A case study of analyzing student teachers' concept images of the definite integral

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Abstract
This paper presents a tool for analyzing student teachers' concept images of the definite integral. The tool shows the basic concepts that underpin the concept of the definite integral and displays them in terms of process and object conceptions within various representations in the context of area under a curve. The use of the tool is exemplified by an analysis of three student teachers' concept images exhibited during an interview that I held with them at the end of one-semester first-year calculus course. The tool can enable mathematics educators to analyze student teachers' concept images of the definite integral. The findings also can orient them in revisiting their teaching strategies in order to improve student teachers' concept images developed during a given period.

Key words: Definite integral, concept images, process conceptions, object conceptions, operational conceptions.

Introduction

Kigali Institute of Education is a Rwandan Higher Learning Institution that implements mainly Pre-service Teacher Education Programmes for teachers of Secondary schools and Teacher Training Colleges. During their Secondary Education, the mathematics student teachers are taught the notions of functions, limits, derivatives and integrals. However in most of secondary schools, the teaching and learning of mathematics is of the format "*definition-theorem-proof-applications*" (Delice & Sevimli, 2010), a format which does not generally favour the conceptual understanding but rather the instrumental one. Mathematics Educators and other researchers have undergone various researches aiming at finding out how calculus can be taught with understanding.

Research on Teaching and Learning Calculus

Four categories of studies deal with the teaching and learning of specific key concepts of calculus. The first category contains studies about the concept of a limit and the notion of infinity (Bezuidenhout, 2001; Dubinsky, Weller, Mcdonald, & Brown, 2005a, 2005b; Sierpinska, 1987). The second category concerns studies about the concept of the derivative (Likwambe & Christiansen, 2008; Ndlovu, Wessels, De Villiers, 2011; Orton, 1983a; Zandieh, 2000). The third category regards studies about the concept of the integral. This category can be subdivided into three subcategories, namely studies regarding the concept of the definite integral (Burn, 1999; Delice & Sevimli 2010; Nguyen & Rebero, 2011; Orton, 1983b; Rasslan & Tall, 2002; Rosken & Rolka, 2007; Sevimli & Delice, 2011), studies regarding the concept of the indefinite integral (Koepf & Ben-Israel, 1994), and studies regarding the Fundamental Theorem of Calculus (Thompson, 1994). The fourth category comprises studies that deal with the whole range of the key concepts of calculus (Artigue, 1991, 1996; Carlson, Oehrtman & Engelke, 2010; Ferrini-Mundy & Gaudard, 1992; Lam, 2009; Naidoo & Naidoo, 2007; Tall, 1992, 1996). However, the boundaries of the above categories are not firmly fixed since a study of a given concept can touch on any other concept for the sake of coherence among the concepts of calculus. Moreover, given that the idea of a function underlies almost all the concepts of calculus, it is important to mention here studies that dealt specifically with the concept of a function (e.g., Aspinwal, Shaw, & Presmeg, 1997; Breidenbach, Dubinsky, Hawks, & Nichols, 1992; Vinner & Dreyfus, 1989). All these researches focus on difficulties met by students and teachers while learning and teaching calculus topics and strive for finding ways and means to overcome them. The current research would like to contribute to this endeavour.

Theoretical frameworks

The tool that I am presenting in this paper rests on my own conceptual analysis of the definite integral and on three existing theoretical frameworks in mathematics education, namely, the notion of the concept image defined by Tall & Vinner (1981), the operational and structural conception described by Sfard (1991), and the Zandieh' s (2000) framework for analyzing students' understanding of the concept of the derivative.

Concept image

According to Tall and Vinner (1981), an individual's concept image for a given concept is "the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes" (p.152). They added that the individual's concept image evolves as the individual becomes more experienced. This experience can be gained from personal observations, other people or books. They used the expression "evoked concept image" to indicate the portion of the concept image that the individual exhibit at a particular time (ibid.).

