Parallel Vector Fields and Einstein Equations of Gravity

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Abstract

In this paper, we prove that no nontrivial timelike or spacelike parallel vector field exists in a region where the gravitational field created by macroscopic bodies and governed by Einstein's equations does not vanish.

In other words, we prove that the existence of such vector fields in a region implies the vanishing of the Riemann curvature tensor in that region.

To prove this statement, we reduce the 4-dimensional problem to a 3-dimensional one. This enables us to use a link existing between the Riemann curvature tensor and the Ricci tensor in a 3-dimensional Riemannian manifold.

1. Introduction

This paper deals with the existence of covariantly constant smooth vector fields in a gravitational field. I call them parallel vector fields because, when those vector fields exist, they are parallel with respect to any smooth curve of space-time. Stephan Waner [2] uses the same terminology.

M. Ray [1] calls them uniform fields, but he states that the construction of a uniform vector field is possible if and only if the Riemann-Christoffel curvature tensor vanishes, which is wrong.

Through an affine connection defined on a differentiable manifold, it is possible to define a parallel displacement of vectors which, at the infinitesimal level, has the same properties as the parallel displacement in linear spaces.

But the integrability of this parallel transport for any vector is an intrinsic property of linear spaces.

On one hand, it is an interesting mathematical problem to try to find for which kind of vectors the parallel transport is an integrable process.

On the other hand, Einstein's equations of General Relativity constitute a highly sophisticated system of partial differential equations, and it is interesting to know which kind of solutions we will obtain by imposing the existence of a parallel vector field.

This paper uses Einstein equations of General Relativity as presented for example in *Landau and Lifchitz* and standard theorems of Differential Geometry as presented, for example in Sternberg [3].

2. Parallel vector fields on Riemannian manifolds

Let M be an n-dimensional Riemannian manifold with metric tensor g given by its components g_{ij} in a local coordinate system $\left(x^{i}\right) 1 \leq i \leq n$.

A vector field X defined by its contravariant components $\left(X^{i}\right)$ or its covariant components $\left(X_{i}\right)$ is called parallel if it is parallel with respect to any smooth curve on M.

Then X will be parallel if it satisfies the following equations

$$X^{i}_{;j} \equiv \frac{\partial X^{i}}{\partial x^{j}} + \Gamma^{i}_{jk} X^{k} = 0 \tag{1}$$

the Γ^{i}_{jk} being the Christoffel symbols associated with g in the coordinate system $\left(x^{i}\right)$.

The covariant form of Eqs. (1) reads

$$X_{i;j} \equiv \frac{\partial X_{i}}{\partial x^{j}} - \Gamma^{k}_{ij} X_{k} = 0 \tag{2}$$

From this equation it is clear that

$$\frac{\partial X_{i}}{\partial x^{j}} - \frac{\partial X_{j}}{\partial x^{i}} = 0 \tag{3}$$

and then, by Poincaré theorem we conclude that :

Any parallel vector field is a gradient field.

From Eqs. (1), it is easy to deduce that a necessary condition for X to be parallel is that

$$R_{ijkl} X^{l} = 0 \tag{4}$$

As consequence of Eqs. (4), we have also the relation

$$R_{jl} X^{l} = 0 \tag{5}$$

R_{ijkl} and R_{jl} are respectively the components of the Riemann curvature tensor and of the Ricci tensor associated with g_{ij}.

3. Einstein equations of gravity, parallel vector field and energy - momentum tensor

In General Relativity, gravitational fields are governed by Einstein's field equations

$$R_{ij} - \frac{1}{2} R\, g_{ij} = \kappa\, T_{ij} \tag{6}$$

where

- $T_{ij} \equiv (p+\varepsilon) u_i\, u_j - p\, g_{ij}$ is the energy-momentum tensor of the macroscopic body creating the gravitational field ;
- $R = g^{ij} R_{ij}$ is the scalar curvature ;
- g_{ij} is the metric tensor of space- time with respect to the coordinates (x^i);
- κ is the gravitational constant whose value depends, of course, on the choice of the units system ;
- p is the pressure and ε is the proper energy density [4] ;
- u_i are the components of the 4-velocity.

The signature of the metric tensor g_{ij} is (1, 3) or equivalently $(+,-,-,-)$.

This signature corresponds to the relation $g_{ij}\, u^i\, u^j = 1$ for the 4-velocity.

