp 204-210
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