Operational and structural conceptions

According to Sfard (1991), an individual is said to have a structural conception of a mathematical concept when he or she conceives the mathematical concept as if it is an abstract object in a static way whereas the individual is said to have an operational conception of the same concept when he or she focus the thinking on the processes, algorithms and actions contained in the concept. Sfard (as quoted by Zandieh, 2000, p.107) added that an individual is said to have a pseudo structural conception when the object conception manifested by that individual does not refer to the objects of lower levels and to the processes that led to it. Zandieh (2000), on her side, uses the term "pseudo-object" (p.107) instead of the term pseudostructural object and described it as an object conception which does not refer to or not imply the underlying process. Alternatively to the term pseudostructural conceptions, Sfard and Linchevski (1994) used the term "semantically debased conceptions" (p. 220) because "the new knowledge remains detached from its operational underpinnings and from the previously developed system of concepts" (p. 221). The meaning of the equality symbol as described by Kieran (1981) is of great importance in facilitating the distinction between the two conceptions.

Description of the tool for analysing students' concept images

Inspired by the above mentioned theoretical frameworks, a conceptual analysis of the concept of the definite integral led me to conceive a three dimensional matrix with process-object layers in rows, and with context and representations – where unlike Zandieh, I make a distinction between the two - in columns. In the resulting cells, the various aspects of students' evoked concept images will be illustrated in diagrams by different symbols that I will present in the next section.

To determine the layers of the definite integral, I consulted textbooks and mathematicians. We find variations of the following definition of the definite integral in textbooks; however, all of them represent the same idea. I used the following one in the construction of my tool.

The definite integral of f from a to b is

$$\int_a^b f(x)dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(c_i)\Delta x$$

for any f defined on $[a,b]$ for which the limit exist and is the same for any choice of evaluation points $c_1, c_2, \ldots\ldots, c_n$. (Smith & Minton, 2002)

An analysis of the components of the above formula led me to identify the layers of processes and objects of the definite integral. These process-object layers are given in the first column of the mentioned illustrative diagrams 1 to 4. The other columns contain the context and the representations. Below, I present each of the components of the tools.

Context

A context is a 'text' that comes before, after or with the 'text' expressing the concept at stake. It can be that the evoked context is appropriate or inappropriate with regard to the concept, or it can be that no context is evoked to accompany the expression of concept. In the case of the concept of the definite integral, there are a variety of appropriate contexts. Firstly, there are contexts of a pure mathematical nature: an area under a curve of a function, a volume of a solid with a certain base and height, a length of a curve of a function from a point to another point, an area between two curves from a point to another, a volume of a solid of revolution, and the like. Secondly, the context can be within the probability and statistics domain: the triad probability, probability density function and the random variable; the mean of a random variable with a given probability density function on a given interval. Thirdly, the following triads in physics constitute appropriate contexts for the definite integral: distance-velocity-time, velocity-acceleration-time, Energy-power-time, work-force-distance, and mass-density-length.

Other contexts could be population growth at a given rate in time, disintegration or elimination of a substance (chemical or biological) at a given rate in time. In summing up, an appropriate context for the definite integral involves a triad of a quantity to be determined, a quantity expressing a certain relation between variables, and an interval of the independent variable. An inappropriate context for the definite integral is one which does not fit in the above mentioned structure.

Representations

A representation of a concept is a way used by people to express it. In the case of the concept of the definite integral, at least four representations are likely to be used. These representations are symbols, graphics/diagrams, numbers, and words.

Symbolical representation is the way people combine letters and other symbols specific to mathematics to express the concept at stake. Numbers can also be used in this representation but not for the sake of calculations. These symbols form a specific mathematical language to express the definite integral. In the case of the definite integral, the symbolical representation is $\int_a^b f(x)dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(c_i)\Delta x$

Graphical/visual representation involves graphics or diagrams to express the mathematical concept at stake. In the context of area, the graphical representation of the definite integral is a diagram that represents the area under a given curve for a given interval.

Numerical representation refers to the use of numbers, generally presented in tables after some calculations, to express the mathematical concept at stake.

In the verbal representation, people use words to express the mathematical concept at stake. In this case of the definite integral, a verbal representation can be to express the definite integral as the overall impact on an entity of the rate at which it changes. Obviously, the verbal representation will also depend on the context, so that in relation to distance-velocity-time, it could be the overall distance covered with the changing velocity in the given time interval.