The following proposition is the main statement of this paragraph

Proposition 3.1 *If the metric g_{ij} satisfies Einstein Eqs. (6) and admits a non-trivial parallel vector field X, the energy- momentum tensor T_{ij} is identically zero.*

Proof: From Eqs. (6), we can write

$$g^{ij}\left(R_{ij} - \frac{1}{2} R\, g_{ij}\right) = \kappa\, g^{ij}\, T_{ij}$$

with $T_{ij} = (p+\varepsilon) u_i\, u_j - p\, g_{ij}$, and then we obtain

$$R = -\kappa(\varepsilon - 3\,p) \tag{7}$$

Replacing in Eqs. (6) R by its value (7), we get

$$R_{ij} + \frac{1}{2}\kappa(\varepsilon - 3\,p)\, g_{ij} = \kappa(p+\varepsilon) u_i\, u_j - p\, g_{ij} \tag{8}$$

Then

$$R_{ij}\, u^i\, X^j + \frac{1}{2}\kappa\left(\varepsilon - 3\,p\right) g_{ij}\, u^i\, X^j = \kappa\left(p+\varepsilon\right) u_i\, u_j\, u^i\, X^j - \kappa\, p\, g_{ij}\, u^i\, X^j \qquad (9)$$

The equality $u_i\, u^i = 1$ for the 4- velocity and the vanishing of

$$R_{ij}\, u^i\, X^j \equiv \left(R_{ij}\, X^j\right) u^i = 0$$

for a parallel vector field reduce Eq. (9) to

$$\frac{1}{2}\kappa(\varepsilon - 3p) u_j X^j = \kappa(p+\varepsilon) u_j X^j - \kappa p u_j X^j,$$

which yields

$$\left(\varepsilon + p\right) u_j\, X^j = 0 \qquad (10)$$

The meaning of Eq. (10) is that $\varepsilon + p = 0$ or $u_j\, X^j = 0$.

i) Let us assume that $u_j\, X^j = 0$.

Then from Eqs. (8) we write

$$R_{ij}\, X^j + \frac{1}{2}\kappa\left(\varepsilon - 3\,p\right) g_{ij}\, X^j = \kappa\left(p + \varepsilon\right) u_i\, u_j\, X^j - \kappa\, p\, g_{ij}\, X^j$$

Since $R_{ij}\, X^j = u_j\, X^j = 0$, we get

$$\frac{1}{2}\kappa\left(\varepsilon - 3\,p\right) g_{ij}\, X^j = -\kappa\, p\, g_{ij}\, X^j$$

and finally

$$\frac{1}{2}\kappa\left(\varepsilon - p\right) g_{ij}\, X^j = 0 \qquad (11)$$

Since g_{ij} is regular and X is non zero, $\varepsilon - p = 0$.

Generally, for macroscopic bodies, $\varepsilon - 3p \geq 0$ [4].

Therefore, $\varepsilon = p$ implies $\varepsilon = p = 0$ and the energy momentum tensor is identically zero.

ii) If $u_j\, X^j \neq 0$, then $\varepsilon + p = 0$ and $\varepsilon = p = 0$, which implies $T_{ij} = 0$

This ends the proof of the proposition (3.1)

4. Einstein equations' solutions with parallel vector fields

The parallel vector field can be timelike, spacelike or lightlike.
The 3 cases have to be considered separately. In this paper we deal only with the 2 first cases.

In the sequel, if an index can take 4 values 0, 1, 2, 3 it is represented by a latin letter, if it takes 3 values 1, 2, 3, it is represented by a greek letter.

4.1 Solutions with timelike parallel vector field *X*

Since the vector field X never takes the value 0 and is of constant length, we can take X as corresponding to the coordinate x^0 of a local coordinate system $\left(x^0, x^1, x^2, x^3\right)$ of space-time [3]

$$X = \frac{\partial}{\partial x^0}, \quad g\left(X, X\right) = g_{00} = 1 \tag{12}$$

Since $X^0 = 1,\ X^\alpha = 0, \quad 1 \leq \alpha \leq 3,$ Eqs. (1) of parallel vector fields read

$$\Gamma^i_{jk} X^k \equiv \Gamma^i_{j0} = 0 \tag{13}$$

i.e

$$g^{ik}\left(\frac{\partial g_{j0}}{\partial x^k} + \frac{\partial g_{jl}}{\partial x^0} - \frac{\partial g_{0k}}{\partial x^l}\right) = 0 \tag{14}$$

The covariant components of X are given by

$$X_i = g_{ij} X^j = g_{i0} \tag{15}$$

Then since X is a gradient field, there exists a differentiable function f such that

$$g_{i0} = \frac{\partial f}{\partial x^i} \tag{16}$$

Equation (13) reads

$$g^{ik}\left(\frac{\partial g_{jk}}{\partial x^0} + \frac{\partial g_{0k}}{\partial x^j} - \frac{\partial g_{0j}}{\partial x^k}\right) = 0$$