However in practice, none of these representations stand alone; students or other mathematicians often move through representations to express their ideas, and one could argue that it is in the linking of representations that a concept 'comes alive' to us. The tool shows which representations are used by students. However, it fails to show the hierarchy of the representations, that is, which comes before the other in the students' evocation of the concept. This issue is discussed by Sevimli and Delice (2011) in terms of what representation is preferred by the students.

Process-object layers of the definite integral

During my conceptual analysis of the definite integral concept, I identified four process-object layers, namely, the partition, the product, the sum, and the limit. Each process aspect and object aspect for each layer can be described in each representation.

I say that a student demonstrates the process aspect of the evoked concept in a given layer when the student evokes the corresponding operations. Those operations are dividing or partitioning for the partition layer, multiplying for the product layer, adding for the sum layer, and 'limiting' for the limit layer. I understand the process of limiting as the operation of repeating the previous calculations in order to better and better approximate values and finally seeing the value to which the approximate values tend. Notice the use of the gerund to express process in contrast to the names of the process-object layers. Also, the use of the symbol of equality (Kieran, 1981) between the two types of conceptions is an apparent manifestation of the exhibition of both types of conceptions of a given layer.

When the student evokes the operation as said above and the result of the operation, I said that the student possesses both the process and the object conceptions of the evoked concept in the given layer. For some examples in the graphical/visual representation, the objects are subintervals for the partition layer, areas for the product layer, total approximate area for the sum layer, and the exact area for the limit layer.

The possession of the two conceptions implies a highly developed concept image. In fact, to conclude that an individual has a personal concept image in agreement with the formal concept image, the individual has to demonstrate the two conceptions. Demonstrating the object conception only leads to the presumption that the individual has only a pseudo-object conception. The same observation of partial comprehension of the concept can be made for a person that displays only the process conception and appears to have failed to reify that process into an object. This person will struggle to proceed in conceptual learning within calculus because he or she will not have an object on which to act. Therefore, in order to have all the interrelations that link the underlying elements of the concept, the ideal is for the individual to have both process and the object conceptions.

Diagrams to illustrate student teachers' conceptions

Like Zandieh (2000), I will use a three-dimensional matrix in which in the rows of the matrix, I will put the successive processes and objects which form the layers of the definite integral, whereas in the columns I will put the contexts and the representations. To illustrate the evocation of both the process and the object conceptions I will put a shaded circle (●) in the intersection of the row of the concerned layer and the column of the representation in which the evocation is made. When a student teacher mentions only the result of the operation without evoking that operation, which means that the students demonstrated a pseudo-object conception of the evoked layer, I will illustrate this concept image by an empty circle (○) placed in the intersection of the relevant cell. When a student teacher evokes only the operational conception, I will use a crossed circle (⊗).

Research question

The use of the above-described tool helps to answer the following research question: What concept images do mathematics student teachers evoke after a one-semester calculus course?

Research Methodology

For this research, I prepared and conducted a teaching experiment on a first year calculus course at Kigali Institute of Education, Rwanda. Before, during and after the course, I interviewed the eleven student teachers that were registered for the course in the Department of Integrated Science Education. These student teachers were an opportunistic sample for my research as they were students for whom their first year university coincided with my period of collecting data.

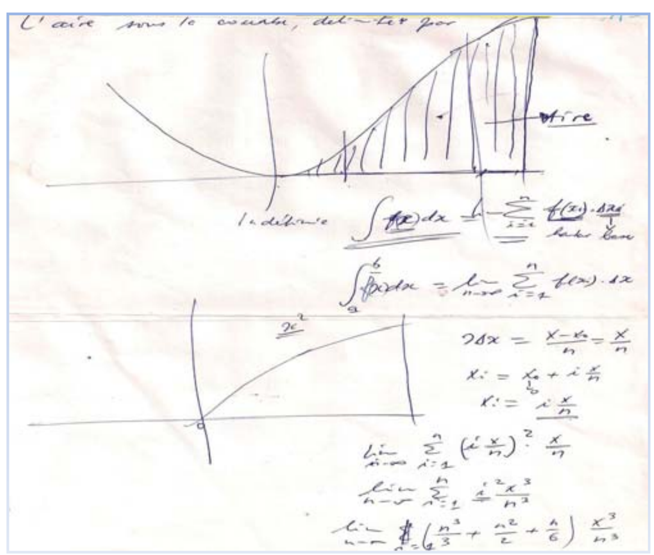
I spent 30 hours on teaching and reviewing functions, limits, and derivatives, and 32 hours on teaching integrals. At the end of the experiment, I held an interview with the student teachers to identify what concept images they would then exhibit at that time. In the interview room, student teachers were given papers, pencils, and a ruler in case they would like to write and draw. All the interviews were audio taped. They were conducted and transcribed in French. Later on, they were translated into English. This is the reason the students' scripts are in French. The analysis shown in this paper is based on students' scripts and translation of transcripts. The coding of the transcripts is based on the meaning of the evoked words in context rather than simply choice of vocabulary. In the next section, I present the findings of this research. At the same time, I exemplify the use of the tool.

Findings

To identify the student teachers, I used pseudonyms of alphabetical letters: B, C, D, E, F, G, H, J, K, M and N. Three of student teachers have been selected to exemplify the analysis and the illustrations that I did using the previously described tool. The student teacher B has been selected as an average student, the student F as a very good one and the student teacher K as the weakest one. The interviewer is identified by the pseudonym FH. In the extracts from the interviews I used the following notations [ ], {.}, (....) and (*with words*) to respectively mean comments or words added by the researcher, short pause, irrelevant or empty words skipped by the researcher and words that were articulated by the interviewee to explain or to change representation.

Concept images evoked by the student teacher B

During the interview that I held at the end of the teaching experiment, student B exhibited the following concept image of the definite integral illustrated in diagram 1.



1. FH: And if you were a teacher and your pupil asked you to explain to him/her what an integral is; what will you tell him/her?
2. B: I will tell him/her that the integral is defined as the area which is under the curve and that area is delimited by the axis x and thus it is the area which delimited by the x-axis and the curve.
3. FH: And how will you explain him/her that it is the area under the curve?
4. B: By sketching a diagram [student B sketches the following diagram].

Here I chose a curve that has a form of a parabola that passes through the point zero. Then considering the part which is shaded, it is that part that I will consider as being the area under the curve.

5. FH: And how will you tell him/her that the integral is the area?

6. B: In view of that this part seems not to have any form with regard to geometric figures, I will explain that this part can be subdivided into geometric figures for which one can calculate areas. Those figures are rectangles. Then calculating area of each rectangle and summing these areas, one obtains the area under the curve. The area being the summation of product of heights and bases of these rectangles {.} and from here, I will explain the two types of integrals, the indefinite and the definite integral. (…)

7. FH: and for the definite integral?

8. B: For the definite integral, one has also the elongated S always with the same function; here we have the lower limit and the upper limit. This will be equal to the limit of the summation of f of xi multiplied by the variation when n tends to infinity. [Bernard writes $\int_a^b f(x)dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i).\Delta x$]. Here concerning this definite integral, one obtains a value because one fixes the limits…the boundaries, the lower limit and the upper limit. In this case the area will be delimited by the curve, the x-axis and the two lines which are determined by the boundaries, thus the vertical lines that pass by the limits.

In line 2, student B mentioned an appropriate context for the definite integral context when he said that the integral is the area under the curve and delimited by the x-axis.

In line 4, student B sketched a parabola, shaded the area under the parabola and used that shaded part to provide explanation in that graphical representation. In line 6, he talked about subdividing this part into geometric figures in order to calculate the areas and he continued by saying that those figures are rectangles ("…subdivided into geometric figures for which one can calculate areas. Those figures are rectangles"). Therefore I put a shaded circle in the cell intersecting the row of partition with the column of graphical representation.

In line 6, student B evoked the process of "summing [adding]" and the resulting object ("area under the curve"). For that reason I put a shaded circle in the cell intersecting the row of the sum layer and the graphical representation, as well as a shaded circle in the cell intersecting the row of the product layer and the column of the graphical representation because, in line 6, student B said that "the area being the product of the heights and the bases" which evokes the process of multiplying and the resulting object of area.