On the other hand, Eqs. (16) give

$$\frac{\partial g_{0k}}{\partial x^j} - \frac{\partial g_{0j}}{\partial x^k} \equiv \frac{\partial^2 f}{\partial x^j \partial x^k} - \frac{\partial^2 f}{\partial x^k \partial x^j} = 0$$

Then (14) reduces to

$$g^{ik} \frac{\partial g_{jk}}{\partial x^0} = 0$$

and since g_{ik} is regular, we obtain

$$\frac{\partial g_{jk}}{\partial x^0} = 0 \tag{17}$$

which means that the metric tensor g_{ij} depends only on the variables x^1, x^2 and x^3 .

This circumstance will allow us to transform our 4-dimensional problem into a 3-dimensional one as follows :

Since $g_{00}=1\neq 0$, the symmetric matrix $\left(g^{\alpha\beta}\right) 1\leq\alpha,\beta\leq 3$ is regular and we can consider the 3-dimensional Riemannian space whose metric tensor $\gamma_{\alpha\beta}$ satisfies

$$g^{\alpha\lambda}\,\gamma_{\alpha\beta}=\delta^{\alpha}_{\beta} \tag{18}$$

We obtain $\gamma_{\alpha\beta}$ by the following considerations :

From $g^{\alpha j}\,g_{j0}\equiv g^{\alpha 0}\,g_{00}+g^{\alpha\lambda}g_{\lambda 0}=\delta^{\alpha}_{0}=0$ and $g_{00}=1$,

we deduce

$$g^{\alpha 0}=-g^{\alpha\lambda}\,g_{\lambda 0} \tag{19}$$

Then the formula

$$g^{\alpha j}\,g_{j\beta}\equiv g^{\alpha 0}\,g_{0\beta}+g^{\alpha\lambda}g_{\lambda\beta}=\delta^{\alpha}_{\beta}$$

takes the form

$$-g^{\alpha\lambda}g_{\lambda 0}g_{0\beta}+g^{\alpha\lambda}g_{\lambda\beta}=g^{\alpha\lambda}\left(g_{\lambda\beta}-g_{0\lambda}g_{0\beta}\right)=\delta^{\alpha}_{\beta}$$

Therefore, the 3-dimensional metric tensor γ is given by

$$\gamma_{\alpha\beta}=g_{\alpha\beta}-g_{0\alpha}g_{0\beta} \tag{20}$$

It satisfies the condition $g^{\alpha\beta}=\gamma^{\alpha\beta}$.

Our aim is to show that the existence of a timelike parallel vector field and the vanishing of the Ricci tensor in space- time implies the vanishing of the Riemann curvature tensor.

- First we note that, from Eqs. (4), $R_{ijkl}\,X^{l}=0$, with $X^{0}=1$ and $X^{\alpha}=0,\quad 1\leq\alpha\leq 3$,

It is clear that

$$R_{ijk0}=0,\,R_{j0}=0,\quad 0\leq i,j,k\leq 3 \tag{21}$$

- It remains to prove that $R_{\alpha\beta\lambda\delta}=0$ for $1\leq\alpha,\beta,\gamma,\delta\leq 3$. We proceed as follows :

Let us denote by $\Lambda^{\alpha}_{\beta\gamma}, P_{\alpha\beta\gamma\delta}, P_{\alpha\beta}$ and *P* respectively the Christoffel symbols, the Riemann curvature tensor, the Ricci tensor and the scalar curvature corresponding to the metric tensor $\gamma_{\alpha\beta}$, the analogous quantities for g_{ij} being denoted respectively by $\Gamma^{i}{}_{jk}$, R_{ijkl}, R_{ij} and *R*.

Then we prove the following equalities:

$$\Lambda^{\alpha}_{\beta\gamma}=\Gamma^{\alpha}_{\beta\gamma},\,R_{\alpha\beta\gamma\delta}=P_{\alpha\beta\gamma\delta},\,R_{\alpha\beta}=P_{\alpha\beta},\,R=P,\quad 1\leq\alpha,\beta,\gamma,\delta\leq 3$$

1. The Christoffel symbols corresponding to $\gamma_{\alpha\beta}$:

$$\Lambda^{\alpha}_{\beta\gamma}=\frac{1}{2}\gamma^{\alpha\lambda}\left(\frac{\partial\gamma_{\alpha\lambda}}{\partial x^{\beta}}+\frac{\partial\gamma_{\gamma\lambda}}{\partial x^{\beta}}-\frac{\partial\gamma_{\beta\gamma}}{\partial x^{\lambda}}\right)$$