In line 8, student B evoked the formula that expresses the definite integral concept by saying and writing the symbols $\int_a^b f(x)dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i).\Delta x$ and finally said in the same line 8, that "one obtains a value". Therefore, he expressed both the operational and the structural aspects for the layer of limit in symbolical representation. To indicate the conceptions expressed by student B, a shaded circle, (●), is put in the cell intersecting the layer of the limit and the column of the symbolical representation. For the layer of partition student B evoked a pseudo-object by writing the variation Δx without evoking the underpinning operations. Then I put an empty circle (○) in the corresponding cell. In the contrary for the layers of product and sum, Bernard evoked only the operations ("…the summation of f of xi multiplied by the variation…", line 8). Therefore, I put a crossed circle (⊗) in the appropriate cells.

Diagram 1: Concept images of student B evoked after the teaching experiment

Process – Object Layers	Context: Area under a curve ✓			
	Representations			
	Verbal (oral and written words)	Graphical/Visual (geometrical figures)	Numerical (numerical application)	Symbolical (generalization)
Partition		●		○
Product		●		⊗
Sum		⊗		⊗
Limit				●

On this diagram, we notice the evocation of layers in the graphical and the symbolical representations as shown on the student's script.

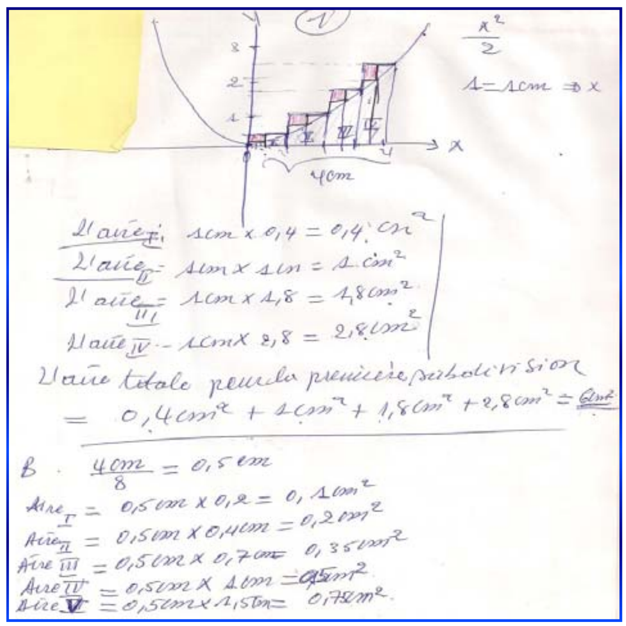
Concept images evoked by the student F

1. FH: If you were a teacher and your pupil asks you what integral is, what would you tell him?
2. F: In explaining the integral I will tell him that the integral is the area; it is a summation of areas from any limit to another one.
3. FH: How will you explain that?
4. F: I could sketch a function, for example x square over two ($\frac{x^2}{2}$) [Ferdinand draws the diagram below].
5. F: As it is practicable, I will start by subdividing the area, for example, the pupil will ask what integral is, I will tell him that it is the area delimited by the curve, from the lower limit which is observed on the x-axis until any upper limit. For example I can point the lower limit as equal to 0 and the upper limit as 4. And then I delimit and I shade the area and I proceed as we learned. I can do the first subdivision.
6. FH: You explain to your pupil who is supposed not to know.
7. F: Yes. Firstly, I will tell him that the integral is area. Secondly I will show him how to calculate this area. (…) but after having done a subdivision one noticed that we subdivided the interval into 4 subintervals with length of 1cm each and I will show how one can calculate. This is called part one (I), part Two (II), part | Three (III), part Four (IV). [Showing the 4 parts on the diagram]. (…)

8. F: (…) Thus I will show how to calculate area I. The base is equal to 1, the height will be for example 0.4. Then area I is equal to the base 1cm times the height which is equal to 0.4 and it will be equal to 0.4 ($1cmx0.4cm = 0.4cm^2$). Area II. Approximately its height is equal to 1 then area II is equal to 1cm, the base does not change, times the height equals to 1cm it will 1 cm ($1cmx1cm = 1cm^2$). Area III. If one projects the height, we see that it will be equal to almost 1.8 thus the area will be equal to 1cm times 1.8 cm square ($1cmx1.8cm = 1.8cm^2$). If one projects for the fourth rectangle one will find the image is equal to almost 2.8, thus the area IV will be equal to 1cm times the height which is almost 2.8cm which will be equal to 2.8 cm square ($1cmx2.8cm = 2.8cm^2$).

9. F: And then I will show him that the sum of our rectangles will be the sum of the parts for which I have found the area above. It will be equal to the total area for the first subdivision is equal to $0.4cm^2 + 1cm^2 + 1.8cm^2 + 2.8cm^2 = 6cm^2$. I will do another subdivision.

10. F: (…) I will explain to him that even if we have calculated this area I will show him that the area is not the exact area because we included the parts above the curve. Thus I will tell him that if one makes another subdivision, in subdividing each interval into two parts and proceeding in the same way of taking heights as the right side, the area will decrease. To show him how the area varies I will shade the area which has been retrenched (lost), I will show that these areas are not included (showing diagram) with respect to the first subdivision. At the first subdivision we had four rectangles, thus at the second subdivision, we will have eight rectangles. To find the area of each rectangle we will take 4cm divided by the number of rectangles equals 8 and then find 0.5 cm which is the base for the 2nd subdivision. Then we will proceed in the same ways taking the height of each rectangle and calculating area of each rectangle.



11. F: The total area for the 2nd subdivision will be equal to sum of areas that we calculated:

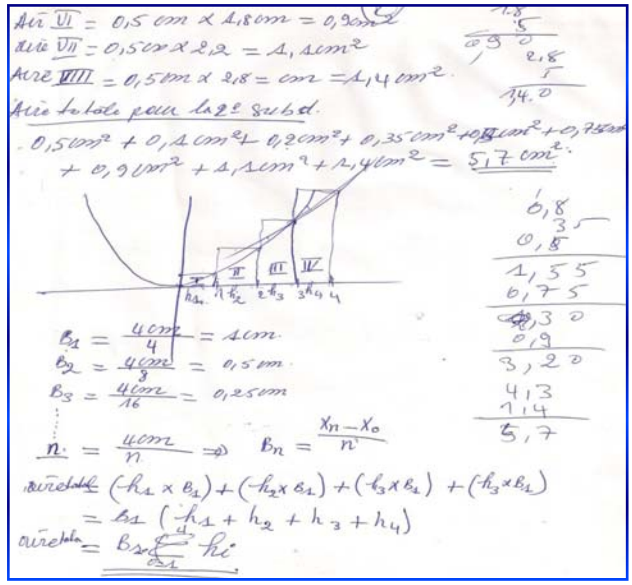
$$0.5cm^2 + 0.1cm^2 + 0.2cm^2 + 0.35cm^2 + 0.5cm^2 + 0.75cm^2 + 0.9cm^2 + 1.1cm^2 + 1.4cm^2 = 5.7cm^2$$

12. Then we will proceed in the same way in subdividing.

F: I will try to compare the area that I found in the 1st subdivision, and the area found in the 2nd subdivision;

13. F: Then I will try to show him how the area decreases as long as we subdivide. for example as one can see, for the 1st subdivision the total area was 6cm^2, for the 2nd subdivision the total area was 5.7cm^2, thus the area is decreasing; because as long as one subdivides, one reduces the supplementary that remains above the curve.

14. F: Then I will try to show how one can find a general formula which can be used to calculate areas. For example, if one considers the 1st subdivision, one sees that one has a curve like this one (see graph on script). As we have said $B_1 = 1cm$, $B_2 = 0.5cm$, and $B_3 = 0.25cm$. One will continue the subdivision at the infinity, for example to the subdivision n the base will be $B_n = \frac{4}{n} cm$, then as a general formula to find the base at nth subdivision we will have $B_n = \frac{x_n - x_0}{n}$ where n is the number of rectangles.