Taking account of the fact that $\gamma_{\alpha\beta}=g_{\alpha\beta}-g_{0\alpha}\,g_{0\beta}$ and that $g_{0\alpha}=\dfrac{\partial f}{\partial x^{\alpha}}$, a straightforward calculation gives :

$$\Lambda^{\alpha}_{\beta\gamma}=\frac{1}{2}g^{\alpha\lambda}\left(\frac{\partial g_{\beta\lambda}}{\partial x^{\gamma}}+\frac{\partial g_{\gamma\lambda}}{\partial x^{\beta}}-\frac{\partial g_{\beta\gamma}}{\partial x^{\lambda}}\right)=\Gamma^{\alpha}_{\beta\gamma}$$

2. The Riemann curvature tensor associated with $\gamma_{\alpha\beta}$:

$$P_{\alpha\beta\gamma\delta}=\frac{1}{2}\left(\frac{\partial^{2}\gamma_{\alpha\delta}}{\partial x^{\beta}\partial x^{\gamma}}+\frac{\partial^{2}\gamma_{\beta\gamma}}{\partial x^{\lambda}\partial x^{\delta}}-\frac{\partial^{2}\gamma_{\alpha\gamma}}{\partial x^{\beta}\partial x^{\delta}}-\frac{\partial^{2}\gamma_{\beta\delta}}{\partial x^{\alpha}\partial x^{\gamma}}\right)+\gamma_{\lambda\mu}\left(\Gamma^{\lambda}_{\alpha\delta}\Gamma^{\mu}_{\beta\gamma}-\Gamma^{\lambda}_{\alpha\gamma}\Gamma^{\mu}_{\beta\delta}\right)$$

$$=\frac{1}{2}\left(\frac{\partial^{2}g_{\alpha\delta}}{\partial x^{\beta}\partial x^{\gamma}}+\frac{\partial^{2}g_{\beta\gamma}}{\partial x^{\lambda}\partial x^{\delta}}-\frac{\partial^{2}g_{\alpha\gamma}}{\partial x^{\beta}\partial x^{\delta}}-\frac{\partial^{2}g_{\beta\delta}}{\partial x^{\alpha}\partial x^{\gamma}}\right)+\frac{\partial^{2}f}{\partial x^{\alpha}\partial x^{\delta}}.\frac{\partial^{2}f}{\partial x^{\beta}\partial x^{\gamma}}-\frac{\partial^{2}f}{\partial x^{\beta}\partial x^{\delta}}.\frac{\partial^{2}f}{\partial x^{\alpha}\partial x^{\gamma}}$$

$$+g_{kl}\left(\Gamma^{k}_{\alpha\delta}\Gamma^{l}_{\beta\gamma}-\Gamma^{k}_{\alpha\gamma}\Gamma^{l}_{\beta\delta}\right)-g_{0k}\,g_{0l}\left(\Gamma^{k}_{\alpha\delta}\Gamma^{l}_{\beta\gamma}-\Gamma^{k}_{\alpha\gamma}\Gamma^{l}_{\beta\delta}\right),\text{ with}$$

$0\le k,l\le 3,\quad 1\le\lambda,\mu\le 3$.

Straightforward calculations yield

$$g_{0k}\,g_{0l}\left(\Gamma^{k}_{\alpha\delta}\Gamma^{l}_{\beta\gamma}-\Gamma^{k}_{\alpha\gamma}\Gamma^{l}_{\beta\delta}\right)=\frac{\partial^{2}f}{\partial x^{\alpha}\partial x^{\delta}}.\frac{\partial^{2}f}{\partial x^{\beta}\partial x^{\gamma}}-\frac{\partial^{2}f}{\partial x^{\beta}\partial x^{\delta}}.\frac{\partial^{2}f}{\partial x^{\alpha}\partial x^{\gamma}}$$

Therefore

$$P_{\alpha\beta\gamma\delta}=R_{\alpha\beta\gamma\delta}+\frac{\partial^{2}f}{\partial x^{\alpha}\partial x^{\delta}}.\frac{\partial^{2}f}{\partial x^{\beta}\partial x^{\gamma}}-\frac{\partial^{2}f}{\partial x^{\alpha}\partial x^{\gamma}}.\frac{\partial^{2}f}{\partial x^{\beta}\partial x^{\delta}}-\frac{\partial^{2}f}{\partial x^{\alpha}\partial x^{\delta}}.\frac{\partial^{2}f}{\partial x^{\beta}\partial x^{\gamma}}+\frac{\partial^{2}f}{\partial x^{\alpha}\partial x^{\gamma}}.\frac{\partial^{2}f}{\partial x^{\beta}\partial x^{\delta}}$$

i.e $\qquad P_{\alpha\beta\gamma\delta}=R_{\alpha\beta\gamma\delta}$

3. The Ricci tensor associated with $\gamma_{\alpha\beta}$:

The relations $R_{0\alpha 0\beta}=R_{0\alpha\lambda\beta}=R_{\alpha\lambda 0\beta}=0$ imply,

$R_{\alpha\beta}=g^{\lambda\mu}R_{\lambda\alpha\mu\beta}=g^{\lambda\mu}P_{\lambda\alpha\mu\beta}=P_{\alpha\beta}$.

4. The scalar curvature associated with $\gamma_{\alpha\beta}$:

It is obvious that $P=R$

The following proposition is useful for our purpose and is valid only for 3-dimensional Riemannian manifolds.

Proposition 4.1 *Let $g_{\alpha\beta}$ be the metric tensor of a 3-dimensional Riemannian manifold.*

Let the definition of Ricci tensor

$$R_{\alpha\beta} = g^{\lambda\mu} R_{\lambda\alpha\mu\beta} \tag{22}$$

be considered as a linear system on a 6-dimensional vector space with $R_{1212}, R_{1213}, R_{1223}, R_{1313}, R_{1323}, R_{2323}$ *as unknowns. The system (22) admits the unique solution*

$$R_{\alpha\beta\gamma\delta} = R_{\alpha\gamma} g_{\beta\delta} - R_{\alpha\delta} g_{\beta\gamma} + R_{\beta\delta} g_{\alpha\gamma} - R_{\beta\gamma} g_{\alpha\delta} + \frac{1}{2} R \left(g_{\alpha\delta} g_{\beta\gamma} - g_{\alpha\gamma} g_{\beta\delta} \right)$$

The proof of this proposition is easy but will not be given here.

As a consequence, if the Ricci tensor of a 3-dimensional Riemannian manifold is zero, the Riemann curvature tensor is zero.

We have seen that if a metric g_{ij} satisfies Einstein's equations and admits a parallel vector field, then its Ricci tensor is identically zero. The reduction of the previous 4-dimensional problem to a 3-dimensional one leads to the following conclusion :

If a metric tensor g_{ij} *satisfies Einstein's equations of General Relativity and admits a non trivial timelike parallel vector field, the* g_{ij} *is the metric of a flat space-time.*

4.2 Solutions with Spacelike parallel vector field *X*

In that case, we can consider that $X = \frac{\partial}{\partial x^1}$ with $g(X, X) = -1 = g_{11}$.

Mathematically, there is no inconsistency if we use a non physical reference frame and put $X = \frac{\partial}{\partial x^0}$ with $g(X, X) = g_{00} = -1$.

Then $g^{\alpha\beta}, \quad 1 \le \alpha, \beta \le 3$ will be a regular matrix and $\gamma_{\alpha\beta} = g_{\alpha\beta} + g_{0\alpha} g_{0\beta}$ is its inverse.

The metric tensor of space-time will not depend on x^0 and $\gamma_{\alpha\beta}$ can be considered as the metric tensor of a 3-demensional Riemannian space.

The same reasoning as in the case **1** shows that the existence of spacelike parallel vector field implies that the metric g_{ij} corresponds to a flat space-time.

5. Conclusion

The considerations presented in this paper refer systematically to open submanifolds corresponding to a coordinate system (x^i).

Stricto sensu, the conclusion of our analysis is that space-time admits timelike and spacelike parallel vector fields only on open submanifolds where the gravitational field vanishes.

For example, in the case of a central symmetric gravitational field, if the body responsible of the gravitational field possesses an empty cavity, the region of the world restricted to this cavity constitutes a spatially bounded flat space-time, while the region outside the body corresponds to a nontrivial Riemannian manifold.

The conclusions concerning timelike and spacelike parallel vector fields cannot be extended automatically to the case of lightlike vector fields.

In that case, if we consider $X \equiv \frac{\partial}{\partial x^0}$ as a lightlike parallel vector field, $g\left(X, X\right) = g_{00} = 0$.

Therefore, the submatrix $\left(g^{\alpha\beta}\right)$ $1 \le \alpha, \beta \le 3$ of the inverse matrix g^{ij} of g_{ij} is necessarily singular. The process which allowed us to reduce our 4-dimensional problem to a 3-dimensional one is not valid any more. However, work is in progress concerning this case also.

References

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