15. F: As we saw that the area tends to decrease as long as one does subdivision and that it decreases approaching the exact area; thus after having done many subdivisions, the pupil will discover that as long as one continues to subdivide the area will tend to the exact area.

16. F: Then I will explain the notion of limit. I will show him that the exact area will be reached when the number of subdivision will be to the infinity. Then I will introduce the notion of limit. I will tell him that the exact area will be equal to the limit when n tends to infinity, thus when the number of subdivision tends to infinity, of the summation of height thus the number of rectangles ($Exactarea = \lim_{n\to\infty} \sum_{i=1}^{n} hi.Bn$). Here also as it is visible, one sees that heights hi times base Bn because it is the nth subdivision; as it is visible on the graph, one sees that hi corresponds to f of xi.

17. F: Thus saying that limit when n tends to infinity of the summation of heights times base at the nth subdivision i varying from 1 to n ($\lim_{n\to\infty} \sum_{i=1}^{n} hi.Bn$) it is the same like to say that it is the limit as n tends to infinity of the summation i varying from 1 to n of f of xi multiplied by the base at the nth subdivision ($\lim_{n\to\infty} \sum_{i=1}^{n} f(xi).Bn$. Now then, it will be the question of calculating the xi. (….).

18. F: Thus I will show him that except the integral one can calculate area under the curve using the application of limits. This is an application of limits.

An analysis similar to the one used while analysing the concept images of the student B leads to the following diagram 2 illustrating the concept images of student F in this concept images we notice the evocation of many layers in various representations.

Diagram 2: concept images of student F evoked after the teaching experiment

Process – Object Layers	Context: Area under a curve ✓			
	Representations			
	Verbal (oral and written words)	Graphical/Visual (geometrical figures)	Numerical (numerical application)	Symbolical (generalization)
Partition		●	●	●
Product		●	●	⊗
Sum		●	●	⊗
Limit		●	⊗	⊗

Concept images evoked by student K at the end of teaching experiment

19. FH: If you were a teacher and your pupil asks to explain to him what integral is, what will you tell him?
20. K: I will tell him that it is the summation of subdivisions of the area.
21. FH: How is that?
22. K: For example if I have the area delimited by a line, I will explain to him first of all that there are two types of integrals, the definite integral and the indefinite integral. For the definite integral I will tell him that it is an integral that possesses limits; a lower limit and an upper limit. Then to find the area, one proceeds to subdivisions to find the actual values. Then the sum of those subdivisions to find the area is the integral.
23. FH: The sum?
24. K: Yes; or the summation.

The student K at the time of the interview did exhibit neither a graphic nor a symbol; I considered that his concept image was evoked in the verbal representation. The words of summation in line 2 and the words "one proceeds to subdivisions" in line 4 refer to the processes of the layers of sum and of partition. Hence I used a crossed circle to illustrate this evoked concept image. Also the context of area he referred to in line 2 was not clearly articulated as a context to introduce the concept of the definite integral. Thus in the corresponding cell I put a tree-like symbol in the illustrative diagram below.

Diagram 3: Concept images of student teacher K evoked after the teaching experiment

Process – Object Layers	Context: Area under a curve 			
	Representations			
	Verbal (oral and written words)	Graphical/Visual (geometrical figures)	Numerical (numerical application)	Symbolical (generalization)
Partition	⊗			
Product				
Sum	⊗			
Limit				

Synopsis of the concept images evoked by the eleven student teachers

An analysis similar to the one used above was applied to the interviews of all the eleven student teachers. The following diagram illustrates their evoked concept images. The first and the last rows show the pseudonyms of the students, the second row shows the context used by the students to explain the definite integral and the remaining rows display the representations in which the layers were evoked.

Diagram 4: Synopsis of 11 student teachers' evoked concept images at the end of the teaching experiment

Student teachers		B	C	D	E	F	G	H	J	K	M	N
Area under a curve		✓	✓	✓	✓	✓	✓	✓	✓		✓	✓
VERBAL	Partition									⊗		
	Product											
	Sum									⊗		
	Limit					○						
GRAPHIC	Partition	●	●	●	●	●	●	●	●		●	●
	Product	●	●	●		●		●				
	Sum	⊗		⊗		●		⊗				⊗

A L	Limit		⊗	●	●	●	○		●			●
N U	Partition					●						
M E	Product					●						
R I	Sum					●						
C A L	Limit					⊗						
S Y M	Partition	○	●	●	●	●	●	●	●		●	○
B O	Product	⊗	⊗	⊗	⊗	⊗	⊗	⊗	⊗		⊗	⊗
L I C	Sum	⊗	⊗		⊗	⊗	⊗	⊗	⊗		⊗	⊗
A L	Limit	●	●	 ○	●	⊗	●	●	●		●	⊗
		B	C	D	E	F	G	H	J	K	M	N

In the next section, I will give the interpretation of the above diagrams 1 to 4.

Interpretation of the findings

Considering all the diagrams above, I notice that the student teachers evoked different concept images. This is in accordance with the understanding of the concept image as described by Tall and Vinner (1981) where they said that the concept image develops as the individual becomes more experienced through personal observations, other people or books. Also, the evoked concept image depends also on the prompting conditions and circumstances. It can also be noted that all the student teachers evoked the context of area under a curve. This was in fact the context that I used during my teaching experiment. Considering the diagram 4, the process and the object of the layer of partition has been evoked by 10 out of 11 student teachers in the graphical and the symbolical representations. This may mean that the student teachers have started to understand that this layer is fundamental for the explanation of the definite integral.

Regarding the layer of limit, 7 out of 11 student teachers evoked the process conception and the object conception in the symbolical representation as manifested by the evocation of the symbol of the definition of the definite integral as a limit of sum.

As said in the introduction of the presentation of findings, student B has been selected as an average student teacher. Like most of the student teachers, the student B evoked his concept image in two representations, namely, the graphical representation and the symbolical representation as it can be seen from the script he produced during the interview. Some words were also used to articulate some layers in these representations but I consider that the main ideas were expressed in the other representations and not in words. According to Anderson et al. (2001), translating from one representation to another is a cognitive process of the category of understanding in the Bloom's taxonomy.

Regarding student H whom I selected as a very good student, I noticed that in addition to using geometrical and symbolical representations in his explanation, he is the only one who evoked in numerical representation the processes and the objects conceptions of the layers of partition, production, and sum.

Concerning student K, at the time of the interview he evoked his concept images in words only; he did not use either a graph or a symbol; he used only the verbal representation. In sum, he exhibited an under developed concept image compared to the ones exhibited by other student teachers.

As noted earlier, all student teachers used words to articulate their conceptions. I considered that their verbalisation was for supporting their conceptions in other representations. This is the reason why there is no illustration in the verbal representation for most of the student teachers, except the student K and the student F. The latter linked the verbal representation to the symbolical representation when he wrote the symbol:

$$Exact\ area = \lim_{n\to\infty}\sum_{i=1}^{n} hi.Bn \quad .$$

The issue of which representation is preferred by students has been raised by Sevimli and Delice (2011). My tool does not allow judging what the preferred representation is. As said above, the translation from one representation to another is considered as an indicator of understanding according to the Bloom's taxonomy (Anderson et al., 2001). However, the issue of mixing representations needs further research to reveal the status of such an evoked concept image.

Finally, the above-displayed diagrams show many empty cells and many crossed circles in the matrices, this means that the student teachers' evoked concept images need further developments. The teacher researcher should take into account this situation to improve his strategies of teaching calculus in order to help new student teachers produce more enriched concept images. Among other things the teacher researcher should diversify the contexts of learning the definite integral.

Conclusion

In this paper, I described a tool for analysing student teachers' concept images of the definite integral. I exemplified its use by analysing the case of eleven student teachers to identify their concept images after a one-semester calculus course. Findings show that the student teachers exhibited different concept images of the definite integral. This is in accordance with the fact that the concept image develops as the individual becomes experienced through a number of observations and circumstances. As an implication to education, basing on the evoked concept images, the teacher researcher may strengthen or reorganise his teaching strategies in order to help the student teachers develop more enriched concept images.

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